Good Lie Brackets for classical and quantum harmonic oscillators

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Abstract

We study the small-time controllability problem on the Lie groups $SL_2(\mathbb{R})$ and $SL_2(\mathbb{R}) \ltimes H_d(\mathbb{R})$ with Lie bracket methods (here $H_d(\mathbb{R})$ denotes the $(2d+1)$-dimensional real Heisenberg group). Then, using unitary representations of $SL_2(\mathbb{R}) \ltimes H_d(\mathbb{R})$ on $L^2(\mathbb{R}^d, \mathbb{C})$ and $L^r(T^*\mathbb{R}^d, \mathbb{R})$, $r \in [1, \infty]$, we recover small-time reachability properties of the Schrödinger PDE for the quantum harmonic oscillator, and find new small-time reachability properties of the Liouville PDE for the classical harmonic oscillator.

Keywords: Geometric control, unitary representations, harmonic oscillator, Schrödinger PDE, Liouville PDE.

1 Introduction

1.1 The model

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. In this paper, we study left-invariant control affine systems of the form

$$\dot{q}(t) = q(t)\left(a + \sum_{i=1}^{k} u_i(t) b_i\right) \in T_{q(t)}G, \quad q(t) \in G, a, b_i \in \mathfrak{g}, t \in [0, T]. \tag{1}$$

The controls are real-valued and piece-wise constant, $u = (u_1, \dots, u_k) \in \mathrm{PWC}([0, T], \mathbb{R}^k)$. The solution of (1) associated with the initial condition q_0 and the control u at time t is denoted by $q(q_0, u, t)$. When the initial condition $q_0 = \mathrm{id}_G$ is the identity of G, we drop it from the notation and simply write the associated solution as $q(u, t)$.

Definition 1. — *An element $q \in G$ is **reachable** by system (1) if there exist $T > 0$ and $u \in \mathrm{PWC}([0, T], \mathbb{R}^k)$ such that $q(u, T) = q$. The set of reachable elements is denoted by $\mathscr{A}$.*

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— *An element $q \in G$ is **approximately reachable** by system (1) if for every $\varepsilon > 0$ there exist $T \geq 0$ and $u \in \mathrm{PWC}([0,T],\mathbb{R}^k)$ such that $|q(u,T) - q| < \varepsilon$. The set of approximately reachable elements is denoted by $\overline{\mathscr{A}}$.*

— *An element $q \in G$ is **small-time reachable** by system (1) if for every $T > 0$, there exist $\tau \in [0,T]$ and $u \in \mathrm{PWC}([0,\tau],\mathbb{R}^k)$ such that $q(u,\tau) = q$. The set of small-time reachable elements is denoted by $\mathscr{A}_{st}$.*

— *An element $q \in G$ is **small-time approximately reachable** by system (1) if for every $\varepsilon > 0$ there exist $\tau \in [0,\varepsilon]$ and $u \in \mathrm{PWC}([0,\tau],\mathbb{R}^k)$ such that $|q(u,\tau) - q| < \varepsilon$. The set of small-time approximately reachable elements is denoted by $\overline{\mathscr{A}_{st}}$.*

The system is said to be *controllable* (respectively *approximately controllable*) if $\mathscr{A} = G$ (resp. if $\overline{\mathscr{A}} = G$). The system is said to be *small-time controllable* (respectively *small-time approximately controllable*) if $\mathscr{A}_{st} = G$ (resp. if $\overline{\mathscr{A}_{st}} = G$).

Note that the set $\mathscr{A}$ presented in Definition 1 is the attainable set from the identity. Since system (1) is left invariant, it is controllable from the identity if and only if it is controllable from any point of the group. The same statement holds true for small-time controllability.

Note that one could also define the reachable sets w.r.t. L^∞ controls. Then, the (small-time) approximately reachable sets w.r.t. piecewise constant controls, or L^∞ controls, are the same. This is a consequence of the density of piecewise constant functions in L^∞, and the continuity of the endpoint map $L^\infty([0,\tau],\mathbb{R}^k) \ni u \mapsto q(u,\tau) \in G$.

1.2 Good Lie brackets

In this work we use the following notions of geometric control theory.

Definition 2. *An element $X \in \mathfrak{g}$ is said to be a* good Lie bracket *for system (1) if $e^{vX} \in \overline{\mathscr{A}_{st}}$ for every $v \in \mathbb{R}$.*

Definition 3. *An element $X \in \mathfrak{g}$ is said to be* compatible *with system (1) (at time t) if $e^X \in \overline{\mathscr{A}}$ for (1) (at time t). Another control system is said to be compatible with system (1) if its approximately reachable set is contained in $\overline{\mathscr{A}}$.*

It is well-known that every element in the Lie algebra $\mathfrak{l} := \mathrm{Lie}\{b_1, \ldots, b_k\}$, generated by $b_1, \ldots, b_k$, is a good Lie bracket (see e.g. [10, Lemma 6.2] and [8, Theorem 3.3]).

It is also well-known that, if the drift of (1) is recurrent or Poisson stable, then $-va$, $v > 0$, is compatible with the system for a time large enough (see e.g. [3, Proposition 8.2]). In this work, we shall study systems where the drift a is a good Lie bracket, a stronger property than compatibility, since the system is able to displace along the drift not only with positive and negative coefficients, but also in arbitrarily small times. The notion of good Lie bracket was recently introduced by the first author for more general control affine systems [4], and studied also in the context of bilinear Schrödinger PDEs by Karine Beauchard and the third author [6, 5].

1.3 A small-time controllability result on $SL_2 \ltimes H_d$

We introduce the following Lie algebras:

— $\mathfrak{sl}_2(\mathbb{R}) := \{M \in \mathrm{Mat}_2(\mathbb{R}) \mid \mathrm{tr}(M) = 0\}$;

— $\mathfrak{h}_d(\mathbb{R}) := \left\{ \begin{pmatrix} 0 & \vec{x} & z \\ 0 & 0_d & \vec{y}^\dagger \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{d+2}(\mathbb{R}) \mid \vec{x}, \vec{y} \in \mathbb{R}^d, z \in \mathbb{R} \right\}$.

Here, 0_d denotes the d-dimensional square zero matrix, $\vec{x}$ is a row d-dimensional vector and $\vec{y}^\dagger$ is a column d-dimensional vector. To ease notations, we shall simply denote them as $\mathfrak{h}_d$ and $\mathfrak{sl}_2$. Let $\mathfrak{Der}(\mathfrak{h}_d)$ denote the derivations over the algebra $\mathfrak{h}_d$. We define a homomorphism $\rho : \mathfrak{sl}_2 \to \mathfrak{Der}(\mathfrak{h}_d)$ as

$$\rho\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\begin{pmatrix} 0 & \vec{x} & z \\ 0 & 0_d & \vec{y}^\dagger \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha\vec{x} + \beta\vec{y} & 0 \\ 0 & 0_d & \gamma\vec{x}^\dagger - \alpha\vec{y}^\dagger \\ 0 & 0 & 0 \end{pmatrix}.$$

It induces a semi-direct product structure on $\mathfrak{sl}_2 \oplus \mathfrak{h}_d$, called $\mathfrak{sl}_2 \ltimes_\rho \mathfrak{h}_d$, with bracket

$$[(a,X),(b,Y)] = ([a,b],[X,Y] + \rho(a)Y - \rho(b)X), \quad \forall a,b \in \mathfrak{sl}_2, X,Y \in \mathfrak{h}_d. \tag{2}$$

We shall simply denote this Lie algebra by $\mathfrak{sl}_2 \ltimes \mathfrak{h}_d$, and its associated Lie group by $SL_2(\mathbb{R}) \ltimes H_d(\mathbb{R})$ or simply $SL_2 \ltimes H_d$. We also fix basis $\{a,b,c\}$ of $\mathfrak{sl}_2$ and $\{X_1, Y_1, \dots, X_d, Y_d, Z\}$ of $\mathfrak{h}_d$ in such a way that the following commutation relations are satisfied

$$[b,a] = c, [a,c] = 2a, [b,c] = -2b,$$

$$[X_i, Y_i] = Z, [X_i, Z] = [Y_i, Z] = 0, \quad i = 1, \dots, d.$$

We first study the controllability of the following left-invariant control-affine system:

$$\frac{d}{dt}q(t) = q(t)\left(a + u_0(t)b + \sum_{i=1}^{d} u_i(t)X_i + r(t)Z\right), \quad q(t) \in SL_2 \ltimes H_d. \tag{3}$$

System (3) can be lifted to $M_n \ltimes H_d$ for every $n \in \mathbb{N} \cup \{\infty\}$, where M_n denotes the covering space of SL_2 of degree $n \in \mathbb{N}$, and M_∞ denotes the universal cover (see Section 3 for the definition of the covering spaces).

Our first result is the following small-time controllability property.

Theorem 1. *Equation* (3) *is small-time controllable on $SL_2 \ltimes H_d$ and also on $M_n \ltimes H_d$, for any $n \in \mathbb{N}$. Moreover, equation* (3) *with $u_i, r \equiv 0$ for $i \in \{1, \dots, d\}$ is small-time controllable on SL_2 and also on M_n, for any $n \in \mathbb{N}$.*

As it will be clear from the proof given in Section 4, both statements of Theorem 1 are false on the universal covers $M_\infty \ltimes H_d$ and M_∞. We remark that the second part of the theorem states the small-time controllability of a scalar-input system with drift on a non-compact connected Lie group, SL_2: such a result would be impossible on compact Lie groups such as $SU_N(\mathbb{C}), N \geq 2$, where scalar-input systems with drift are never small-time controllable (see, e.g., [2, 1] and we refer also to [9] for recent advances on the subject). To the best of our knowledge, Theorem 1 is new. The large-time controllability of system (3) could also be proved by evoking the Lie bracket generating condition in combination with the periodicity of $e^{t(a-b)}$, but the small-time controllability property is more subtle and necessitates a different proof inspired by the recently developed techniques of good Lie brackets [4].

1.4 Consequences for harmonic oscillator Schrödinger equation

The choice of studying $SL_2 \ltimes H_d$ in this article is motivated by the description of two physical systems, namely the quantum and the classical harmonic oscillators. In this section we state the consequences for the quantum harmonic oscillator. The relation between SL_2 and the Schrödinger equation for the harmonic oscillator is known as the Weil representation (see e.g. [11, Chapter XI]).

The Schrödinger equation for the quantum harmonic oscillator is given by

$$i\partial_t\psi(t,x) = \left(-\frac{\Delta}{2} + u_0(t)\frac{|x|^2}{2} + \sum_{j=1}^{d} u_j(t)x_j\right)\psi(t,x), \qquad \psi(t=0) = \psi_0 \in L^2(\mathbb{R}^d,\mathbb{C}), \tag{4}$$

with control $u = (u_0,\dots,u_d) \in \mathrm{PWC}([0,T],\mathbb{R}^{d+1})$. The control u_0 models the frequency tuning of the trapping potential, and the $u_j, j = 1,\dots,d$ model the dipolar interaction with the spacial positions of the oscillator. Such system is among the most relevant quantum dynamics, widely used in physics and chemistry to model a variety of situations, such as atoms trapped in optical cavities or vibrational dynamics of molecular bonds.

The Lie algebra generated by the linear operators $\mathrm{Lie}\{\frac{i\Delta}{2}, \frac{i|x|^2}{2}\}$ equipped with the commutator is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, the algebra of real 2×2 traceless matrices, and the Lie algebra generated by $\mathrm{Lie}\{\frac{i\Delta}{2}, \frac{i|x|^2}{2}, ix_1,\dots,ix_d\}$ is isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{h}_d$. We thus have introduced infinite-dimensional representations of the algebras $\mathfrak{sl}_2, \mathfrak{sl}_2 \ltimes \mathfrak{h}_d$ acting on $L^2(\mathbb{R}^d,\mathbb{C})$.

Quantum states are defined up to global phases, hence we shall say that a state ψ_1 is reachable if there exists a number $\theta \in [0,2\pi)$ such that $e^{i\theta}\psi_1$ is reachable. Since the generator $-\frac{1}{2}\Delta + u_0\frac{|x|^2}{2} + \sum_{j=1}^d u_j x_j$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d,\mathbb{C})$ for any $u_0,\dots,u_d \in \mathbb{R}^{d+1}$ (see, e.g., [13, Corollary page 199]), equation (4) is globally-in-time well-posed, meaning that for any $\psi_0 \in L^2(\mathbb{R}^d,\mathbb{C}), u \in \mathrm{PWC}(\mathbb{R},\mathbb{R}^{d+1})$ there exists a unique mild solution $(t \mapsto \psi(t,u,\psi_0)) \in C(\mathbb{R}, L^2(\mathbb{R}^d,\mathbb{C}))$ of (4).

Definition 4. *A state $\psi_1 \in L^2(\mathbb{R}^d,\mathbb{C})$ is said to be small-time reachable from $\psi_0 \in L^2(\mathbb{R}^d,\mathbb{C})$ for system* (4) *if for every time $T > 0$ there exist $\tau \in [0,T]$, a control $u \in \mathrm{PWC}([0,\tau],\mathbb{R}^{d+1})$ and a global phase $\theta \in [0,2\pi)$ such that the solution ψ of* (4) *satisfies $\psi(\tau,u,\psi_0) = e^{i\theta}\psi_1$. The set of small-time reachable states from $\psi_0 \in L^2(\mathbb{R}^d,\mathbb{C})$ is denoted by $\mathscr{R}_{st}(\psi_0)$.*

As a first consequence of Theorem 1, we have the following characterization of the small-time reachable set of (4).

Theorem 2. *For any* $\psi_0 \in L^2(\mathbb{R}^d, \mathbb{C})$, *system* (4) *satisfies*

$$\mathscr{R}_{st}(\psi_0) = \{e^{is(\Delta - |x|^2)} e^{i(\alpha|x|^2 + px)} \sigma^{d/2} \psi_0(\sigma x + \beta), s, \alpha \in \mathbb{R}, p, \beta \in \mathbb{R}^d, \sigma > 0\}.$$

Intuitively, the control of each component p_j and β_j is obtained thanks to the presence of the control operator x_j, while the control of σ, α, and s follows from the presence of $|x|^2$.

Such a result was already found in [6, Theorem 23] with analytical methods, and we propose here a geometric proof, based on an infinite-dimensional representation of $SL_2 \ltimes H_d$. The statement of this theorem tells that we can control in small-times some physically relevant quantities, such as position, momentum, and spread of the initial state. We cannot control the global phase of the wavefunction with system (4). However, in order to recover the control on the global phase, it suffices to add an additional constant-in-space control of the form $u_{d+1}(t) \cdot 1$ to (4). The corresponding controllability analysis for system (4) with $u_0 \equiv 1$ was performed in [12], and a weaker statement was obtained in [14]. More general small-time controllability properties of Schrödinger PDEs were recently obtained in [5].

1.5 Consequences for harmonic oscillator Liouville equation

In this section we state the consequences for the classical harmonic oscillator. The Liouville equation for the classical harmonic oscillator is given by the transportation of a density $\rho \in L^r(T^*\mathbb{R}^d, \mathbb{R})$ along a Hamiltonian vector field

$$\partial_t \rho(t, q, p) = \overrightarrow{H}_u \rho(t, q, p), \quad \rho(t = 0) = \rho_0 \in L^r(T^*\mathbb{R}^d), \tag{5}$$

where the Hamiltonian function is given by

$$H_u(q, p) = \frac{|p|^2}{2} + u_0(t)\frac{|q|^2}{2} + \sum_{j=1}^{d} u_j(t) q_j, \tag{6}$$

and the Hamiltonian vector field is the first-order differential operator defined as

$$\overrightarrow{H}_u := \{H_u, \cdot\} = p \cdot \nabla_q - u_0(t) q \cdot \nabla_p - \sum_{j=1}^{m} u_j(t) \partial_{p_j}, \tag{7}$$

and $\{\cdot, \cdot\}$ denotes the Poisson bracket on $T^*\mathbb{R}^d$ (that is, $\{f, g\} = \nabla_p f \cdot \nabla_q g - \nabla_q f \cdot \nabla_p g$ for any $f = f(q, p), g = g(q, p) \in C^\infty(T^*\mathbb{R}^d)$). The Lie algebra of smooth functions generated by Lie $\{\frac{|p|^2}{2}, \frac{|q|^2}{2}, q_1, \ldots, q_d\}$, equipped with the Poisson bracket, is also isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{h}_d$. Since the vector field (7) is globally Lipschitz, the solution of the Liouville equation $\rho(t)$ is globally-in-time well-posed, meaning that for any $\rho_0 \in L^r(T^*\mathbb{R}^d, \mathbb{R}), u \in \mathrm{PWC}(\mathbb{R}, \mathbb{R}^{d+1})$ there exists a unique mild solution $(t \mapsto \rho(t, u, \rho_0)) \in C(\mathbb{R}, L^r(T^*\mathbb{R}^d, \mathbb{R}))$ of (5). Such a solution writes as $\rho(t, u, \rho_0) = \rho_0 \circ \Phi^t_{H_u}(q, p)$ where $\Phi^t_{H_u}$ is the flow solving the Hamiltonian equations on $T^*\mathbb{R}^d$,

$$\frac{d\Phi^t_{H_u}(q, p)}{dt} = \begin{pmatrix} \nabla_p H_u(\Phi^t_{H_u}(q, p)) \\ -\nabla_q H_u(\Phi^t_{H_u}(q, p)) \end{pmatrix}, \quad \Phi^{t=0}_{H_u}(q, p) = (q, p). \tag{8}$$

Notice that $\Phi^t_{H_u}$ is an Hamiltonian diffeomorphism hence orientation- and volume-preserving.

Definition 5. *A state* $\rho_1 \in L^r(T^*\mathbb{R}^d, \mathbb{R})$, $r \in [1, \infty]$, *is said to be small-time reachable from* $\rho_0 \in L^r(T^*\mathbb{R}^d)$ *for system* (5) *if for every* $T > 0$ *there exist* $\tau \in [0, T]$ *and* $u \in \mathrm{PWC}([0,\tau], \mathbb{R}^{d+1})$ *such that the solution* ρ *of* (5) *satisfies* $\rho(\tau, u, \rho_0) = \rho_1$. *The set of small-time reachable states from* $\rho_0 \in L^r(T^*\mathbb{R}^d, \mathbb{R})$ *is denoted by* $\mathscr{R}_{st}(\rho_0)$.

As a second consequence of Theorem 1, we have the following characterization of the small-time reachable sets of (5).

Theorem 3. *For any* $\rho_0 \in L^r(T^*\mathbb{R}^d, \mathbb{R})$, *system* (5) *satisfies*

$$\begin{aligned}&\mathscr{R}_{st}(\rho_0) = \\ &\{\rho_0(\alpha(q+s)\cos(t) + \alpha^{-1}(p + rq + w)\sin(t), -\alpha(q+s)\sin(t) + \alpha^{-1}(p + rq + w)\cos(t)) \mid \\ &\alpha > 0, t, r \in \mathbb{R}, s, w \in \mathbb{R}^d\}.\end{aligned}$$

To the best of our knowledge, Theorem 3 is new, and gives some first insights on the controllability of Liouville Hamiltonian equations, a research direction almost unexplored. It is obtained thanks to the specific nature of the Hamiltonian function (6). Extensions of such technique to more general Hamiltonians will be the subject of future investigations.

A study on the relation between classical and quantum controllability appeared in [7]; however, their work deals with the control of a single classical particle following Hamilton equations, while here we consider the control of a density of classical particles described by ρ following the Liouville equation. In particular, the harmonic oscillator equation of a single classical particle (described by (8)) is clearly globally controllable, while the associated Liouville equation is not (as we can only reach the states described in Theorem 3). In this perspective, our geometric approach based on the representations of $SL_2 \ltimes H_d$ highlights an analogy between classical and quantum controllability of quadratic Hamiltonians.

The article is organised as follows: in Section 2 we recall a useful compatibility property for system (1); in Section 3 we recall the covering spaces of $SL_2(\mathbb{R})$; in Section 4 we prove a slightly more general version of Theorem 1; in Sections 5 and 6 we prove resp. Theorems 2 and 3.

2 Compatible elements on Lie groups

In this section we recall a property (found by the first author in [4, Corollary 1] for general control affine systems) in the specific Lie group framework introduced above.

We associate to the Lie algebra $\mathfrak{l} := \mathrm{Lie}\{b_1, ..., b_k\}$ its Lie group

$$L := \{e^{t_1 X_1} ... e^{t_m X_m} \mid m \in \mathbb{N}^*, X_1, ..., X_m \in \mathfrak{l}, t_1, ..., t_m \in \mathbb{R}\}.$$

The following result is a particular case of [4, Corollary 1].

Theorem 4. *For* $\tau \geq 0$, *every convex combination of elements of the form* $Ad_L(\tau a) + \mathfrak{l}$ *is compatible with system (1) at time* τ.

Proof. The driftless system

$$\dot{q}(t) = q(t) \sum_{i=1}^{k} u_i(t) b_i \qquad q(t) \in G, b_i \in \mathfrak{g}, u_i(t) \in \mathbb{R}, \tag{9}$$

is compatible with system (1). More precisely, every $q \in L$ is small-time approximately reachable by (1): indeed, thanks to [10, Lemma 6.2], $q = e^{s_N b_{i_N}} \dots e^{s_1 b_{i_1}}$ for some $s_1, \dots, s_N \in \mathbb{R}, N \in \mathbb{N}$, $\{i_1, \dots, i_N\} \subset \{1, \dots, k\}$. Consider then a piecewise constant control u defined as $u_{i_1}(t) \equiv s_1/t_1$ for a time interval $[0, t_1]$, ..., $u_{i_N}(t) \equiv s_N/t_N$ for a time interval $[0, t_N]$: the associated solution writes $q(t_1 + \dots + t_N, u) = e^{t_N a + s_N b_{i_N}} \dots e^{t_1 a + s_1 b_{i_1}}$ hence for $t_1, \dots, t_N$ small enough $q(t_1 + \dots + t_N, u)$ is close to q.

Moreover, $e^{\tau a}$ is reachable in time $T = \tau$, using a free evolution of the system (i.e. a control $u = 0$ on a time interval of size τ). Let $q \in L$, then $q e^{\tau a} q^{-1}$ is approximately reachable in time $T = \tau + \varepsilon$ for every $\varepsilon > 0$. Thanks to the properties of the exponential map on a Lie group,

$$q e^{\tau a} q^{-1} = e^{\mathrm{Ad}_q(\tau a)}.$$

Thus $\mathrm{Ad}_q(\tau a)$ is compatible with (1). Finally, according to standard relaxation technique, every convex combination of compatible vector fields is also compatible (see e.g. [3, Proposition 8.1]). □

To pass from approximate to exact controllability, we shall need the following corollary of Krener's theorem (see e.g. [3, Corollary 8.1]).

Proposition 1. *If a system defined on a finite-dimensional manifold is approximately controllable and Lie Bracket Generating, then it is controllable.*

3 Covering spaces of $SL_2(\mathbb{R})$

In this section we recall for completeness some properties of $SL_2(\mathbb{R})$ that we shall need. Their proofs are standard and we thus omit them (see, e.g., [11, Chapters II & III] for details).

The representation formula given in the Proposition below is known as the Iwasawa decomposition.

Proposition 2. *Every element $g \in SL_2(\mathbb{R})$ has a unique representation $g = kan$, $k \in K$, $a \in A$, $n \in N$, where*

$$K = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \theta \in [0, 2\pi) \right\}, \quad A = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, r \in \mathbb{R}_{>0} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{R} \right\}.$$

The covering spaces of $SL_2(\mathbb{R})$ are defined in terms of the Iwasawa decomposition.

Proposition 3. *The function $\chi : \mathbb{R}^3 \to SL_2(\mathbb{R})$ defined as*

$$\chi(\theta, r, x) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{10}$$

is the universal cover of $SL_2(\mathbb{R})$.

Definition 6. *The covering space M_n of degree $n \in \mathbb{N}$ of $SL_2(\mathbb{R})$ is obtained by identifying $(\theta + 2k\pi, r, x) \in \mathbb{R}^3$ with $(\theta, r, x) \in \mathbb{R}^3$ for every $k \in n\mathbb{Z}$. The cover M_2 of degree 2 of $SL_2(\mathbb{R})$ is called the metaplectic group. The cover M_1 of degree 1 is identified with $SL_2(\mathbb{R})$, and the universal cover M_∞ is identified with $\mathbb{R}^3$.*

4 Proof of Theorem 1

4.1 Proof of the second statement of Theorem 1

Owing to (2), we have

$$[a, X_i] = Y_i, [b, X_i] = 0, [c, X_i] = X_i, [a, Y_i] = 0, [b, Y_i] = X_i, [c, Y_i] = -Y_i, \quad i = 1, \dots, d,$$

$$[a, Z] = [b, Z] = [c, Z] = 0,$$

where we simply denote a and X the elements $(a, 0)$ and $(0, X)$ for any $a \in \mathfrak{sl}_2, X \in \mathfrak{h}_d$. Thanks to Proposition 2, every element of SL_2 can be written in the form $e^{t_1(a-b)}e^{t_2c}e^{t_3b}$ where $t_1, t_2, t_3 \in \mathbb{R}$, and this decomposition is unique.

System (3) can then be lifted to every covering space of $SL_2 \ltimes H_d$ (notice that H_d is simply connected).

Theorem 5. *The elements b and c are good Lie brackets and for every $t_1 \geq 0$, $e^{t_1(a-b)}$ is small-time approximately reachable for system (3) defined on any covering space $M_n, n \in \mathbb{N} \cup \{\infty\}$, of SL_2. System* (3) *with $u_i, r \equiv 0, i \in \{1, \dots, d\}$ is small-time controllable on every covering space $M_n, n \in \mathbb{N}$, of SL_2 but not on the universal one M_∞.*

Proof. We apply Theorem 4 with $\mathfrak{l} = \text{Span}\{b\}$ and $L = \{e^{vb} \mid v \in \mathbb{R}\}$. Every element of the form $\text{Ad}_{e^{vb}}(\tau a) + ub$ with $u, v \in \mathbb{R}, \tau > 0$ is compatible. Thanks to the properties of the exponential map on a Lie group,

$$\text{Ad}_{e^{vb}}(\tau a) = e^{\text{ad}_{vb}}(\tau a) = \sum_{k=0}^{+\infty} \frac{v^k}{k!}(ad_b)^k(\tau a) = \tau a + v\tau c - v^2\tau b.$$

By taking $v = r/\tau$, the elements $\tau a + rc$ are compatible for every $r \in \mathbb{R}$, hence c is a good Lie bracket. Then, the following system

$$\dot{q} = q(a + ub + vc) \qquad u, v \in \mathbb{R}, \tag{11}$$

is compatible with system (1). We apply Theorem 4 to the system (11), with $\mathfrak{l} = \text{Span}\{b, c\}$. Then every element of the form $\text{Ad}_{e^{ub+vc}}(a) + sb + tc$, $u, v, s, t \in \mathbb{R}$ is compatible. In particular, $\text{Ad}_{e^{vc}}(a) = \exp(\text{ad}_{vc}(a)) = e^{-2v}a$ is compatible for every $v \in \mathbb{R}$. Thus wa is also compatible for every $w > 0$. Then every element of the form $e^{t_1(a-b)}e^{t_2c}e^{t_3b}$, $t_1 \geq 0$, $t_2, t_3 \in \mathbb{R}$ is small-time approximately reachable by the compatible system, thus it is small-time approximately reachable by system (1). If the considered covering space is not the universal one, for every $t \in \mathbb{R}$ there exists $t' \geq 0$ such that $e^{t(a-b)} = e^{t'(a-b)}$. Hence (1) is small-time approximately controllable on every covering space $M_n, n \in \mathbb{N}$, but not on the universal one M_∞. Moreover, system (3) is Lie Bracket Generating so, according to Proposition 1, it is small-time controllable on every covering space $M_n, n \in \mathbb{N}$, of SL_2, but not on the universal one M_∞. □

4.2 Proof of the first statement of Theorem 1

Thanks to [10, Lemma 6.2], and since (3) is Lie Bracket Generating, we are left to prove that $X_i, Y_i, i = 1,\dots,d$ and Z are good Lie brackets. It is clear that $X_i, i \in \{1,\dots,d\}, Z$ are good Lie brackets. Thanks to Theorem 4, every element of the form $\mathrm{Ad}_{e^{vX_i}}(a) + uZ$ with $u, v \in \mathbb{R}$ is compatible. Thanks to the properties of the exponential map on a Lie group,

$$\mathrm{Ad}_{e^{vX_i}}(\tau a) = e^{\mathrm{ad}_{vX_i}}(\tau a) = \sum_{k=0}^{+\infty} \frac{v^k}{k!}(ad_{X_i})^k(\tau a) = \tau a + v\tau Y_i - \frac{v^2\tau}{2} Z.$$

By taking $v = r/\tau$, $u = v^2\tau/2$, the elements $\tau a + rY_i$ are compatible for every $r \in \mathbb{R}, \tau > 0$, hence by taking τ small enough Y_i is a good Lie bracket.

5 Proof of Theorem 2

For any $\psi \in C_c^\infty(\mathbb{R}^d)$, we compute

$$[i\Delta, i|x|^2]\psi := -\Delta(|x|^2\psi) + |x|^2(\Delta\psi) = -2d\psi - 4x\cdot\nabla\psi = (-2d - 4x\cdot\nabla)\psi.$$

Analogously, one checks that $[i\Delta, \frac{d}{2} + x\cdot\nabla]$ is proportional (as an operator acting on test functions) to $i\Delta$ and $[i|x|^2, \frac{d}{2} + x\cdot\nabla]$ is proportional to $i|x|^2$. Hence,

$$\mathrm{Lie}\left\{\frac{i\Delta}{2}, \frac{i|x|^2}{2}\right\} = \mathrm{span}\left\{\frac{i\Delta}{2}, \frac{i|x|^2}{2}, \frac{d}{2} + x\cdot\nabla\right\}.$$

Every element of this Lie algebra is an essentially skew-adjoint operator on $L^2(\mathbb{R}^d, \mathbb{C})$ with domain $C_c^\infty(\mathbb{R}^d, \mathbb{C})$. Moreover, the application $\pi : \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{Lie}\{\frac{i\Delta}{2}, \frac{i|x|^2}{2}\}$ defined by $\pi(a) = \frac{i\Delta}{2}$, $\pi(b) = \frac{i|x|^2}{2}$, $\pi(c) = \frac{d}{2} + x\cdot\nabla$ and extended by linearity, is a Lie algebra isomorphism.

In Theorem 5 we showed the small-time controllability on every covering space of SL_2 (except the universal one). In the following section we translate this property for the Schrödinger equation (4). We denote with $\mathbb{U}(X)$ the group of linear unitary operators on a normed space X.

5.1 Unitary representation of SL_2

The construction of the following group morphism, sometimes called the Weil representation, is standard hence we omit the details (see, e.g., [11, Chapter XI]).

Proposition 4. *There exists a group morphism f from the universal cover M_∞ of SL_2 to $\mathbb{U}(L^2(\mathbb{R}^d, \mathbb{C}))$ such that $f(e^a) = e^{\pi(a)}$ for every $a \in \mathfrak{sl}(2)$.*

The following lemma tells that it is possible to restrict the representation from the universal cover to lower degree covers.

Lemma 1. *If d is even, there exists a unitary representation $f : M_n \to \mathbb{U}(L^2(\mathbb{R}^d))$ such that $f(e^a) = e^{\pi(a)}$ for every covering space $M_n, n \in \mathbb{N} \cup \{\infty\}$, of SL_2. If d is odd, such a unitary representation f is obtained if M_{2n} is a covering space of even degree $2n, n \in \mathbb{N}$, or the universal cover M_∞. In particular, if d is odd, we have a unitary representation from the metaplectic group M_2.*

Proof. We denote $f : M_\infty \to \mathbb{U}(L^2)$ the unitary representation of the universal cover, which exists according to Proposition 4 and which verifies $f(e^a) = e^{\pi(a)}$. In order to obtain a representation of a covering space $M_m, m \in \mathbb{N}$, of SL_2, we have to check that f is well-defined on every homotopy class of loops. Every loop $t \mapsto \gamma(t) \in SL_2$ is homotopic to one of the loops $[0,1] \ni t \mapsto w_k(t) = e^{2\pi k t(a-b)}, k \in \mathbb{N}$. If $L^2(\mathbb{R}^d, \mathbb{C}) \ni \varphi = \sum_{j \in \mathbb{N}^d} c_j \varphi_j$ is decomposed on the Hilbert basis of Hermite functions $\{\varphi_j\}_{j \in \mathbb{N}^d}$, then

$$e^{\frac{it(\Delta - |x|^2)}{2}} \varphi = \sum_{(j_1,\dots,j_d) \in \mathbb{N}^d} e^{-it(j_1 + \dots + j_d + \frac{d}{2})} c_j \varphi_j. \tag{12}$$

According to (12), we obtain $f(w_k(1)) = e^{i\pi k(\Delta - |x|^2)} = (-1)^{kd} I$. If d is even, then the definition of f coincide on every $w_k, k \in \mathbb{N}^*$. If d is odd, this is true for the loops of even degree, and so we obtain a representation of the covering space of degree $2m$, $m \in \mathbb{N}^*$. □

5.2 Conclusion of the proof of Theorem 2

Let M_2 be the metaplectic group, for which there exists a unitary representation whatever the dimension d of the space is (cf. Lemma 1). Since H_d is simply connected, the strongly continuous unitary representation f can be extended in a standard way to $M_2 \ltimes H_d$. It satisfies $f(e^{tX_i}) = e^{itx_i}, f(e^{tY_i}) = e^{t\partial_{x_i}}, e^{tZ} = e^{it}, t \in \mathbb{R}$.

Given any ψ_1 of the form $e^{is(\Delta - |x|^2)} e^{i(\alpha|x|^2 + px)} \sigma^{d/2} \psi_0(\sigma x + \beta)$, for some $s, \alpha \in \mathbb{R}, p, \beta \in \mathbb{R}^d, \sigma > 0$, there exists an element g in $M_2 \ltimes H_d$ such that $\psi_1 = f(g)\psi_0$. Since the element g is small-time reachable for system (3) (cf. Theorem 1), for any $T > 0$ there exists a control $u \in \mathrm{PWC}([0,\tau], \mathbb{R}^{d+1}), \tau \in [0, T]$, such that $q(u,\tau) = g$. Since f is a morphism and u is piecewise constant, $\psi(u,\tau,\psi_0) = f(q(u,\tau))\psi_0 = f(g)\psi_0 = \psi_1$, hence $\psi_1 \in \mathscr{R}_{st}(\psi_0)$. Conversely, given any $\psi_1 \in \mathscr{R}_{st}(\psi_0)$ there exists $n \in \mathbb{N}$ and $t_1, \dots, t_n \geq 0, u_i^1, \dots, u_i^n \in \mathbb{R}, i = 0, \dots, d$ such that $\psi_1 = \prod_{j=1}^n e^{i(t_j \Delta/2 - u_0^j |x|^2/2 - \sum_{k=1}^d u_k^j x_k)} \psi_0$. Hence, $\psi_1 = f(g)\psi_0$ where $g = \prod_{j=1}^n e^{-t_j a + u_0^j b + \sum_{k=1}^d u_k^j X_k}$. Since g can be written as in Proposition 2, ψ_1 has the form $e^{is(\Delta - |x|^2)} e^{i(\alpha|x|^2 + px)} \sigma^{d/2} \psi_0(\sigma x + \beta)$, for some $s, \alpha \in \mathbb{R}, p, \beta \in \mathbb{R}^d, \sigma > 0$.

The proof of Theorem 2 is concluded.

6 Proof of Theorem 3

For proving Theorem 3, one follows the same exact lines of the proof of Theorem 2; we only point out the needed modifications. First of all a direct computation shows that

$$\mathrm{Lie}\{|p|^2/2, |q|^2/2, q_1, \dots, q_d\} = \mathrm{span}\{|p|^2/2, |q|^2/2, q_1, \dots, q_d, p \cdot q, p_1, \dots, p_d, 1\}.$$

We thus have an isomorphism of Lie algebras $\pi : \mathfrak{sl}_2 \ltimes \mathfrak{h}_d \to \mathrm{Lie}\{|p|^2/2, |q|^2/2, q_1, \dots, q_d\}$, and a group morphism $f : M_\infty \ltimes H_d \to \mathbb{U}(L^r(T^*\mathbb{R}^d))$ satisfying $f(e^a) = \Phi^1_{\pi(a)}$ for any $a \in \mathfrak{sl}_2 \ltimes \mathfrak{h}_d$, and $\Phi^t_{H(p,q)}$ acts on $\rho_0 \in L^r(T^*\mathbb{R}^d)$ as $\rho_0 \circ \Phi^t_{H(p,q)}$. Finally, as a slight difference w.r.t. to Lemma 1, notice that $\Phi^t_{(p^2-q^2)/2}(q_0, p_0) = (q_0 \cos(t) + p_0 \sin(t), -q_0 \sin(t) + p_0 \cos(t))$ hence $\Phi^{2\pi k}_{(p^2-q^2)/2} = I$, irrespectively of d being odd or even; the corresponding Lemma 1 for the Liouville representation $f : M_n \ltimes H_d \to \mathbb{U}(L^r(T^*\mathbb{R}^d))$ hence holds for covering spaces of even and odd dimensions $n \in \mathbb{N} \cup \{\infty\}$.

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