Optimal Control for Linear Systems with L^1-norm Cost

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Abstract

We study L^1-optimal stabilization of linear systems with finite and infinite horizons. Main results concern the existence, uniqueness and structure of optimal solutions, and the robustness of optimal cost.

1 Introduction

The L^1-norm of control as a cost attracted much less attention in the mathematical optimal control theory than the L^2-norm or the traditional time-optimal problem. This cost has interesting peculiarities and is very relevant in common situations where optimal behavior should contain periods of the movement with a switched off control. An important motivation is the aerospace navigation where this cost is proportional to the fuel consumption (see [2, 4]).

In this paper, we consider the most simple setting: optimal stabilization of a linear system with control taking values in the unit ball. We will see that the absolute value of optimal control is either 0 or 1 at every moment of time. This means that the cost is equal to the time spent with the activated control, while in the time-optimal problem the cost is the whole time of movement. We study both finite and infinite horizon problems. Main results concern the existence, uniqueness and structure of optimal solutions, and robustness of the optimal cost. The time-optimal control comes as a special case when the horizon is minimal possible to arrive to the target; in this case, the control is always activated.

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2 Finite Horizon

We consider optimal control problem for linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathbb{R}^n,\ u(t) \in U \subset \mathbb{R}^m,\ 0 \le t \le T. \qquad (1)$$

Here $A : \mathbb{R}^n \to \mathbb{R}^n,\ B : \mathbb{R}^m \to \mathbb{R}^n$, are linear maps, $U = \{u \in \mathbb{R}^m : |u| \le 1\}$ is a unit ball. We would like to minimize the cost $J(u) = \int_0^t |u(t)|\, dt$ among all pairs (u, x) which satisfy (1) and boundary conditions $x(0) = x_0,\ x(T) = 0$. The time segment T and the initial condition x_0 are parameters of the problem and we are seeking for optimal solutions for all values of these parameters.

Let $E \subset \mathbb{R}^n$ be an invariant subspace of the operator A and $Bv \in E,\ \forall\, v \in \mathbb{R}^m$. Then E is an invariant subspace of system (1) and the condition $x(T) = 0$ implies that $x(t) \in E,\ 0 \le t \le T$. Minimal invariant subspace of A which contains $\{Bv : v \in \mathbb{R}^m\}$ is the linear hull of $A^i Bv,\ i = 0, 1, \dots, n-1,\ v \in \mathbb{R}^m$. Indeed, A^n is a linear combination of $A^i,\ 0 \le i \le n-1$, according to the Cayley–Hamilton theorem.

We do not lose generality if we restrict our study to this invariant subspace or simply assume that this subspace is the whole $\mathbb{R}^n$. It is why in this paper we always assume that A, B satisfy the Kalman rank condition:

$$span\{A^i Bv : v \in \mathbb{R}^m,\ i = 0, 1, \dots, n-1\} = \mathbb{R}^n. \qquad (2)$$

Admissible controls $u(t),\ 0 \le t \le T$ are measurable vector-functions with values in U. In other words, the space of admissible controls is the unit ball in $L^\infty([0, T]; \mathbb{R}^m)$. We denote the space of admissible controls by $\mathcal{U}_T$ and we equip $\mathcal{U}_T$ with the $*$-weak topology: $u_n \rightharpoonup \bar{u}$ as $n \to \infty$ if and only if

$$\int_0^T \langle u_n(t) - \bar{u}(t), v(t)\rangle\, dt \to 0, \quad \forall\, v \in L^1([0, T]; \mathbb{R}^m).$$

Then $\mathcal{U}_T$ is a compact topological space.

We denote by $x(t; u),\ 0 \le t \le T$, the solution of (1) with the initial condition $x(0) = x_0$. Cauchy formula for solutions of linear ordinary differential equations gives:

$$x(t; u) = e^{tA}\left(x_0 + \int_0^t e^{-\tau A} Bu(\tau)\, d\tau\right).$$

Hence $u \mapsto x(t; u)$ is a continuous linear map from $\mathcal{U}_T$ to $\mathbb{R}^n$.

Let

$$\mathcal{A}_T = \{x_0 \in \mathbb{R}^n : \exists\, u \in \mathcal{U}_T \text{ such that } x(T; u) = 0\};$$

then

$$\mathcal{A}_T = \left\{ \int_0^T e^{-tA} Bu(t)\, dt : u \in \mathcal{U}_T \right\}.$$

It is easy to see that $\mathcal{A}_T,\ T > 0$, is a monotone growing family of compact convex and symmetric with respect to the origin subsets of $\mathbb{R}^n$.

Proposition 1. *The origin of $\mathbb{R}^n$ is an interior point of $\mathcal{A}_T$ for any $T > 0$.*

Proof. Assume that $0 \in \partial\mathcal{A}_T$; the convexity of $\mathcal{A}_T$ implies that there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\langle \xi, x\rangle \leq 0$ for any $x \in \mathcal{A}_T$. Hence $\langle \xi, \alpha x\rangle \leq 0$ for any $\alpha > 0,\ x \in \mathcal{A}_T$. On the other hand, the rank condition (2) implies that

$$\bigcup_{\alpha \geq 1} \alpha\mathcal{A}_T = \left\{ \int_0^T e^{-tA} Bu(t)\, dt : u \in L^\infty([0,T];\mathbb{R}^m) \right\} = \mathbb{R}^n.$$

This contradiction completes the proof.

Corollary 1. $\mathcal{A}_T \subset int\mathcal{A}_{T+\varepsilon},\ \forall\, \varepsilon > 0.$

Indeed, $\mathcal{A}_{T+\varepsilon} = \mathcal{A}_T + e^{-TA}\mathcal{A}_\varepsilon$.

Proposition 2. *Let $x_0 \in \mathcal{A}_T$; then there exists $\bar{u} \in \mathcal{U}$ such that*

$$J(\bar{u}) = \min\{J(u) : u \in \mathcal{U}_T,\ x(T;u) = 0\}.$$

Proof. The desired result follows from the compactness of $\mathcal{U}_T$ and low semi-continuity of the functional $\mathcal{U}_T \to \mathbb{R}$. $\square$

The optimal control $\bar{u}$ may be not unique even for very simple systems. Indeed, let us consider the system $\begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = u \end{cases}$, which describes a free particle on the line controlled by the external force.

Let $u \in \mathcal{U}_T,\ x(0;u) = x_0,\ x(T;u) = 0$, where $x_0 = (x_0^1, x_0^2)$. We have

$$x_0^2 = -\int_0^T u(t)\, dt, \quad x_0^1 = \int_0^T tu(t)\, dt, \tag{3}$$

$$J(u) = \int_0^T |u(t)|\, dt \geq \left| \int_0^T u(t)\, dt \right| = |x_0^2|.$$

Moreover, $J(u) = |x_0^2|$ if and only if control $u(t),\ 0 \leq t \leq T$, does not change sign. Hence any function from $\mathcal{U}_T$ which does not change sign is optimal for some initial condition; namely, for the initial condition (3).

Now assume that $\varepsilon \le u(t) \le 1-\varepsilon$, for some $\varepsilon > 0$ and let $v \in L^\infty([0,T];\mathbb{R})$ has zero average and zero first momentum:

$$\int_0^T v(t)\,dt = \int_0^T tv(t)\,dt = 0,$$

then $u+sv$ is optimal control for our problem with the same initial condition for any s sufficiently close to 0.

Coming back to the general case, we are going to characterize optimal controls by the Pontryagin maximum principle (see [3, 1] and to impose natural conditions on the pair (A,B) which guarantee the uniqueness and simple structure of the optimal control. Pontryagin maximum principle is a universal necessary optimality condition but it is also sufficient if the system is linear and if the constrains and the cost are convex, i. e. in our framework.

In order to formulate the maximum principle, we introduce the Hamiltonians of the system. Normal Hamiltonian:

$$h_u(p,x) = pAx + pBu - |u|,$$

where $p \in \mathbb{R}^{n*}$ is a row, $p = (p_1, \dots, p_n)$, $px = p_1x^1 + \dots + p_nx^n$ (the product of a row and a column), and abnormal Hamiltonian: $h_u^0 = pAx + pBu$.

Both Hamiltonians depend on the parameter u. The Hamiltonian system for both Hamiltonians reads:

$$\begin{cases} \dot{p} = -pA, \\ \dot{x} = Ax + Bu. \end{cases} \tag{3}$$

We say that $u \in \mathcal{U}_T$ is a normal extremal control if $x(T;u) = 0$ and there exists a solution of the system $\dot{p} = -pAx$ such that

$$h_{u(t)}(p(t), x(t;u)) = \max_{v \in U} h_v(p(t), x(t;u)), \quad \forall\, t \in [0,T].$$

We say that u is an abnormal extremal control if there exists a nonzero solution of the system $\dot{p} = -pAx$ such that

$$h^0_{u(t)}(p(t), x(t;u)) = \max_{v \in U} h^0_v(p(t), x(t;u)), \quad \forall\, t \in [0,T].$$

The Pontryagin maximum principle states that $u \in \mathcal{U}_T$, is optimal if and only if $x(T;u) = 0$ and u is a normal or abnormal extremal control.

Abnormal extremal controls do not depend on the cost J and are actually extremal controls of the very well studied time-optimal problem. We briefly summarize there properties and devote the rest of the paper to the normal extremal controls.

Theorem 1. *Let $u \in \mathcal{U}_T,\ x(T;u) = 0$. The control u is an abnormal extremal control if and only if $x_0 \notin \mathcal{A}_t,\ \forall\, t < T$. Moreover, any abnormal extremal control u has the following properties:*

1. *u is a unique solution of the equation $x(T;v) = 0,\ v \in \mathcal{U}_T$;*

2. *u is a piece-wise analytic vector-function with isolated jump discontinuities;*

3. *$|u(t)| = 1,\ 0 \le t \le T$, and $u(t+0) = -u(t-0)$ in the discontinuity points.*

Proof. We start from the proof of the statements *2.* and *3.* The covector $p(t)$ from the maximality condition has a form $p(t) = p_0 e^{-tA}$, it is an analytic vector-function. Moreover, $u(t)^* = \frac{1}{|p(t)B|} p(t)B$, where * is the transposition, it transforms columns in rows and vice versa. The vector-function $p(t)B$ is analytic, it has only isolated zeros if it is not identically zero. The identically zero case is excluded by the Kalman rank condition.

We have shown that u is piece-wise analytic and takes values in the sphere ∂U. It remains to understand the structure of singularities. Let $p(\bar t)B = 0$ and $k = \min\{i : \frac{d^i}{dt^i} p(\bar t) \neq 0\}$. We have:

$$p(t)B = (t - \bar t)^k e + O\left((t - \bar t)^{k+1}\right).$$

Then:

$$u(t) = sign(t - \bar t)^k \frac{e}{|e|} + O(t - \bar t).$$

Now we prove the first statements of the theorem. First of all, the Pontryagin maximum principle for the time-optimal problem implies that any time-optimal control must be an abnormal extremal control in our sense. Moreover, the equality $x_0 = -\int_0^T e^{-tA} Bu(t)\,dt$ implies:

$$p(0)x_0 = -\int_0^T p(t)Bu(t)\,dt = -\int_0^T |p(t)B|\,dt < -\int_0^T p(t)Bv(t)\,dt,$$

for any $v \in \mathcal{U}_T$ which differs from u on positive measure subset of $[0,T]$, and

$$p(0)x_0 < -\int_0^{t_1} p(t)Bv(t)\,dt, \quad \forall\, t_1 < T, v \in \mathcal{U}_T.$$

It follows that u is a unique control which transfers x_0 to the origin in time T and that $x_0 \notin \mathcal{A}_{t_1},\ \forall\, t_1 < T$. $\square$

Now we analyse the maximality condition (3) for the normal extremal controls. We have:

$$h_{u(t)}(p(t),x) = \begin{cases} p(t)Ax + |p(t)B| - 1, & \text{if } |p(t)B| \geq 1; \\ p(t)Ax, & \text{if } |p(t)B| \leq 1. \end{cases}$$

If $|p(t)B| > 1$, then $u(t)^* = \frac{1}{|p(t)B|}p(t)B$ like in the abnormal case; if $|p(t)B| < 1$, then $u(t) = 0$. If $|p(t)B| = 1$, then we can only say that $u(t)^* = \frac{s}{|p(t)B|}p(t)B$, for some $s \in [0,1]$.

Recall that $p(t) = p(0)e^{-tA}$; then either $|p(t)B| \equiv 1$ or the equation $|p(t)B| = 1$ with unknown $t \in [0,T]$ has only a finite number of solutions. If the first possibility is not realized, then optimal control is piece-wise analytic with jump discontinuities: it simply switches from $\frac{1}{|p(t)B|}(p(t)B)^*$ to 0 and back when $|p(t)B| - 1$ changes the sign.

Moreover, optimal control is unique in this case as we see in the next proposition.

Proposition 3. *Assume that for any $p^0 \in \mathbb{R}^{n*}$ there exists $t_0 \in \mathbb{R}$ such that $|p^0 e^{tA}| \neq 1$; then optimal control is unique for any T, x_0.*

Proof. Optimal control is either normal or abnormal extremal control. As we know (see Th. 1), abnormal extremal control is a unique control which transfers x_0 to the origin in time T. It corresponds to the pairs T, x_0 such that $x_0 \in \partial\mathcal{A}_T$. For all other pairs T, x_0 optimal controls are normal.

Assume that $u, w \in \mathcal{U}_T$ are two optimal controls for the same T, x_0. Then

$$\int_0^T e^{-tA}Bu(t)\,dt = \int_0^T e^{-tA}Bw(t)\,dt, \qquad \int_0^T |u(t)|\,dt = \int_0^T |w(t)|\,dt. \quad (4)$$

Moreover, there exists $p^0 \in \mathbb{R}^{n*}$ such that

$$p^0 e^{-tA}Bu(t) - |u(t)| = \max_{v \in U}(p^0 e^{-tA}Bv - |v|), \quad \forall\, t \in [0,T], \quad (5)$$

and u is a unique element of U, which satisfies maximality condition (5).

In particular,

$$p^0 e^{-tA}Bu(t) - |u(t)| > p^0 e^{-tA}Bw(t) - |w(t)|$$

for any $t \in [0,T]$ such that $w(t) \neq u(t)$ and $|p_0 e^{-tA}B| \neq 1$.

It follows that $u(t) = w(t)$ almost everywhere. $\square$

The assumption of Proposition 3 is violated for the considered above system $\begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = u \end{cases}$, where, as we know, optimal control is not unique.

Lemma 1. *Let $m = 1$. the assumption of Proposition 3 is valid if and only if* $\det A \neq 0$.

Proof. We have $B : \mathbb{R} \to \mathbb{R}^n,\ B1 = b \in \mathbb{R}^n$. In this case, the assumption of Proposition 3 can be rewritten as follows: $p^0 e^{tA} b \not\equiv const$ for any $p^0 \in \mathbb{R}^n * \setminus \{0\}$ or, equivalently, $p^0 A e^{tA} b \not\equiv 0$. In other words, this assumption is valid if and only if $A(span\{e^{tA}b : t \in \mathbb{R}\}) = \mathbb{R}^n$.

The Kalman rank condition guaranties that $span\{e^{tA}b : t \in \mathbb{R}\} = \mathbb{R}^n$. Hence the assumption of Proposition 3 is valid if and only if the operator A is not degenerate.

Unfortunately, the last test does not work for $m \geq 2$. Here is a simple counterexample for $n = m = 2$. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Complex numbers are identified with linear operators on $\mathbb{C} = \mathbb{R}^2$ acting by the complex multiplication. We set $A = i,\ B = 1$; then $e^{tA}B = e^{ti}$. If $p^0 = (1, 1) \in \mathbb{R}^2*$, then $|p^0 e^{tA} B| = |e^{ti}| = 1$.

To make things working for any m, we require the hyperbolicity of the operator A. Recall that A is called hyperbolic if any eigenvalue of A has a nonzero real part. A key property of the hyperbolic operators is as follows: if A is hyperbolic and $b \in \mathbb{R}^n \setminus \{0\}$, then either $|e^{tA}b| \to \infty$ as $t \to +\infty$ or $|e^{tA}b| \to \infty$ as $t \to -\infty$. In the first case, $|e^{tA}b| \to 0$ as $t \to -\infty$ and in the second case, $|e^{tA}b| \to 0$ as $t \to +\infty$.

Lemma 2. *If A is hyperbolic, then the assumption of Proposition 3 is valid and optimal controls are unique.*

Proof. Let $e_1, \ldots, e_m$ be an orthonormal basis of $\mathbb{R}^m$, $b_i = Be_i,\ i = 1, \ldots, m$. We have: $|p^0 e^{tA} B|^2 = \sum\limits_{i=1}^{m} |p^0 e^{tA} b_i|^2$. The key property of the hyperbolic operators implies that $|p^0 e^{tA} B|^2 = 0$ if and only if $p^0 e^{tA} b_i = 0,\ i = 1, \ldots, m$. The differentiation of the last identities gives: $p^0 A^j b_i = 0, \forall\, i, j$, but this is not possible for $p^0 \neq 0$ due to the Kalman rank condition. $\square$

Let $x_0 \in \mathcal{A}_T$ and

$$\mu_T(x_0) = \min\{J(u) : u \in \mathcal{U}_T,\ x(T; u) = 0\}$$

be the optimal cost. In what follows, we are interested not only in the dependence of $\mu_T(x_0)$ on T and x_0 but also on the matrices A, B and we use notations $\mathcal{A}_T = \mathcal{A}_T(A, B),\ \mu_T(x_0) = \mu_T(x_0; A, B),\ (T, x_0; A, B) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \mathbb{R}^{nm}$.

Theorem 2. *The set*

$$\{(T, x_0; A, B) : x_0 \in int\mathcal{A}_T(A, B)\} \tag{6}$$

is an open subset of $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \mathbb{R}^{nm}$ *and the function*

$$(T, x_0; A, B) \mapsto \mu_T(x_0; A, B)$$

is continuous on (6).

Proof. In what follows, we assume that $\mathcal{U}_T \subset \mathcal{U}_{T'}$ for any $T' > T$, where $u \in L^\infty([0,T];\mathbb{R}^m)$ is extended to the interval $(T, T']$ by zero.

First we prove the openness of set (6). Let $x_0 \in int\mathcal{A}_T(A,B)$; take a simplex with vertices $y_0, \dots, y_n \in \mathcal{A}_T(A,B)$ such that x_0 is an interior point of this simplex. We have:

$$y_i = \int_0^T e^{-tA} B u_i \, dt, \quad u_i \in \mathcal{U}_T, \ i = 0, \dots, n.$$

If T', A', B' are close to T, A, B, then $y_i' = \int_0^{T'} e^{-tA'} B' u_i \, dt$ are close to $y_i, \ i = 0, \dots, n$, and x_0 is an interior point of the simplex with vertices y_i', i. e. $x_0 = \sum\limits_{i=0}^{n} \alpha_i y_i$, where $\alpha_i \in (0,1)$. Hence

$$x_0 = \int_0^{T'} e^{-tA} B \sum_{i=0}^{n} \alpha_i u_i(t) \, dt$$

and $x_0 \in int\mathcal{A}_{T'}(A', B')$.

The next step is the continuity of $\mu_T(x_0; A, B)$ with respect to T. First of all, $T \mapsto \mu_T(x_0)$ is a monotone decreasing function. Indeed, let $u \in \mathcal{U}_T$ be the minimizing control, $\mu_T(x_0) = J_T(u)$; then

$$\mu_{T'}(x_0) \le J_{T'}(u) = J_T(u) = \mu_T(x_0), \quad \forall\, T' \ge T.$$

The monotony implies the existence of the right and left limits:

$$\lim_{T' \searrow T} \mu_{T'}(x_0) = \mu_{T+0}(x_0), \quad \lim_{T' \nearrow T} \mu_{T'}(x_0) = \mu_{T-0}(x_0).$$

It is easy to see that $\mu_{T+0}(x_0) = \mu_T(x_0)$. Indeed, let $T_n \searrow T$ and $\mu_{T_n}(x_0) = J_{T_n}(u_n)$ We may assume that $u_n \rightharpoonup \bar{u}$ as $n \to \infty$ by taking a subsequence if necessary; then $\lim\limits_{n\to\infty} \mu_{T_n}(x_0) = J_T(\bar{u}) \ge \mu_T(x_0)$. On the other hand, $\mu_{T+0}(x_0) \le \mu_T(x_0)$; hence $\mu_{T+0}(x_0) = \mu_T(x_0)$.

It remains to prove that $\mu_{T-0}(x_0) = \mu_T(x_0)$. This is more complicated. Let $\tilde{u} \in \mathcal{U}_T$ be the optimal control: $J(\tilde{u}) = \mu_T(x_0)$ and $-\int_0^T e^{-tA} B\tilde{u}(t)\, dt = x_0$. We have to show that there exists an arbitrarily close to $\tilde{u}$ in the norm L^1 control $u' \in \mathcal{U}_T$ such that $-\int_0^{T'} e^{-tA} B\tilde{u}'(t)\, dt = x_0$ for some $T' < T$.

Recall that $\mathcal{A}_T$ is a full-dimensional convex and symmetric with respect to the origin compact. Moreover, $x_0 \in int\mathcal{A}_T$; hence $x_0 \in \nu_0\mathcal{A}_T$ for some $\nu_0 \in (0,1)$. Let $v_0 \in \mathcal{U}_T$ be such that $-\nu \int_0^T e^{-tA}Bv_0(t)\,dt = x_0$. Then

$$-\int_0^T e^{-tA}B(s\nu_0 v_0(t) + (1-s)\tilde{u}(t)\,dt = x_0, \quad \forall\, s \in [0,1].$$

Let $v = s\nu_0 v_0(t) + (1-s)\tilde{u}(t)$, where $s > 0$ is so small that the norm L^1 of $v - \tilde{u}$ is smaller than a preliminary chosen $\varepsilon > 0$.

By construction, $v \in \nu\mathcal{U}_T$ for some $\nu \in (0,1]$ and $-\int_0^T e^{-tA}Bv(t)\,dt = x_0$.

Let $\tau \in [0,T)$, we set $\mathcal{A}_{\tau,T} = \left\{\int_\tau^T e^{-tA}Bu(t)\,dt : u \in \mathcal{U}_T\right\}$; then $\mathcal{A}_{\tau,T}$ is a full-dimensional convex and symmetric with respect to the origin compact. We have:

$$x_0 + \int_0^\tau e^{-tA}Bv(t)\,dt = -\int_\tau^T e^{-tA}Bv(t)\,dt \in \nu\mathcal{A}_{\tau,T}.$$

Hence $-\int_\tau^T e^{-tA}Bv(t)\,dt \in int\mathcal{A}_{\tau,T}$. I claim that any interior point of $\mathcal{A}_{\tau,T}$ belongs to $\mathcal{A}_{\tau,T_\tau}$ for some $T_\tau < T$. This fact can be proved in the same way as the openness of set (6), We leave details to the reader.

Now we know that $x_0 + \int_0^\tau e^{-tA}Bv(t)\,dt \in \mathcal{A}_{\tau,T_\tau}$ for some $T_\tau \in [\tau,T)$. Hence

$$x_0 + \int_0^\tau e^{-tA}Bv(t)\,dt = -\int_\tau^{T_\tau} e^{-tA}Bv_\tau(t)\,dt,$$

for some $v_\tau \in \mathcal{U}_T$. We set

$$u_\tau(t) = \begin{cases} v(t), & \text{if } 0 \le t \le \tau; \\ v_\tau(t), & \text{if } \tau < t \le T. \end{cases}$$

Then $-\int_0^{T_\tau} e^{-tA}Bu_\tau(t)\,dt = x_0$ and u_τ tends to v in the norm L^1 as $\tau \to T$. These relations complete the proof of the continuity of $\mu_T(x_0;A,B)$ with respect to T.

The continuity with respect to T will help us to prove continuity with respect to all variables. Let $(T_n, x_n; A_n, B_n) \to (T, x_0; A, B)$ as $n \to \infty$. We will separately prove the inequalities:

$$\mu_T(x_0;A,B) \le \liminf_{n\to\infty} \mu_{T_n}(x_n;A_n,B_n) \quad \mu_T(x_0;A,B) \ge \limsup_{n\to\infty} \mu_{T_n}(x_n;A_n,B_n).$$

The first inequality is easy. Let

$$J_{T_n}(u_n) = \mu_{T_n}(x_n;A_n,B_n), \quad -\int_0^{T_n} e^{-tA}Bu_n(t)\,dt = x_n.$$

Let u_{n_k} be a convergent subsequence, $u_{n_k} \rightharpoonup \bar{u}$ as $k \to \infty$, then

$$J_T(\bar{u}) = \lim_{k\to\infty} \mu_{T_{n_k}}(x_{n_k}; A_{n_k}, B_{n_k}), \quad -\int_0^T e^{-tA} B\bar{u}(t)\, dt = x_0$$

and $\mu_T(x_0; A, B) \le J_T(\bar{u})$. Hence $\mu_T(x_0; A, B) \le \liminf_{n\to\infty} \mu_{T_n}(x_n; A_n, B_n)$.

Let us prove the second inequality. Take a small $\delta > 0$; let

$$J_{T-\delta}(u_\delta) = \mu_{T-\delta}(x_0; A, B), \quad -\int_0^{T-\delta} e^{-tA} Bu_\delta(t)\, dt = x_0.$$

We set $y_n = x_n + \int_0^{T-\delta} e^{-tA} Bu\delta(t)\, dt$; then $|T_n - T| < \delta$ and $y_n \in \mathcal{A}_{T-\delta, T_n}$ for all big enough n. In particular $y_n = -\int_{T-\delta}^{T_n} e^{-tA} Bv_n(t)\, dt$ for some $v_n \in \mathcal{U}_{T+\delta}$. Let

$$\hat{u}_n(t) = \begin{cases} u_\delta(t), & \text{if } 0 \le t \le T - \delta; \\ v_n(t), & \text{if } T - \delta < t \le T_n. \end{cases}$$

Then $-\int_0^{T_n} e^{-tA} B\hat{u}_n(t)\, dt = x_n$ and

$$\mu_{T_n}(x_n; A_n, B_n) \le J(\hat{u}_n) \le \mu_{T-\delta}(x_0; A, B) + 2\delta.$$

Hence $\limsup_{n\to\infty} \mu_{T_n}(x_n; A_n, B_n) \le \mu_{T-\delta}(x_0; A, B) + 2\delta$, for any $\delta > 0$. It remains to go to the limit as $\delta \to 0$. $\square$

3 Infinite Horizon

Let

$$\mathcal{U}_\infty = \{u \in L^1([0,\infty); \mathbb{R}^k) : |u(t)| \le 1,\ \forall\, t \ge 0\}, \quad J_\infty(u) = \int_0^\infty |u(t)|\, dt,$$

$\mathcal{A}_\infty = \bigcup_{T>0} \mathcal{A}_T$; then $\mathcal{A}_\infty$ is an open convex subset of $\mathbb{R}^n$. Given $x_0 \in \mathcal{A}_\infty$, we set

$$\mu_\infty(x_0) = \inf \left\{ J_\infty(u) : u \in \mathcal{U}_\infty,\ \lim_{t\to\infty} x(t; u) = 0 \right\}.$$

Proposition 4. $\mu_\infty(x_0) = \inf_{T>0} \mu_T(x_0)$.

Proof. We have: $\mu_T(x_0) = \inf\{J_T(u) : u \in \mathcal{U}_T,\ x(T; u) = 0\}$. The inclusion $\mathcal{U}_T \subset \mathcal{U}_\infty$ implies that $\mu_T(x_0) \ge \mu_\infty(x_0), \forall\, T > 0$. On the other hand, for any $\varepsilon > 0$ and $u \in \mathcal{U}_\infty$, the relation $x(t; u) \to \infty$ $(t \to \infty)$ implies there existence of $T > 0$ such that $x(T-\varepsilon; u) \in \mathcal{A}_{T-\varepsilon,T}$. It follows that $x(T-\varepsilon; u) = -\int\limits_{T-\varepsilon}^{T} e^{-tA}Bv(t)\,dt$ for some $v \in \mathcal{U}_T$. We set:

$$\hat{u}_\varepsilon(t) = \begin{cases} u(t), & \text{if } 0 \le t \le T-\varepsilon; \\ v(t), & \text{if } T-\varepsilon < t \le T. \end{cases}$$

Then $x(T; u_\varepsilon) = 0$, $J_T(u_\varepsilon) \le J_\infty(u) + \varepsilon$. Hence $\mu_T(x_0) \le \mu_\infty(x_0) + \varepsilon$. $\square$

In general, we cannot substitute the inf by the min in the definition of μ_∞ (see the next section for a simple counterexample). We can do it if A is a hyperbolic operator. Moreover, the infinite horizon problem is reduced to a finite horizon one in this case.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be hyperbolic, then $\mathbb{R}^n = E^+ \oplus E^-$, where $AE^\pm \subset E^\pm$ and invariant subspace E^+ $(E-)$ corresponds to the eigenvalues of A with positive (negative) real parts. We have $e^{tA}E^\pm = E^\pm$; moreover, there exist $\alpha > 0$ such that

$$|e^{tA}x^+| \ge c_+e^{t\alpha}|x^+|, \quad |e^{tA}x^-| \le c_-e^{-t\alpha}|x^-|,$$

for some $c_\pm > 0$ and any $x^\pm \in E^\pm$, $t \ge 0$.

Given $x \in \mathbb{R}^n$, we set $x = x^+ + x^-$, where $x^\pm \in E^\pm$ and define a linear map $B^+ : \mathbb{R}^m \to E^+$ by the formula $B^+u = (Bu)^+$. Finally, we set $\mu_T^+(x_0^+) = \mu_T(x_0^+; A, B^+)$ and $\mu_\infty^+(x_0) = \mu_\infty(x_0^+; A, B^+)$, the optimal cost for the system $\dot{x}^+ = Ax^+ + B^+u$ on E^+.

Theorem 3. *Let A be hyperbolic; then for any $x_0 \in \mathcal{A}_\infty$ there exists a unique optimal control $u \in \mathcal{U}_\infty$ for the infinite horizon problem with the initial condition x_0. Moreover, there exists $T > 0$ such that $\mu_\infty(x_0) = \mu_T^+(x_0^+) = J_T(u)$ and $u(t) = 0$ for any $t > T$.*

Proof. It is easy to see that $t \mapsto x(t; u)^+$ is a solution of the system $\dot{x}^+ = Ax^+ + B^+u$ on E^+, for any $u \in \mathcal{U}_\infty$. Hence $\mu_\infty^+(x_0^+) \le \mu_\infty(x_0)$. Moreover, if $x(T; u)^+ = 0$, then $x(T; u)$ belongs to the asymptotically stable subspace of the system $\dot{x} = Ax$ and zero control transfers $x(T; u)$ to the origin in infinite time without augmentation of the cost. Hence $\mu_\infty^+(x_0^+) \le \mu_\infty(x_0) \le \mu_T^+(x_0^+)$.

It remains to show that for any $x_0^+ \in E^+$ there exists $T > 0$ such that $\mu_\infty^+(x_0^+) = \mu_T^+(x_0^+)$. To do that, we may assume that $E^+ = \mathbb{R}^n$ in order to simplify notations. We fix $x_0 \in \mathcal{A}_{T_0}$ and take $u_T \in \mathcal{U}_T$ such that $J_T(u_T) = \mu_T(x_0)$, for any $T > T_0$. Then u_T is a normal extremal control and there exists $p_T \in \mathbb{R}^{n*}$ such that:

$$|u_T(t)| = \begin{cases} 1, & \text{if } |p_Te^{-tA}B| > 1; \\ 0, & \text{if } |p_Te^{-tA}B| < 1, \end{cases} \qquad 0 \le t \le T.$$

Recall that $|p_T e^{-tA}B| \leq ce^{-\alpha t}|p_T|$, for some positive constants α and c.

If p_T are uniformly bounded, i. e $|p_T| \leq c',\ \forall T > T_0$, then $u_T(t) = 0$ for any $t > \frac{1}{\alpha}(\ln c + \ln c')$ and we obtain that

$$\mu_\infty(x_0) = \mu_T(x_0), \quad \forall T > \frac{1}{\alpha}(\ln c + \ln c').$$

If $|p_T|$ is not uniformly bounded, then there exists a sequence $T_k \to \infty,\ (k \to \infty)$ such that $|p_{T_k}| \to \infty$; moreover, we may assume that $\frac{1}{|p_{T_k}|}p_{T_k} \to \xi$, where $|\xi| = 1$. Let us show that this is not possible.

We consider the set $\mathcal{T} = \{t \in [0, 2T_0] : \xi e^{-tA}B = 0\}$; this is a finite subset of $[0, 2T_0]$. Let $O_\varepsilon\mathcal{T}$ be the radius ε neighborhood of $\mathcal{T}$, this is the union of $\#\mathcal{T}$ intervals where each interval has length 2ε. We fix a small enough ε to guarantee that the measure of $O_\varepsilon\mathcal{T}$ is smaller than T_0.

There exists $\delta > 0$ such that $|p_{T_k} e^{-tA}B| \geq \delta|p_{T_k}|$ for any $t \in [0, 2T_0] \backslash O_\varepsilon\mathcal{T}$ and any $k > \frac{1}{\delta}$. Hence $|p_{T_k} e^{-tA}B| \geq 1$ for all $t \in [0, 2T_0] \setminus O_\varepsilon\mathcal{T}$ if k is sufficiently big. It follows that $|u_{T_k}(t)| = 1,\ \forall t \in [0, 2T_0] \setminus O_\varepsilon\mathcal{T}$ and $\mu_{T_k}(x_0) = J(u_{T_k}) > T_0$.

On the other hand, $\mu_{T_k}(x_0) \leq \mu_{T_0}(x_0) \leq T_0$. This contradiction proves that $|p_T|$ is uniformly bounded and thus completes the proof of the theorem.

Suppose that $u_1, u_2 \in \mathcal{U}_\infty$ are optimal. Then there exist $T_1, T_2 > 0$ such that $\mu_\infty(x_0) = J_{T_i}(u_{i|[0;T_i]})$ and $u_i(t) = 0, \forall t \geq T_i$. Assume $T_1 < T_2$. The origin is an equilibrium point of the system so $x(T_2; u_1)^+ = 0$ and $J_{T_2}(u_{1|[0;T_2]}) = J_{T_1}(u_{1|[0;T_1]}) = J_{T_2}(u_{2|[0;T_2]})$. Applying Lemma 2 in the hyperbolic case, the optimal control for the finite horizon problem in time T_2 is unique. Then $u_{2|[0;T_2]} = u_{1|[0;T_2]}$ and $u_1 = u_2$ in $\mathcal{U}_\infty$.

Corollary 2. *If A is hyperbolic, then $\mu_\infty : \mathcal{A}_\infty \to \mathbb{R}$ is a continuous function.*

Indeed, according to Theorem 3, optimal control has a compact support. This support depends on x_0 but we see from the proof of the theorem that it remains uniformly bounded if x_0 runs a compact subset of $\mathcal{A}_\infty$. Continuity of μ_∞ now follows from Theorem 2.

Theorem 4. *Assume that $E^+ = \mathbb{R}^n$. A control $u \in \mathcal{U}_\infty$ such that $x(T; u) = 0$ and $x(t; u) \neq 0$ for any $t < T$ is optimal for the infinite horizon problem if and only if $x(.; u)$ can be complemented by $p(\cdot)$ in such a way that the Pontryagin maximum principle is satisfied, and, moreover, $|p(T)B| = 1$ and $|p(T)e^{-\tau A}B| \leq 1, \forall \tau \geq 0$.*

Proof. We apply the Pontryagin maximum principle. In this case, the transversality condition states that $h_{u(t)}(p(t), x(t; u)) = p(t)Ax(t; u) + p(t)Bu(t) -$

$|u(t)| = 0, \forall t \geq 0$. There exists $\varepsilon > 0$ such that $u(t) = \pm 1, T - \varepsilon \leq t \leq T$, otherwise $x(T-\varepsilon; u) = 0$. Then $p(t)Bu(t) - |u(t)| = |p(t)B| - 1, T - \varepsilon \leq t \leq T$ (maximality condition), and because $x(T; u) = 0$, we obtain $|p(T)B| = 1$.

Assume that u is optimal for the free time problem and that $x(T; u) = 0$, we may assume that $u(t) = 0, \forall t \geq T$. Then the control $u(t), 0 \leq t \leq T+s$ is optimal for the free time problem for any $s > 0$. Hence $x(t; u),\ 0 \leq t \leq T+s$, can be complemented by $p_s(\cdot)$ in such a way that the Pontryagin maximum principle is satisfied. Then $|p_s(T)B| = 1$ and $|p_s(T+\tau)B| = |p_s(T)e^{-\tau A}B| \leq 1,\ \forall \tau \in [0, s]$.

Lemma 3. *Let $s > 0$; then the set $\{\xi \in \mathbf{R}^{n*} : |\xi e^{-\tau A}B| \leq 1, 0 \leq \tau \leq s\}$ is bounded.*

Proof of the lemma. By contradiction assume that there exists a sequence (ξ_k) in the previous set such that $|\xi_k| \to +\infty$ as $k \to \infty$. We may assume that $\frac{\xi_k}{|\xi_k|} \to \eta \in \mathbf{R}^{n*}$, $|\eta| = 1$. Then passing to the limit for $0 \leq \tau \leq s$, we have $\eta e^{-\tau A}B = 0$, which contradicts the Kalman condition. $\square$

Applying this lemma, the family $\{p_s(T), s > 0\}$ is bounded so it has limiting points as $s \to \infty$. Any limiting point satisfies the conditions of the theorem. Hence these are necessary conditions for optimality.

Moreover, the Pontryagin maximum principle is a sufficient optimality condition for the fixed time problem. It follows that our extremal is optimal for every time $T+s, s \geq 0$, i.e. for any arbitrarily big fixed time. Hence this extremal is optimal for the free time problem and conditions of the theorem are sufficient for optimality.

4 Two-dimensional systems

4.1 Classical examples

In this section we study the stabilization to the origin in infinite time horizon of classical two-dimensional systems with a scalar control $|u| \leq 1$: the case of a free particle and the one of a harmonic oscillator.

4.1.1 Free particle

Considering the degenerated case of a free particle, we study the stabilization of the 2-dimensional system to $0_{\mathbf{R}^2}$ in infinite time horizon:

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |u| \leq 1$$

Theorem 5. *For every $x_0 \in \mathbf{R}^2$, $\mu_\infty(x_0) = \inf\limits_{T>0} \mu_T(x_0) = |x_0^2|$.*
If $x_0 \in \{x \mid x^1 \leq -\frac{1}{2}(x^2)^2, x^2 \geq 0\}$ or $x \in \{x \mid x^1 \geq \frac{1}{2}(x^2)^2, x^2 \leq 0\}$, then there exists $0 < T < \infty$ such that $\mu_T(x_0) = |x_0^2|$. Otherwise $\mu_T(x_0) > |x_0^2|$ for all $0 < T < \infty$.

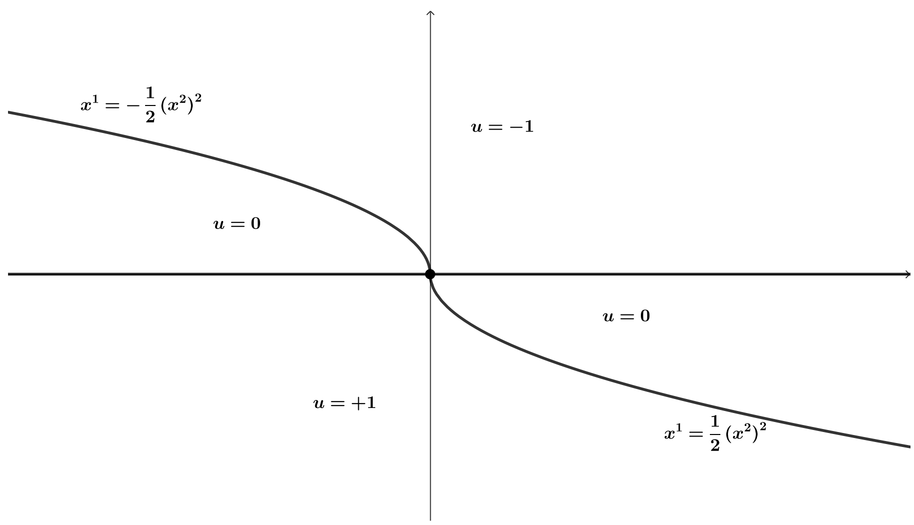

Figure 1: Regions of optimal control at the limit in infinite time horizon

Proof. Noting that $\dot{x}^2 = u$, for every $x_0 \in \mathbf{R}^2$, for every $T > 0$:

$$\mu_T(x_0) = \int_0^T |u(t)|dt \geq |\int_0^T u(t)dt| = |\int_0^T \dot{x}^2(t)dt| = |x^2(0) - x^2(T)| = |x_2^2|$$

Then $\mu_\infty(x_0) = \inf\limits_{T>0} \mu_T(x_0) \geq |x_0^2|$.
We study the form of the trajectory in function of the control.
If $u \equiv 1$ or $u \equiv -1$:

$$\begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = \pm 1 \end{cases} \qquad x_1 = \pm\frac{1}{2}((x^2)^2 - (x_0^2)^2) + x_0^1$$

If $u \equiv 0$:

$$\begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = 0 \end{cases} \qquad \begin{cases} x^1(t) = x_0^2 t + x_0^1 \\ x^2(t) = x_0^2 \end{cases}$$

<u>Region where $\mu_\infty(x_0)$ in reached in finite time:</u>
If $x_0^1 = \frac{1}{2}(x_0^2)^2$ and $x_0^2 \leq 0$ we can take a control $u \equiv +1$ until x reaches

the origin (and then $u \equiv 0$). With this control, we obtain $\mu_T(x_0) = |x_0^2|$ for every $T \geq |x_0^2|$. By symmetry, if $x_0^1 = -\frac{1}{2}(x_0^2)^2$ and $x_0^2 \geq 0$, with a control $u \equiv -1$ until the origin (then $u \equiv 0$), we obtain the same result.

If now $x_0^1 > \frac{1}{2}(x_0^2)^2$ and $x_0^2 \leq 0$, we can take a control $u \equiv 0$ until $x^1 = \frac{1}{2}(x^2)^2$. The portion of the trajectory is a horizontal line $x^2 = x_0^2$. Then we take $u \equiv +1$ and the trajectory is the same as the previous one. The cost does not change by adding a time interval where $u \equiv 0$. By symmetry, we obtain the same result when $x_0^1 < -\frac{1}{2}(x_0^2)^2$ and $x_0^2 \geq 0$.

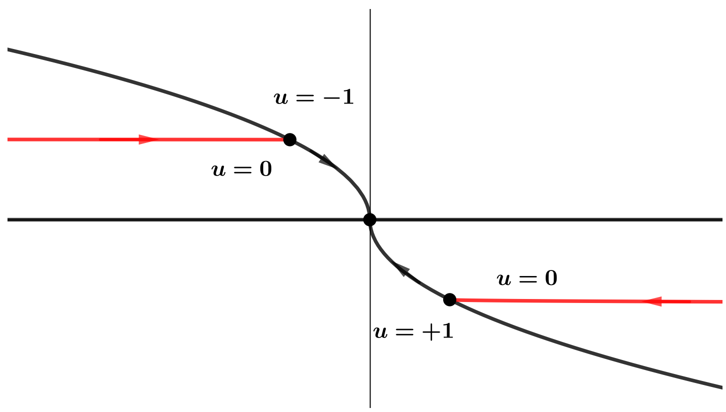

Figure 2: Optimal trajectories in finite time

Region where $\mu_\infty(x_0)$ is not reached in finite time:

We construct a sequence T_k such that $\mu_{T_k}(x_0) \to |x_0^2|$. Applying the PMP, we obtain some extremals, that are all optimal thank to the linearity of the system and the convexity of the constrain set. Solving the Hamiltonian equation $\dot{p} = -pA$, we obtain the switching function:

$$f(t) = p(t) \cdot B = -p_0^1 t + p_0^2$$

Variation of the covector allows the switches $-1 \to 0 \to +1$ and $+1 \to 0 \to -1$ with every lengths for the different time intervals where u is constant.

If $x_0^1 > -\frac{1}{2}(x_0^2)^2$ and $x_0^2 \geq 0$, we consider a control with switches $-1 \to 0 \to +1$. First portion of the trajectory is parabolic with equation $x^1 = -\frac{1}{2}((x^2)^2 - (x_0^2)^2) + x_0^1$. We choose the first time of switch when $x^2(t) = -\frac{1}{k}$. Second portion is then a horizontal line of equation $x^2 = -\frac{1}{k}$. Third portion is parabolic with equation $x^1 = +\frac{1}{2}(x^2)^2$ and arrive at the origin at a time denoted T_k. This control corresponds to a co-vector thus is optimal. Because $x_0^2 \geq 0$:

$$\mu_{T_k}(x_0) = |x_0^2 - (-\frac{1}{k})| + |\frac{1}{k} - 0| = x_0^2 + \frac{2}{k} \underset{k \to +\infty}{\longrightarrow} x_0^2$$

So $\mu_\infty(x_0) = \inf_{T>0} \mu_T(x_0) = |x_0^2|$. If $x_0^1 < \frac{1}{2}(x_0^2)^2$ and $x_0^2 \leq 0$ by symmetry we can construct such a sequence with a control $+1 \to 0 \to -1$ and we obtain also $\mu_\infty(x_0) = |x_0^2|$.

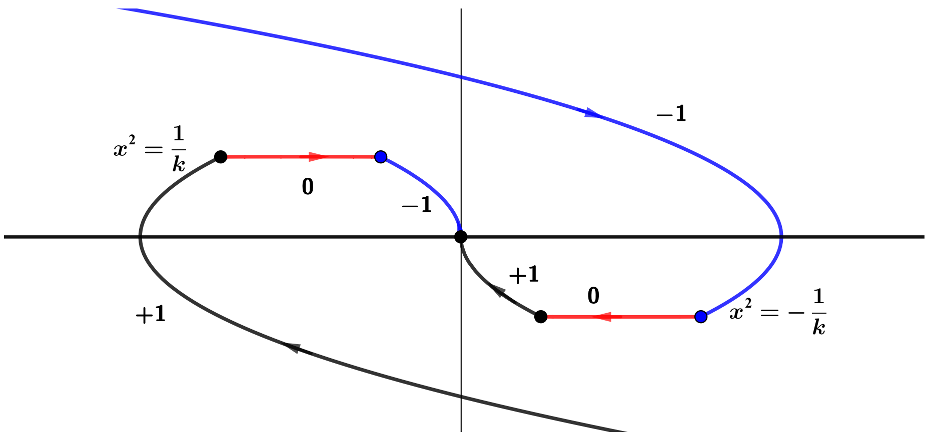

Figure 3: Construction of an optimal trajectory

The trajectory limit when k goes to infinity is not admissible, because if $x^2 = 0$, $x^1 \neq 0$ and $u \equiv 0$, then the point is in an equilibrium of the system and cannot reach the origin. We also see that as soon as the optimal control change of sign, then $\mu_T(x_0) > |x_0^2|$.

4.1.2 Harmonic oscillator

Considering the example of a 2- dimensional controlled harmonic oscillator, we study the stabilization to the origin in infinite time horizon:

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Proposition 5. *In a finite time $T > 0$, the optimal control u is unique and has a mainly periodic structure. Let $\varepsilon \in \{-1; +1\}$. There exists a first switching time $\alpha_0 \in [0; \pi[$ and a period $\delta \in [0; \pi[$such that, for $t \geq \alpha_0$, u takes the values -1, 0 and $+1$ in a periodic way :*

$$-\varepsilon \to 0 \to +\varepsilon \to 0 \to -\varepsilon \to ... \qquad or \qquad 0 \to +\varepsilon \to 0 \to -\varepsilon \to 0 \to ...$$

The time where u is 0 is $\pi - \delta$ and the one where u is $+1$ or -1 is δ.

Proof. Applying Lemma 1, B is a column and $det(A) \neq 0$, the PMP gives the description of the optimal control, which is unique. Solving $\dot{p} = -pA$, we obtain the switching function:

$$f(t) = p(t) \cdot B = a \sin(t + b)$$

Where a and b are real parameters related to p_0^1, p_0^2. Then if $|a| > 1$, the optimal control is not always 0 and takes the mainly periodic structure described in the proposition.

Proposition 6. *The optimal trajectory change periodically between different arc of circles, traveled clockwise:*
Center $(\pm 1, 0)$ when $u = \pm 1$
Center $(0, 0)$ when $u = 0$

Proof. If $u \equiv 0$:

$$\begin{cases} \dot{x}^1(t) = x^2(t) \\ \dot{x}^2(t) = -x^1(t) \end{cases} \qquad \begin{cases} x^1(t) = a_0 \cos(t + b_0) \\ x^2(t) = a_0 \sin(t + b_0) \end{cases}$$

Then:

$$\frac{d}{dt}\{(x^1)^2(t) + (x^2)^2(t)\} = 0$$

So the trajectory is a clockwise circle centered into the origin.
If $u \equiv 1$:

$$\begin{cases} \dot{x}^1(t) = x^2(t) \\ \dot{x}^2(t) = -x^1(t) + 1 \end{cases} \qquad \begin{cases} x^1(t) = a_1 \cos(t + b_1) + 1 \\ x^2(t) = a_1 \sin(t + b_1) \end{cases}$$

Then:

$$\frac{d}{dt}\{(x^1(t) - 1)^2 + (x^2)^2(t)\} = 0$$

So the trajectory is a clockwise circle centered into the point $(1, 0)$.
If $u \equiv -1$, by symmetry the trajectory is a clockwise circle centered into the point $(-1, 0)$.

Theorem 6. *Let $x_0 \in \mathbf{R}^2$. For every $k \in \mathbf{N}$ such that $|k| \geq |x_0|$, there exists a finite time $T(k)$, $(k-1)\pi \leq T(k) \leq k\pi$, such that the optimal trajectory which returns to the origin has its switches on the circles of centers $(0, n)$ for $n \in \mathbf{Z}$, $n \leq k$. The optimal cost of this trajectory is $c(k) = k \cdot \arccos(1 - \frac{|x_0|^2}{2k^2})$. The limit of the optimal cost is $\mu_\infty(x_0) = |x_0|$ and the point-wise limit of the optimal trajectory is the clockwise circle $x_\infty(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} x_0$.*

Proof. The proof is presented in Appendix A.

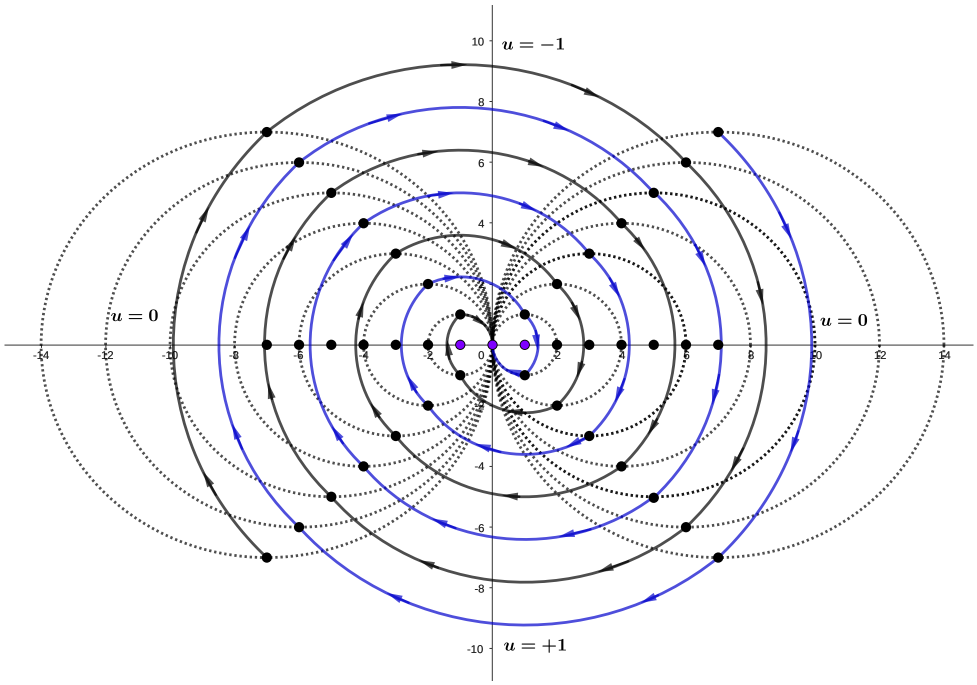

Figure 4: Switching curves and optimal trajectories

4.2 Hyperbolic systems

In this section, we study the stabilization to the origin in infinite time of all real hyperbolic two-dimensional systems $\dot{y} = Ay + Bu$ with a scalar control $|u| \leq 1$. The matrix A is similar to one of the following where $(\lambda_i, \lambda, \alpha, \beta) \in \mathbf{R}^4$:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

As explained in Theorem 3, the $\mathcal{L}^1$-limit cost in infinite time horizon $\mu_\infty(x_0)$ only depends on the part x_0^+ corresponding to the eigenvalues of A with positive real parts. Thus we only consider cases where $\mathbf{R}^2 = E^+$, i.e. where $x_0 = x_0^+$ for all $x_0 \in \mathbf{R}^2 : \lambda, \mu, \alpha > 0$. In order to treat all cases $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ with (A, B) respecting the Kalman condition, we apply a change of basis $x = Py$ such that P and A commute. Thus the system in these new coordinates can be written $\dot{x} = Ax + PBu$.

4.2.1 First hyperbolic case : $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

The Kalman condition is respected if and only if $b_1, b_2 \neq 0$. Applying the change of basis $P = \begin{pmatrix} b_1^{-1} & 0 \\ 0 & b_2^{-1} \end{pmatrix}$, the study reduces to the stabilization of the system:

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + u \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_1, \lambda_2 > 0 \qquad |u| \leq 1$$

Proposition 7. *Let $x_0 \in \mathbf{R}^2$. If $0_{\mathbf{R}^2}$ is reachable from x_0, then the optimal control is unique and has isolated switches with one of the following structure:*

$$\varepsilon \to 0, \qquad 0 \to \varepsilon \to 0, \qquad -\varepsilon \to 0 \to \varepsilon \to 0 \qquad \varepsilon \in \{-1; +1\}$$

Proof. The uniqueness and the structure of the optimal control $u(.)$ are given applying Lemma 2, noting that A is hyperbolic. Solving the Hamiltonian equation $\dot{p} = -pA$, we obtain the switching function:

$$f(t) = p(t) \cdot B = p_0^1 e^{-\lambda_1 t} + p_0^2 e^{-\lambda_2 t}$$

If $f(t)$ is less than -1, then $u \equiv -1$. If it is more than $+1$, then $u \equiv +1$. Else $u \equiv 0$. The function goes to 0 at infinity and has at most two different monotone branches. Thus we obtain one of the structures described in the proposition.

Theorem 7. *The region of initial conditions from which there exists a trajectory that reaches $0_{\mathbf{R}^2}$ is open and delimited by portions of the curves $\mathcal{C}_+$ and $\mathcal{C}_-$ between the points $(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})$ and $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$:*

$$\mathcal{C}_+ \,:\, x^2 = \frac{1}{\lambda_2}(2(\frac{\lambda_1 x^1 + 1}{2})^{\frac{\lambda_2}{\lambda_1}} - 1) \qquad \mathcal{C}_- \,:\, x^2 = \frac{1}{\lambda_2}(1 - 2(\frac{1 - \lambda_1 x^1}{2})^{\frac{\lambda_2}{\lambda_1}})$$

The switching curve $0 \to +1$ is given by the portion of the curve $\mathcal{C}_{0,+1}$ between the points $(0,0)$ and $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$ (respectively $\mathcal{C}_{0,-1}$, $(0,0)$, $(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})$ for the switch $0 \to -1$):

$$\mathcal{C}_{0,+1} \,:\, x^2 = \frac{1}{\lambda_2}((\lambda_1 x^1 + 1)^{\frac{\lambda_2}{\lambda_1}} - 1) \qquad \mathcal{C}_{0,-1} \,:\, x^2 = \frac{1}{\lambda_2}(-(1 - \lambda_1 x^1)^{\frac{\lambda_2}{\lambda_1}} + 1)$$

The switching curve $+1 \to 0$ is given by the portion of the curve $\mathcal{C}_{+1,0}$ between points $(0,0)$ and $(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})$ (respectively $\mathcal{C}_{-1,0}$, $(0,0)$ and $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$ for the switch $-1 \to +1$:

$$\mathcal{C}_{+1,0} \,:\, x^2 = \frac{1}{\lambda_2}(\lambda_1 x^1)^{\frac{\lambda_2}{\lambda_1}} \qquad \mathcal{C}_{-1,0} \,:\, x^2 = -\frac{1}{\lambda_2}(-\lambda_1 x^1)^{\frac{\lambda_2}{\lambda_1}}$$

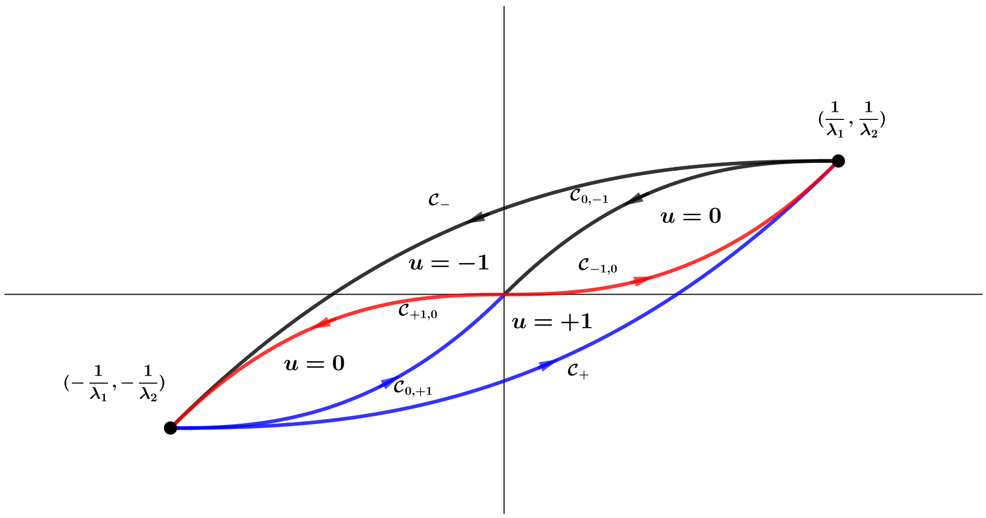

Figure 5: Attainable set and switching curves in the case $\lambda_1 < \lambda_2$

Proof. First we look at the form of the trajectories when the control is constant. If $u \equiv +1$:

$$\begin{cases} \dot{x}^1 = \lambda_1 x^1 + 1 \\ \dot{x}^2 = \lambda_2 x^2 + 1 \end{cases}$$

The only equilibrium point is $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$. Otherwise the solution $|x(t)|$ goes to $+\infty$ in infinite time. Solving the differential system from an initial condition (x_0^1, x_0^2) we obtain a trajectory of the form, traveled in the direction of infinity:

$$\frac{\lambda_2 x^2 + 1}{\lambda_2 x_0^2 + 1} = (\frac{\lambda_1 x^1 + 1}{\lambda_1 x_0^1 + 1})^{\frac{\lambda_2}{\lambda_1}}$$

If $(0,0)$ belongs to the trajectory then (x_0^1, x_0^2) belongs to the portion of $\mathcal{C}_{0,+1}$ between $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$ and $(0,0)$.

Then we look at the trajectories with one switch of type $0 \to +1$. If $u \equiv 0$ on a time interval:

$$\begin{cases} \dot{x}^1 = \lambda_1 x^1 \\ \dot{x}^2 = \lambda_2 x^2 \end{cases}$$

The only equilibrium point is $(0,0)$. Otherwise the trajectory has the following form, traveled in the direction of infinity:

$$\frac{x^2}{x_0^2} = (\frac{x^1}{x_0^1})^{\frac{\lambda_2}{\lambda_1}}$$

In order to arrive at $0_{\mathbf{R}^2}$ with one switch $0 \to +1$, the trajectory must cross the portion of $\mathcal{C}_{0,+1}$ between the points $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$ and $(0,0)$. Because trajectories corresponding to $u \equiv 0$ do not cross except at the origin, the region of initial conditions for which such a switch is possible is delimited by the trajectory with control 0 which pass by the point $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$. We obtain the curve $\mathcal{C}_{-1,0}$ between the points $(0,0)$ and $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$.

By symmetry we obtain the switching curves for $0 \to -1$ between the points $(0,0)$ and $(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})$. Finally in order to obtain a switch of type $-1 \to 0 \to +1$, the trajectory with $u \equiv -1$ at the beginning must cross the portion of $\mathcal{C}_{-1,0}$ between the points $(0,0)$ and $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$. Because the trajectories do not cross except in $(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})$ when $u \equiv -1$, the region of initial conditions for which such a switch is possible is delimited by the one which pass by the point $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$. Thus we obtain the portion of the curve $\mathcal{C}_-$ between points $(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})$ and $(-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2})$.

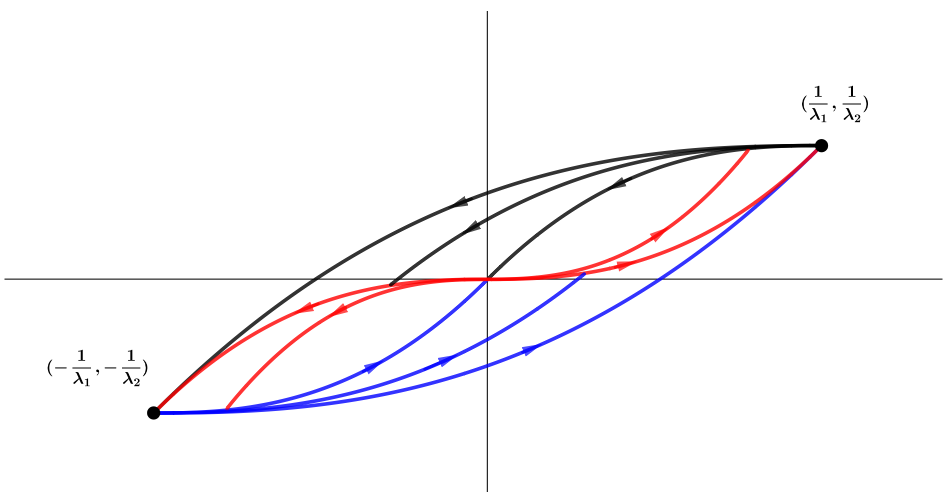

Figure 6: Optimal trajectories

4.2.2 Second hyperbolic case : $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

The Kalman condition is respected if and only if $b_2 \neq 0$. Applying the change of basis $P = \begin{pmatrix} b_2^{-1} & 0 \\ 0 & b_2^{-1} \end{pmatrix}$ and denoting $b = \frac{b_1}{b_2}$, the study reduces to the stabilization of the system:

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + u \begin{pmatrix} b \\ 1 \end{pmatrix} \qquad \lambda > 0 \qquad |u| \leq 1$$

Proposition 8. *Let $x_0 \in \mathbf{R}^2$. If $0_{\mathbf{R}^2}$ is reachable from x_0, then the optimal control is unique and has isolated switches with one of the following structure:*

$$\varepsilon \to 0, \qquad 0 \to \varepsilon \to 0, \qquad -\varepsilon \to 0 \to \varepsilon \to 0 \qquad \varepsilon \in \{-1;+1\}$$

Proof. The proof is the same as in the previous study, we apply the Lemma 2 and we note that the switching function, given by the following formula, has at most two monotone branches and goes to 0 at infinity:

$$f(t) = p(t) \cdot B = (p_0^1 + p_0^2 - p_0^2 t)e^{-\lambda t}$$

Theorem 8. *The region of initial conditions from which there exists a trajectory that reaches $0_{\mathbf{R}^2}$ is open and delimited by portions of the curves $\mathcal{C}_+$ and $\mathcal{C}_-$ between the points $(\frac{1}{\lambda}(b-\frac{1}{\lambda}), \frac{1}{\lambda})$ and $(-\frac{1}{\lambda}(b-\frac{1}{\lambda}), -\frac{1}{\lambda})$:*

$$\mathcal{C}_+ \;:\; x^1 = -\frac{1}{\lambda}(b-\frac{1}{\lambda}) + \frac{1}{\lambda^2}(\lambda x^2+1)(\lambda b - 1 + \ln(\frac{\lambda x^2+1}{2}))$$
$$\mathcal{C}_- \;:\; x^1 = \frac{1}{\lambda}(b-\frac{1}{\lambda}) + \frac{1}{\lambda^2}(\lambda x^2-1)(\lambda b - 1 + \ln(\frac{1-\lambda x^2}{2}))$$

The switching curve $0 \to +1$ is given by the portion of the curve $\mathcal{C}_{0,+1}$ between the points $(0,0)$ and $(-\frac{1}{\lambda}(b-\frac{1}{\lambda}), -\frac{1}{\lambda})$ (respectively $\mathcal{C}_{0,-1}$, $(0,0)$, $(\frac{1}{\lambda}(b-\frac{1}{\lambda}), \frac{1}{\lambda})$ for the switch $0 \to -1$):

$$\mathcal{C}_{0,+1} \;:\; x^1 = -\frac{1}{\lambda}(b-\frac{1}{\lambda}) + \frac{1}{\lambda^2}(\lambda x^2+1)(\lambda b - 1 + \ln(\lambda x^2+1))$$
$$\mathcal{C}_{0,-1} \;:\; x^1 = \frac{1}{\lambda}(b-\frac{1}{\lambda}) + \frac{1}{\lambda^2}(\lambda x^2-1)(\lambda b - 1 + \ln(1-\lambda x^2))$$

The switching curve $+1 \to 0$ is given by the portion of the curve $\mathcal{C}_{+1,0}$ between points $(0,0)$ and $(\frac{1}{\lambda}(b-\frac{1}{\lambda}), \frac{1}{\lambda})$ (and respectively $\mathcal{C}_{-1,0}$, $(0,0)$ and $(-\frac{1}{\lambda}(b-\frac{1}{\lambda}), -\frac{1}{\lambda})$ for the switch $-1 \to +1$:

$$\mathcal{C}_{+1,0} \;:\; x^1 = x^2(1-\frac{1}{\lambda}+\frac{1}{\lambda}\ln(-\lambda x^2)) \qquad \mathcal{C}_{-1,0} \;:\; x^1 = x^2(1-\frac{1}{\lambda}+\frac{1}{\lambda}\ln(\lambda x^2))$$

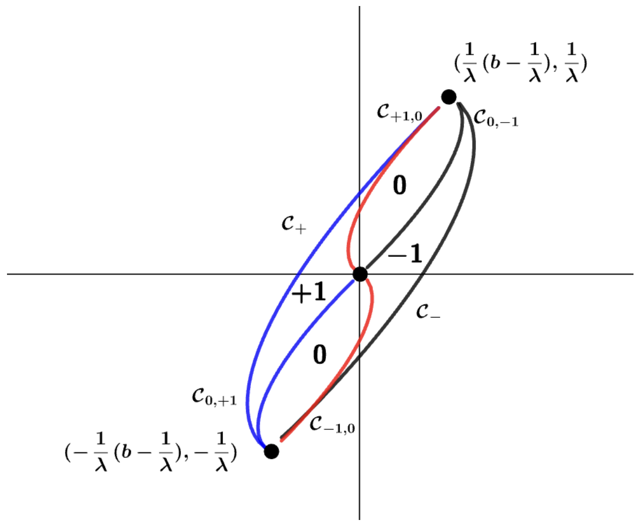

Figure 7: Attainable set and switching curves

Proof. The scheme of the study is exactly the same as in the previous hyperbolic case.

4.2.3 Third hyperbolic case : $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

The Kalman condition is respected if and only if $\beta, (b_1^2 + b_2^2) \neq 0$. Applying the change of basis $P = (b_1^2 + b_2^2)^{-1} \begin{pmatrix} b_2 & -b_1 \\ b_1 & b_2 \end{pmatrix}$, the study reduces to the stabilization of the system:

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \alpha > 0 \qquad \beta \neq 0 \qquad |u| \leq 1$$

We use the complex notation $z(t) = x^1(t) + ix^2(t)$. If $u = 0$ then the trajectory is a logarithmic spiral of equation $z(t) = z(0)e^{(\alpha - i\beta)t}$. If $u \pm 1$ the trajectory is a logarithmic spiral of equation $z(t) = (z(0) \pm \bar{z})e^{(\alpha - i\beta)t} \mp \bar{z}$ with $\bar{z} = \frac{\beta - i\alpha}{\alpha^2 + \beta^2}$.

The domain of initial conditions that can be stabilized to the origin with a bounded control $|u| \leq 1$ is delimited by the parameterized curves $z(t) = \pm z_{lim} e^{(\alpha - i\beta)t} \pm \bar{z}, -\frac{\pi}{\beta} \leq t \leq 0$ with $z_{lim} = \bar{z}(1 + \frac{2}{e^{\frac{\alpha}{\beta}\pi} - 1})$. A simple way to find this domain is to compute time-optimal synthesis. It should be well known and we leave it as an exercise.

Proposition 9. *Let x_0 be an initial condition in the attainable set. There exists a finite time T for which the lowest $\mathcal{L}^1$-cost $\mu_\infty(x_0)$ can be reached. The corresponding optimal control u and optimal trajectory $x(\cdot)$ are unique until $0_{\mathbf{R}^2}$ is reached, and the structure of u is given by a switching function of the form $f(t) = r_0 e^{-\alpha t}\sin(\theta_0 - \beta t), r_0 > 0, \theta_0 \in [0; 2\pi)$. If $|f(t)| \geq 1$, then $u(t) = sign(f(t))$, otherwise $u = 0$.*

Proof. For any nontrivial solution of the adjoined equation $\dot{p} = -pA$, we obtain $p(t)B = p_2(t) = f(t)$, where f is a function of the form described in the statement of the proposition. We see that any switching function has such a form. Note, that this class of functions is invariant with respect to the translation of the argument: if f is in this class then the function $t \mapsto f(t+s)$ also is, $\forall\, s \in \mathbb{R}$. Moreover, for such a function f, there is a unique $s_f \in \mathbb{R}$ such that $|f(s_f)| = 1$ and $|f(t)| \leq 1, \forall\, t \geq s_f$. Given $T > 0$, let

$$u(t) = \begin{cases} sign(f(t + s_f - T)), & \text{if } |f(t + s_f - T)| > 1; \\ 0, & \text{if } |f(t + s_f - T)| \leq 1 \end{cases}$$

and $x(t)$ be the solution of the Cauchy problem

$$\dot{x} = Ax + Bu(t), \qquad x(T) = 0.$$

According to Theorem 4, $x(\cdot)$ is optimal solution for the free time problem with the initial condition $x_0 = x(0)$.

Moreover, any optimal solution has such a form.

In order to obtain images of the optimal synthesis, we use numerical simulation. We compute the trajectory given by some switching functions in reverse time and starting from the origin. The covector follows a trajectory of polar equation $r(\theta) = r_0 e^{-\frac{\alpha}{\beta}\theta_0}(e^{\frac{\alpha}{\beta}})^\theta$. The numerical simulation can reduce to a one parameter simulation. For each spiral of equation $r(\theta) = C_0(e^{\frac{\alpha}{\beta}t})^\theta$, starting from the origin in reverse time and following the spiral curve and using the ordinate of the point as the switching function, one obtain one optimal trajectory in reverse time starting from the origin. It it sufficient to make variations of C_0 in $(0; e^{\frac{\alpha}{\beta}2\pi}]$ to obtain all possible trajectories. In red the part of the trajectory when $u = 0$, in blue when $u = \pm 1$.

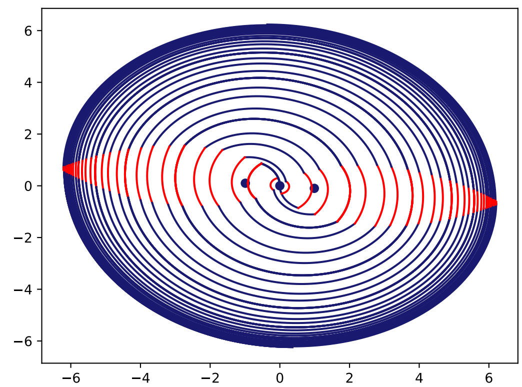

Figure 8: Some optimal trajectories for $\alpha = 0.1, \beta = 1$

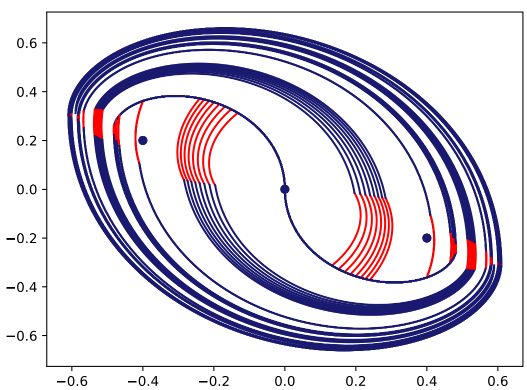

Figure 9: Some optimal trajectories for $\alpha = 1, \beta = 2$

References

[1] A. Agrachev, Yu. Sachkov, *Control Theory from the geometric viewpoint.* Springer Verlag, 2004, 426 p.

[2] Z. Chen, J.-B. Caillau, Y. Chitour, *L^1-minimization for mechanical systems.* SIAM J. Control Optim., 2016, v.54, 1245–1265

[3] L. Pontryagin, V. Boltyanskij, R. Gamkrelidze, E. Mishchenko, *The mathematical theory of optimal processes.* Pergamon Press, 1964

[4] V. Romano, F. Corallo, M. Pontani, P. Teofilatto, *Minimum-fuel orbit transfers using modified equinoctial elements via indirect heuristic method.* 9th Eur. Conf. Aeronaut. Space Sci. (EUCASS), 2022, 13 p.

Appendix

A Harmonic oscillator

Proof of Theorem 5

We consider the optimal control with a mainly periodic structure described in the previous proposition, we denote α_0 the length of the last interval where $u \neq 0$ and we look at the specific case where δ, the length of the time interval where u is equal to $+1$ or -1, is equal to α_0.
If on the last time interval $u \equiv 1$, then because the trajectory arrive at $0_{\mathbf{R}^2}$, the last portion is an arc of the circle of center $(1,0)$ and radius 1, traveled clockwise.
After the interval $]T-\alpha_0, T]$ where $u \equiv 1$, $u \equiv 0$ for a time $\pi - \delta$ and the trajectory is a circle of center $(0,0)$. Because here $\delta = \alpha_0$, applying the inscribed angle theorem, we obtain the upper part of the circle as the previous switching curve.

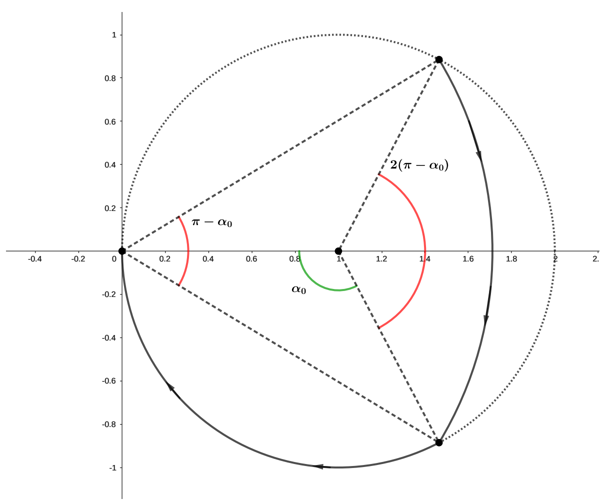

Figure 10: Last switching curve when $\delta = \alpha_0$

In order to construct the last to last switching curve, we use the following geometric construction:

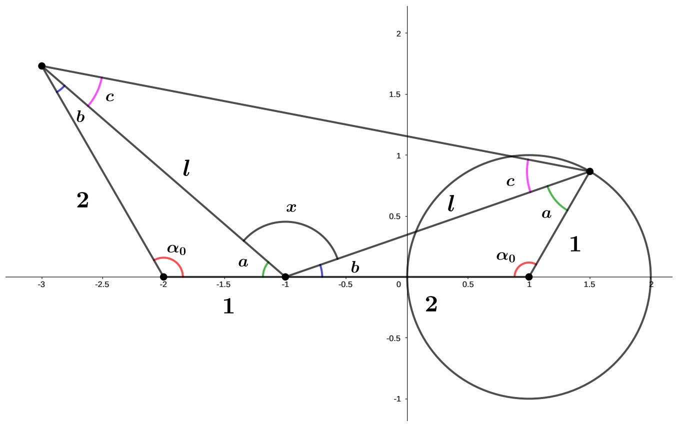

Figure 11: Construction of the last to last switching point

The two triangles that have one side on the abscissa axis are the same triangles but with a different side on this axis. And so the triangle in the middle is isosceles. Then because the sum of the angles of a triangle is π we have:

$$\begin{cases} a + b + \alpha_0 = \pi \\ x + 2c = \pi \end{cases}$$

Because the sum of the angles of a quadrilateral is 2π we have :

$$2\alpha_0 + a + b + 2c = 2\pi$$

Finally we remark also that :

$$a + b + x = \pi$$

Using these equations we obtain $x = \alpha_0$. So this figure corresponds to the construction of the next to next to last switching point. The corresponding switching curve is the upper half circle of center $(-2, 0)$ and radius 2. Repeating this scheme of construction, in the case where $u \equiv 1$ on the last time interval, we obtain all the circles of center $(-2n, 0)$ with radius $2n$ and all those of center $(2n + 1, 0)$ with radius $2n + 1$, where $n \in \mathbf{N}$. By symmetry, if we add the possibility of the value $u = -1$ in the last interval $]T - \alpha_0; T]$ we obtain all the circles of center $(k, 0)$ and radius $|k|$, where $k \in \mathbf{Z}$.

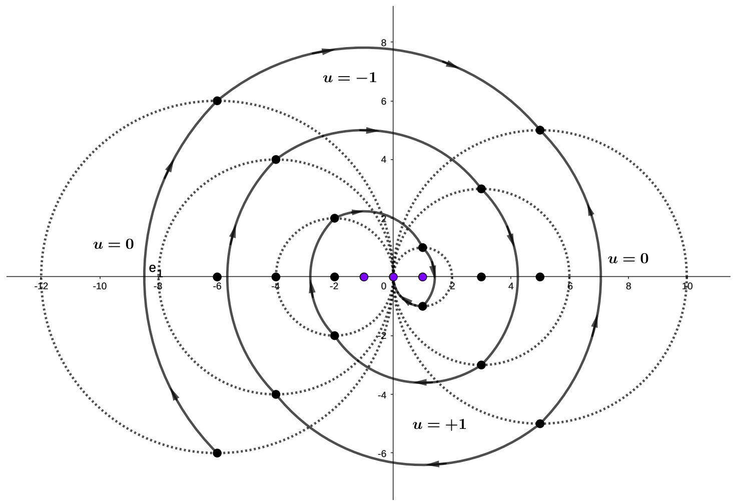

Figure 12: Switching curves when $u \equiv +1$ at the end

Let $x_0 \in \mathbf{R}^2$, $x_0^1 > 0$, $k \in \mathbf{N}, k > |x_0|$. We begin with a control $u \equiv 0$, the trajectory is a circle of center $(0,0)$ until the first switch on the circle of center $(k,0)$ and radius k. Then the trajectory is as described previously. Such a trajectory has k portions where $u \equiv +1$ or $u \equiv -1$, all of length $\delta = \alpha_0(k)$. So the $\mathcal{L}^1$-cost of such a trajectory is equal to $c(k) = k \cdot \alpha_0(k)$. With the notations of the following figure, we compute $\cos(\alpha_0(k)) = 1 - \frac{|x_0|^2}{2k^2}$.

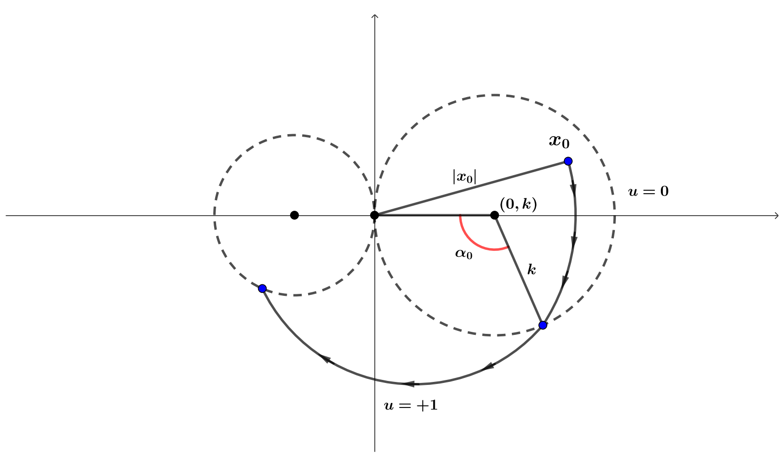

Figure 13: Attainable set and switching curves

The optimal cost $c(k) = k \cdot \arccos(1 - \frac{|x_0|^2}{2k^2})$ goes to $|x_0|$ when k goes to infinity. The functional μ_∞ is decreasing and bounded from below (by 0) so the limit exists : $\mu_\infty(x_0) = \lim_{T_k \to \infty} \mu_{T_k}(x_0) = |x_0|$.

When k goes to infinity, the trajectory has more and more switches, but with portions of circles of centers $(-1, 0)$ and $(1, 0)$ that are smaller and smaller.

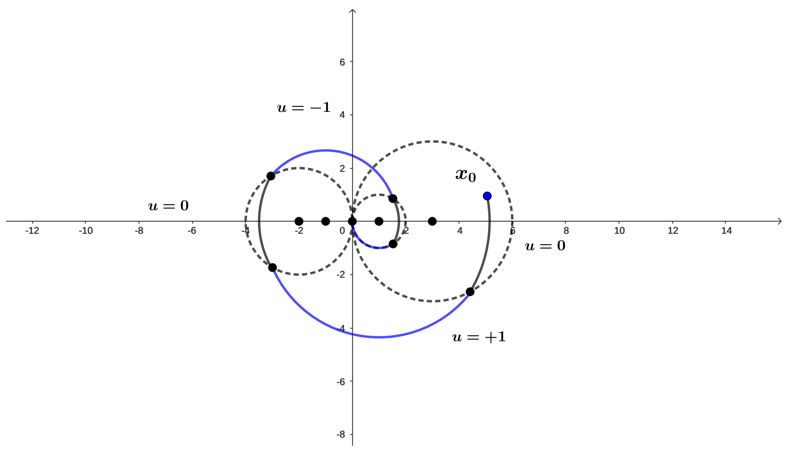

Figure 14: Trajectory for $2\pi \leq T \leq 3\pi$

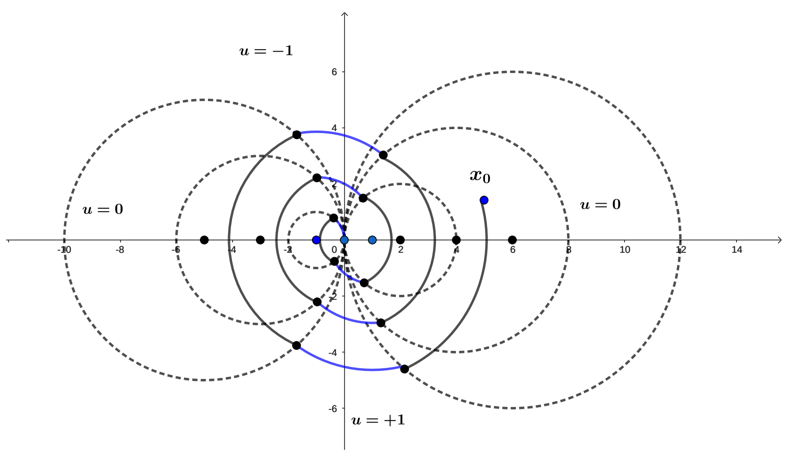

Figure 15: Trajectory for $5\pi \leq T \leq 6\pi$