Докл. Акад. Наук СССР
Том 287 (1986), № 3

Soviet Math. Dokl.
Vol. 33 (1986), No. 2

THE MORSE INDEX AND THE MASLOV INDEX FOR EXTREMALS OF CONTROLLED SYSTEMS

UDC 517.97

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1. Let M be a C^∞-manifold. We consider a controlled system

(1) $$\dot{m} = f_t(m, u), \quad m \in M,\ u \in \mathbf{R}^r,\ t \in \mathbf{R}, \qquad m(0) = m_0,$$

where $f_t(m,u)$ is a family of smooth vector fields on M that depends smoothly on u and piecewise smoothly on t. Moreover, we assume that $f_t(m,0)$ is a complete nonstationary field on M and $p_t\colon M \to M$, $t \in \mathbf{R}$, is the flow on M determined by this field, i.e.,

$$\frac{\partial}{\partial t} p_t(m) = f_t(p_t(m), 0), \qquad p_0(m) = m.$$

Let $t > 0$ and suppose that for all controls $u(\cdot) \in L^r_\infty[0,t]$ sufficiently close to zero the mapping $F_t\colon u(\cdot) \mapsto m(t)$ is defined, where $\dot{m}(\tau) = f_\tau(m(\tau), u(\tau))$, $\tau \in [0,t]$, and $m(0) = m_0$. It is easy to show that F_t is an infinitely differentiable mapping of some neighborhood of zero in $L^r_\infty[0,t]$ into M. The family of mappings F_τ, $\tau \in (0, +\infty)$, determines the behavior of the controlled system (1). Let $G_t = p_t^{-1} \circ F_t$; in studying a controlled system near the zero control it is more convenient to work not with F_t but with the equivalent mapping G_t; in particular, $G_t(0) = m_0$. Assume that zero is a critical point of the mapping G; in this case it is common to call the trajectory $p_\tau(m_0)$, $\tau \in [0,t]$, corresponding to the zero control an *extremal* of the system (1). Let $G'_t\colon L^r_\infty[0,t_0] \to T_{m_0}M$ be the differential of G_t at zero, let $\ker G'_t$ be the kernel of this differential, let $\operatorname{im} G'_t$ be its range, and let $\operatorname{coker} G'_t = T_{m_0}M/\operatorname{im} G'_t$ be its cokernel. Denote by G''_t the Hessian of G_t at zero (recall that the Hessian is a quadratic mapping from the kernel of the differential to the cokernel). If $\dim \operatorname{coker} G'_t = 1$, then G''_t is, in essence, a scalar quadratic form; otherwise, the projections of G''_t onto one-dimensional directions are quadratic forms. Of importance in control theory are the Morse indices of such quadratic forms; for example, the index of extremality of a control can be estimated in terms of them (see [5]). We give explicit expressions for the Morse indices of these forms, connecting them with certain symplectic invariants. It should be emphasized that the formulas obtained are equally suitable both in the regular case and in the singular case. In the case when the controlled system reduces to a standard regular problem in the calculus of variations our approach leads to the well-known identity of Morse and Maslov indices of an extremal (see [1]), and the general case can be regarded as a generalization of this identity. See [2] and [3] for other approaches to the computation of the Morse index in the regular situation.

2. Suppose, as usual, that the symbol $p_{\tau *}$ denotes the differential of the diffeomorphism p_τ. For every $v \in \mathbf{R}^r$ and $\tau \in \mathbf{R}$ let

$$h_\tau(v) = p_{\tau *}^{-1} \left.\frac{\partial^2}{\partial \varepsilon^2}\right|_{\varepsilon=0} f_\tau(p_\tau(m_0), \varepsilon v), \qquad (Z_\tau v)(m) = p_{\tau *}^{-1} \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} f_\tau(p_\tau(m), \varepsilon v);$$

1980 *Mathematics Subject Classification*. Primary 58E05, 58F05; Secondary 58F25.

The mapping $(X, Y) \mapsto \psi[X,Y](m_0)$

then $T_{m_0}M \ni h_\tau(v)$, a tangent vector depending quadratically on v, and $Z_\tau v$ is a vector field on M depending linearly on v. We have that

$$G_t'v(\cdot) \int_0^t (Z_\tau v(\tau))(m_0)\, d\tau \quad \forall v(\cdot) \in L_\infty^r[0,t].$$

Suppose that the covector $\psi \in T_{m_0}^* M$ satisfies the conditions $\psi(Z_\tau v)(m_0) = 0$, $\forall \tau \in (0,t), v \in \mathbf{R}^r$ (i.e., $\psi \perp \operatorname{im} G_t'$). Then

$$\psi G_t''(v(\cdot)) = \int_0^t \left(\psi h_\tau(v(\tau)) + \psi \left[\int_0^\tau (Z_\theta v)\, d\theta, Z_\tau v \right] (m_0) \right) d\tau,$$

where $\int_0^t (Z_\tau v)(m_0)\, d\tau = 0$ (i.e., $v(\cdot) \in \operatorname{Ker} G_t'$), and the square brackets $[\cdot,\cdot]$ denote the commutator of vector fields on M.

Fix once and for all a time $t > 0$ and let $\Pi = \operatorname{im} G'$ and $n = \dim \Pi$; fix also a covector $\psi \neq 0$ with $\psi \perp \Pi$. Let $\mathcal{E}_\Pi$ be the space of all smooth vector fields on M whose values at m_0 lie in Π; it is clear that $Z_\tau v \in \mathcal{E}_\Pi$, $\tau \in (0,t)$, $v \in \mathbf{R}^r$; where X, $Y \in \mathcal{E}_\Pi$, determines a skew-symmetric bilinear form on $\mathcal{E}_\Pi$. Denote by E the quotient space of $\mathcal{E}_\Pi$ by the kernel of this form, and denote by $\sigma(\cdot,\cdot)$ the skew inner product induced on E by this form. Then

$$\psi G_t''(v(\cdot)) = \int_0^t \left(\psi h_\tau(v(\tau)) + \sigma \left(\int_0^\tau z_\theta v(\theta)\, d\theta, z_\tau v(\tau) \right) \right) d\tau, \qquad \int_0^t z_\tau v(\tau)\, d\tau \in \Pi_0,$$

where $E \ni z_\tau v$, the image of the field $Z_\tau v$ upon taking of the quotient, and Π_0 is the corresponding image of the space of fields vanishing at m_0. There is an obvious exact sequence $0 \to \Pi_0 \to E \xrightarrow{\pi} \Pi \to 0$, where π is induced by the mapping associating with a vector field its value at the point m_0. It turns out that Π_0 is a Lagrangian plane in the symplectic space E; in particular, $\dim \Pi_0 = \dim \Pi = n$.

We agree to take all piecewise continuous functions to be left-continuous; with each $\tau \in (0,t]$ we associate an integer $k_\tau \geq 0$ and a quadratic form γ_τ on $\mathbf{R}^r$ as follows (the expression $z_\tau^{(k)} v$ below denotes the derivative of $z_\tau v$ of order $k \geq 0$ with respect to τ): if the form ψh_θ is not identically equal to zero on any interval $\bar\tau < \theta < \tau$, then let $k_\tau = 0$ and $\gamma_\tau = \psi h_\tau$; otherwise, let k_τ be the maximal number k such that $\sigma(z_\theta^{(i)} v_1, z_\theta^{(j)} v_2) \equiv 0$ for $i + j < 2(k-1)$ and v_1, $v_2 \in \mathbf{R}^r$ on some interval $\bar\tau < \theta < \tau$, and let $\gamma_\tau(v) = \sigma(z^{(k_\tau)} v,\ z^{(k_\tau - 1)} v)$, $v \in \mathbf{R}^r$ (if the maximal k exists, then it does not exceed n; if it does not exist, then we set $k_\tau = n+1$ and $\gamma_\tau = 0$). The Morse index of an arbitrary quadratic form Q is denoted by $\operatorname{ind} Q$.

PROPOSITION 1 (cf. [5]). *If* $\operatorname{ind} \psi G_t'' < +\infty$, *then:*
a) $\sigma(z_\tau^{(k_\tau - 1)} v_1, z_\tau^{(k_\tau - 1)} v_2) = 0\ \forall v_1, v_2 \in \mathbf{R}^r$, $\tau \in (0,t]$;
b) $\gamma_\tau \geq 0$, $\tau \in (0,t]$.
Conversely, if a) *holds and* $\gamma_\tau(v) \geq \varepsilon |v|^2$ *for any* $v \in \mathbf{R}^r$ *and* $\tau \in (0,t]$ *and some* $\varepsilon > 0$, *then* $\operatorname{ind} \psi G_t'' < +\infty$.

In what follows we assume the sufficient condition in Proposition 1 for $\operatorname{ind} \psi G_t''$ to be finite (see, however, Remark 2 at the end of the paper). Let $\Gamma_\tau = \operatorname{span}\{z_\tau^{(i)} v,\ 0 \leq i < k_\tau,\ v \in \mathbf{R}^r\}$, an rk_τ-dimensional isotropic subspace of E.

3. The form γ_τ, like every quadratic form on $\mathbf{R}^r$, is determined by some selfadjoint mapping from $\mathbf{R}^r$ to $\mathbf{R}^{r*}$, and the inverse mapping determines a quadratic form γ_τ^{-1} on $\mathbf{R}^{r*}$. Further, for every $x \in E$ the mapping $v \mapsto \sigma(z_\tau^{(k_\tau)} v, x)$ is a linear form on $\mathbf{R}^r$, i.e., $\sigma(z_\tau^{(k_\tau)} \cdot, x) \in \mathbf{R}^{r*}$. We consider on E the nonstationary quadratic Hamiltonian $\frac{1}{2}\gamma_\tau^{-1}(\sigma(z_\tau^{(k_\tau)} \cdot, x))$, $x \in E$, $\tau \in (0,t]$; if $v_1, \ldots, v_r$ is a basis in $\mathbf{R}^r$ such that $\gamma_\tau(v_i, v_j) = 0$

for $i \neq j$, then this Hamiltonian has the form

$$\sum_{i=1}^{r} \frac{\sigma(z_\tau^{(k_\tau)} v_i, x)^2}{2\gamma_\tau(v_i, v_i)}.$$

The corresponding Hamiltonian system is linear and, consequently, determines a one-parameter family of linear symplectic transformations of E.

Let $L(E)$ be the manifold of all Langrangian planes in E: the Lagrange Grassmannian; since symplectic transformations carry Lagrangian planes into Lagrangian planes, we get a one-parameter family of diffeomorphisms of $L(E)$. The infinitesimal generator of this family is a nonstationary vector field on $L(E)$ which we denote by $\frac{1}{2}\gamma_\tau^{-1}(\sigma(z_\tau^{(k_\tau)}\cdot, \Lambda))$, $\Lambda \in L(E)$. This is not simply a conditional notation; in general, for every $\Lambda \in L(E)$ there is a natural identification of the tangent space $T_\Lambda L(E)$ with a space of quadratic forms on Λ: a tangent vector $(\partial\Lambda/\partial s)(s) \in T_{\Lambda(s)}L(E)$ is identified with the quadratic form

$$x(s) \mapsto \sigma\left(\frac{\partial x}{\partial s}(s), x(s)\right),$$

where $x(s) \in \Lambda(s)$. It is not hard to show that under this identification the vector field on $L(E)$ corresponding to a quadratic Hamiltonian $q(x)$ $(x \in E)$ on E is identified with the family of forms $q|_\Lambda$, $\Lambda \in L(E)$.

The differential equation $\dot{\Lambda}_\tau = \frac{1}{2}\gamma_\tau^{-1}(\sigma(z_\tau^{(k_\tau)}\cdot, \Lambda))$ on $L(E)$ is called the Jacobi equation. The solutions of this equation are defined to be not only the continuous but also the piecewise continuous curves on $L(E)$, with the derivative with respect to τ at a point of discontinuity taken to be the limit of the corresponding derivatives from the left. Thus, to uniquely determine a solution it is necessary to specify the jumps at the points of discontinuity in addition to the initial value.

DEFINITION 1. A *Jacobi curve* is defined to be a piecewise smooth curve on $L(E)$ satisfying the Jacobi equation and the conditions $\Lambda_0 = \Pi_0$ and $\Lambda_{\tau+0} = (\Lambda_\tau + \Gamma_\tau) \cap \Gamma_\tau \; \forall \tau \in [0, t)$.

It is easy to see that $\Gamma_\tau \subset \Lambda_\tau \; \forall \tau \in (0, t]$, and Λ_τ is continuous at any point of continuity of Γ_τ, $\tau \in (0, t]$.

Suppose that Λ_i $(i = 0, 1, 2)$ are three Lagrangian planes in E. It is not hard to show that the mapping $\lambda_0 = \lambda_1 + \lambda_2 \mapsto \sigma(\lambda_1, \lambda_2)$, where $\lambda_i \in \Lambda_i$, $i = 0, 1, 2$, unambiguously determines a quadratic form on the space $(\Lambda_1 + \Lambda_2) \cap \Lambda_0 / \bigcap_{i=0}^2 \Lambda_i$. Denote this form by the letter q, and let $\mathrm{ind}_{\Lambda_0}(\Lambda_1, \Lambda_2) = \mathrm{ind}\, q + \frac{1}{2}\dim\ker q$. The index thus defined can without difficulty be expressed in terms of the Maslov index of the triple of Lagrangian planes [4], but for our purposes it is more convenient than the Maslov index. Essential for what follows are its nonnegativeness and the triangle inequality:

$$0 \leq \mathrm{ind}_{\Lambda_0}(\Lambda_1, \Lambda_3) \leq \mathrm{ind}_{\Lambda_0}(\Lambda_1, \Lambda_2) + \mathrm{ind}_{\Lambda_0}(\Lambda_2, \Lambda_3) \quad \forall \Lambda_i \in L(E),\ i = 0, 1, 2, 3.$$

DEFINITION 2. A piecewise smooth curve Λ_τ, $\tau \in [t_0, t_1]$, in $L(E)$ is said to be *simple* if $\exists \Delta \in L(E)$ such that $\Lambda_\tau \cap \Delta = 0$ and $\mathrm{ind}_\Delta(\Lambda_\tau, \Lambda_{\tau+0}) = 0 \; \forall_\tau \in [t_0, t_1]$.

It can be shown that a sufficiently small piece of an arbitrary piecewise smooth curve is simple.

THEOREM 1. *Let Λ_τ, $0 \leq \tau \leq t$, be a Jacobi curve and let $\tau_{l+1} = 0 = \tau_0 < \tau_1 < \cdots < \tau_l = t$ be an arbitrary partition of the interval $[0, t]$. Then*

$$\sum_{i=0}^{l} \mathrm{ind}_{\Pi_0}(\Lambda_{\tau_i}, \Lambda_{\tau_{i+1}}) \leq \mathrm{ind}\, \psi G_t'' + n.$$

But if all the pieces $\Lambda_\cdot|_{[\tau_i, \tau_{i+1}]}$ of the Jacobi curve are simple, $0 \leq i < l$, then the inequality becomes an equality.

There is a simple homotopy interpretation of Theorem 1. Recall that the tangent space $T_\Lambda L(E)$ is identified with the space of quadratic forms on Λ; in particular, it is partially ordered. We say that a curve on $L(E)$ is *nondecreasing* if its velocity at an arbitrary point is a nonnegative quadratic form, and we observe that a Jacobi curve is nondecreasing. Further, it is known that the fundamental group $\pi_1(L(E))$ is isomorphic to the group $\mathbf{Z}$ of integers, and there is a canonical isomorphism of these groups characterized by the fact that the nondecreasing closed curves in $L(E)$ receive positive "indices" in $\mathbf{Z}$. An integer corresponding to a given closed curve under this isomorphism is called the *Maslov index* of the curve.

THEOREM 2. *Let Λ_τ, $\tau \in [0,t]$, be a Jacobi curve, and $t_1, \dots, t_N$ all its points of discontinuity. Suppose that simple continuous nondecreasing curves join Λ_{t_i} with Λ_{t_i+0}, $i = 1, \dots, N$, and Λ_t with Λ_0 (this can always be done). Denote the resulting continuous closed curve by $\overline{\Lambda}$. Then $\operatorname{ind} \psi G''_t = \operatorname{ind} \overline{\Lambda}. - n$, where $\operatorname{ind} \overline{\Lambda}.$ is the Maslov index of $\overline{\Lambda}.$.*

REMARK 1. $\operatorname{ind} \psi G''_t$ is a nondecreasing integer-valued function of t, and the case of a regular variational problem this function has jumps at points conjugate to zero. In the general case the usual concept of a conjugate point does not make much sense, but nevertheless it is easy to deduce an explicit expression for the jumps of $\operatorname{ind} \psi G''_t$ from Theorem 1.

REMARK 2. Assume that the quadratic form γ_τ is nonnegative but degenerate. If $\dim \ker \gamma_\theta$ is constant near the given point τ, then, replacing $\mathbf{R}^r$ by $\ker \gamma_\tau$ in the definition of the number k_τ and the form γ_τ (and parametrizing $\ker \gamma_\theta$ for θ close to τ with the help of this space), we get a number $\hat{k}_\tau > k_\tau$ and a form $\hat{\gamma}_\tau$. If $\operatorname{ind} \psi G''_t < +\infty$, then $\hat{\gamma}_\tau \geq 0$. The procedure can be repeated, and we do this until we either get a strictly positive or a zero form. If a strictly positive form (and uniformly with respect to τ) is obtained as a result for all $\tau \in (0,t]$, then Theorem 1 remains true with the obvious modification of the Jacobi equation. Otherwise, the modified Jacobi equation has singularities. If we assume in addition that z_τ depends on τ in a piecewise analytic manner, then the singularities are concentrated on the union of finitely many isolated points and closed subintervals of $[0,t]$. Let $\delta > 0$; excising from $[0,t]$ the δ-neighborhood of the set of singularities, we get an equation without singularities. Let $i(\delta)$ be the index computed according to the recipe in Theorem 1 in terms of the solution of this "reduced" Jacobi equation. Then $i(\delta) \to \operatorname{ind} \psi G''_t$ as $\delta \downarrow 0$.

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Received 1/MAR/85

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Translated by H. H. MCFADEN