Soft Construction of Floer-type Homologies

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A toy example, the Leray–Schauder degree.

Let $B$ be an infinite-dimensional separable Banach space and $S \subset B$ the unite sphere in $B$. Let $\mathcal{E} = \{ E \subset B : \dim E < \infty \}$ be the ordered by the inclusion directed set of finite-dimensional subspaces of $B$. We set:

$$G_i(S) = H_{i(\dim E - 1)}(S \cap E)$$

and call $G_i(S)$ the Leray–Schauder homology of $S$. 
Now let $\varphi : S \to B$ be a compact map such that $x + \varphi(x) \neq 0$, $\forall x \in S$. $\forall \varepsilon > 0$, $\exists \varepsilon$-close to $\varphi$ finite-dimensional map $\varphi_\varepsilon : S \to E_\varepsilon$. We define a map:

$$
\Phi_\varepsilon^E : S \cap E \to S \cap E, \quad \Phi_\varepsilon^E(x) = \frac{x + \varphi_\varepsilon(x)}{|x + \varphi_\varepsilon(x)|},
$$

for any $E \supset E_\varepsilon$. The degree of this map $d = \text{deg}(\Phi_\varepsilon^E)$ does not depend on $E$ and is the same for all sufficiently good approximations $\varphi_\varepsilon$. This is the Leray–Schauder degree.

The degree is defined by the homomorphism:

$$
\Phi_\varepsilon^E_* : H_{\text{dim} E - 1}(S \cap E) \to H_{\text{dim} E - 1}(S \cap E), \quad \Phi_\varepsilon^E_*(c) = cd,
$$

for any $c \in H_{\text{dim} E - 1}(S \cap E) = \mathbb{Z}$. We may interpret it as a homomorphism $\Phi_* : G_1(S) \to G_1(S)$, where $\Phi = \frac{I + \varphi}{|I + \varphi|}$. 

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Floer Homology

We consider a compact smooth manifold $M$ endowed with a symplectic structure $\sigma$. Let $\tilde{M}$ be the universal covering of $M$ and $\tilde{\sigma}$ the pullback of $\sigma$ to $\tilde{M}$; we assume that $\tilde{\sigma}$ is an exact form: $\tilde{\sigma} = ds$.

We denote by $\Omega$ the space of contractible closed curves in $M$ of class $H^1$. In other words, $\Omega$ consists of contractible maps $\gamma : S^1 \to M$, where $\gamma$ is differentiable almost everywhere with the derivative of class $L^2$. The lifts of $\gamma \in \Omega$ to $\tilde{M}$ are closed curves and we use the same symbol $\gamma$ for any lift of this curve to $\tilde{M}$. 
Let $h_t : M \to \mathbb{R}$ be a measurable bounded w. r. t. $t \in S^1$ family of smooth functions on $M$. The functional $\varphi_h : \Omega \to \mathbb{R}$ is defined by the formula:

$$\varphi_h(\gamma) = \int_{S^1} s(\dot{\gamma}(t)) - h_t(\gamma(t)) \, dt.$$ 

Given $c \in \mathbb{R}$, we denote by $\Omega_h^c$ the Lebesgue set of $\varphi_h$:

$$\Omega_h^c = \{ \gamma \in \Omega : \varphi_h(\gamma) \leq c \}$$

We assume that $M$ is equipped with a Riemannian structure $\langle \cdot, \cdot \rangle$ adapted to the symplectic structure, i.e. $\sigma(\xi, \eta) = \langle J\xi, \eta \rangle$, $\xi, \eta \in TM$, where $J : TM \to TM$ is a quasi-complex structure, $J^2 = -I$. Then:

$$\nabla_\gamma \varphi_h = -J\dot{\gamma} - \nabla_\gamma h.$$
Second variation of $\varphi_0$ at a “constant curve” $q$: 

$$b_q(\xi, \eta) = \int_{S^1} \sigma(\xi(\theta), \dot{\eta}(\theta)) \, d\theta, \quad \xi, \eta \in H^1(S^1; T_qM).$$

We denote by $\iota: H^1(S^1; T_qM) \to H^1(S^1; T_qM)$ the involution defined by the formula $(\iota \xi)(\theta) = \xi(-\theta)$. Then

$$b_q(\iota \xi, \iota \eta) = -b_q(\xi, \eta), \quad \xi, \eta \in H^1(S^1; T_qM).$$
We fix generators $X_1, \ldots, X_l$ of the $C^\infty(M)$-module $\text{Vec} M$ of all smooth vector fields on $M$ and define a linear map $X_q : \mathbb{C}^l \to T_q M$ by the formula

$$X_q u = \sum_{j=1}^l v^j X_j(q) + w^j JX_j(q),$$

where $u = (u^1, \ldots, u^l)$, $u_j = v_j + iw_j \in \mathbb{C}$, $j = 1, \ldots, l$ and

$$\langle \xi, \xi \rangle = \min \{|u|^2 : u \in \mathbb{C}^l, \xi = X_q u\}. $$
Let $W$ be the space of all curves in $M$ of class $H^1$ parameterized by the segment $[0, 1]$. We fix a parametrization of $S^1$ by $[0, 1]$; then $\Omega \subset W$.

We define the map $\phi : M \times L^2([0, 1]; \mathbb{C}^l) \rightarrow W$ as follows. Given $q \in M$ and $u(\cdot) \in L^2([0, 1]; \mathbb{C}^l)$ the curve $\gamma(\cdot) = \phi(q, u(\cdot))$ is the solution of the ordinary differential equation

$$\dot{\gamma}(t) = X_{\gamma(t)} u(t), \quad 0 \leq t \leq 1,$$

with the initial condition $\gamma(0) = q$. We also set $\phi_t(q, u) = (q, \phi(q, u)(t))$ and thus define the map $\phi_t : M \times L^2([0, 1]; \mathbb{R}^l) \rightarrow M \times M$. It is easy to see that $\phi_t$ is a smooth map and $\phi_t$ is a submersion for $0 < t \leq 1$. 
Let $E$ be a finite-dimensional subspace of $L^2([0, 1]; \mathbb{C})$ and $E_0 = \{ v \in E : \int_0^1 v(t) \, dt = 0 \}$. We set:

$$
\mathcal{X}_q(E) = \left\{ \theta : \mapsto \xi_0 + \int_0^\theta X_qu(t) \, dt : \xi_0 \in T_qM, \; u(\cdot) \in E_0^l \right\} \subset H^1(S^1; T_qM).
$$

We say that $E$ is well-balanced if $\mathfrak{v}E = E$ and $\ker b_q|_{\mathcal{X}_q(E)} = \ker b_q$.

**Lemma 1.** Any finite-dimensional subspace of $L^2([0, 1]; \mathbb{C})$ is contained in a well-balanced subspace.
Notations: \( B_r = \{ u \in L^2([0, 1]; \mathbb{C}^l) : \|u\| < r \} \).

\[
U_r(E^l) = \{ (q, u) \in M \times (B_r \cap E^l) : \phi_t(q, u) = (q, q) \}
\]

\[
G_i(E; c, r) = \text{Hom} \left( \phi(U_r(E^l)) \cap \Omega^c_h, \phi(U_r(E^l)) \cap \Omega^{-c}_h \right).
\]

\( j_r^\infty : G_i(E; c, r) \to G_i(E; c, \bar{r}), \bar{r} > r, \) are homology homomorphisms induced by the inclusion.

Finally, \( \mathcal{E} \) is the directed set of well-balanced subspaces ordered by the inclusion.

**Theorem 1.** There exist

\[
\lim_{c \to \infty} \lim_{r \to \infty} \lim_{\bar{r} \to \infty} \mathcal{E} - \text{lim} j_r^\infty \left( G_{i+d_E}(E; c, r) \right) \cong H_i(M),
\]

where \( d_E = \frac{1}{2}(\dim E - 1) \dim M \).
Let $\beta_j(M)$ be the Betti number of $M$ of the dimension $j$ and $C_h$ be the set of all 1-periodic trajectories of the Hamiltonian system. If all 1-periodic trajectories are non-degenerate, then $C_h$ is a finite set.

**Theorem 2** (Morse inequalities). Assume that all 1-periodic trajectories are non-degenerate. Then, for any $k \in \mathbb{Z}$, the following inequality holds:

$$
\sum_{j \leq k} (-1)^{k-j} \beta_j(M) \leq \sum_{\{\gamma \in C_h : i_h(\gamma) \leq k\}} (-1)^{k-i_h(\gamma)},
$$

where

$$
i_h(\gamma) = \frac{1}{2} [\text{sgn}(d_\gamma^2 \varphi_0) - \text{sgn}(d_\gamma^2 \varphi_h)].$$
Step Two Carnot Lie algebras and groups:

\[ g = V \oplus W, \quad [V, V] = W, \quad [g, W] = 0, \quad \mathcal{G} = e^g. \]

To any \( \omega \in W^* \) we associate an operator \( A_\omega \in \text{so}(V) \) by the formula:

\[ \langle A_\omega \xi, \eta \rangle = \langle \omega, [\xi, \eta] \rangle, \quad \xi, \eta \in V. \]

It is easy to see that \( \omega \mapsto A_\omega, \ \omega \in W^* \) is an injective linear map. Moreover, any injective linear map from \( W^* \) to \( \text{so}(V) \) defines a structure of step two Carnot Lie algebra on the space \( V \oplus W \) by the same formula. Hence step two Carnot Lie algebras are in the one-to-one correspondence with linear systems of anti-symmetric operators.
An $H^1$-curve $\gamma : [0, 1] \to \mathcal{G}$ is called \textit{horizontal} if $\dot{\gamma}(t) \in V_{\gamma(t)}$ for a.e. $t \in [0, 1]$.

The following multiplication in $V \times W$ gives a simple realization of $\mathcal{G}$ with the origin in $V \times W$ as the unit element:
\[(v_1, w_1) \cdot (v_2, w_2) = \left( v_1 + v_2, w_1 + w_2 + \frac{1}{2} [v_1, v_2] \right).\]

Starting from the origin horizontal curves are determined by their projection to the first level and have a form:
\[\gamma(t) = \left( \xi(t), \frac{1}{2} \int_0^t [\xi(t), \dot{\xi}(t)] \, dt \right), \quad 0 \leq t \leq 1,\]
where $\xi(\cdot) \in H^1([0, 1]; U)$, $\xi(0) = 0$. 
We set:

\[ \varphi(\xi) = \frac{1}{4\pi} \int_{0}^{1} |\dot{\xi}(t)|^2 \, dt. \]

We focus on the horizontal curves corresponding to closed curves \( \xi \); they connect the origin with the second level. Given \( w \in W \setminus 0 \), let \( \Omega_w \) be the space of horizontal curves connecting \((0,0)\) with \((0,w)\); then

\[ \Omega_w = \left\{ \xi \in H^1([0,1]; V) : \xi(0) = \xi(1) = 0, \frac{1}{2} \int_{0}^{1} [\xi(t), \dot{\xi}(t)] \, dt = w \right\}. \]

For any \( s > 0 \), we set: \( \Omega^s_w = \{ \xi \in \Omega_w : \varphi(\xi) \leq s \} \). Note that central reflection \( \xi \mapsto -\xi \) preserves \( \Omega^s_w \).
Let $E \subset H^1([0, 1]; V)$ be a finite-dimensional subspace and $\bar{E} = (E \setminus 0)/\langle \xi \sim (-\xi) \rangle$ its projectivization. We set $E_w^s = \Omega_w^s \cap E$ and denote by $\bar{E}_w^s$ the image of $E_w^s$ under the factorization $\xi \sim (-\xi)$.

We consider the homology $H.(\bar{E}_w^s; Z_2)$ and its image in $H.(\bar{E}; Z_2)$ by the homomorphism induced by the imbedding $\bar{E}_w^s \subset \bar{E}$. We have:

$$\text{rank}(H_i(\bar{E}_w^s; Z_2)) = \beta_i(\bar{E}_w^s) + \varrho_i(\bar{E}_w^s),$$

where $\beta_i(\bar{E}_w^s)$ is rank of the kernel of the homomorphism from $H_i(\bar{E}_w^s; Z_2)$ to $H_i(\bar{E}; Z_2)$ induced by the imbedding $\bar{E}_w^s \subset \bar{E}$ and $\varrho_i(\bar{E}_w^s) \in \{0, 1\}$ is the rank of the image of this homomorphism.
For given $w, E, s$, we introduce two positive atomic measures on the half-line $\mathbb{R}_+$, the “Betti distributions”:

$$b(\bar{E}_w^s) = \sum_{i \in \mathbb{Z}_+} \frac{1}{s} \beta_i(\bar{E}_q^s) \delta_{\frac{i}{s}}, \quad r(\bar{E}_w^s) = \sum_{i \in \mathbb{Z}_+} \frac{1}{s} \varrho_i(\bar{E}_q^s) \delta_{\frac{i}{s}}.$$ 

Assume that $\dim W = 2$ and let $\mathcal{E}$ be the directed set of all finite-dimensional subspaces of the Hilbert space $H^1([0, 1]); V)$. It appears that there exist limits of these families of measures

$$\lim_{s \to \infty} \mathcal{E}-\lim b(\bar{E}_w^s), \quad \lim_{s \to \infty} \mathcal{E}-\lim r(\bar{E}_w^s)$$

in the weak topology. Moreover, the limiting measures are absolutely continuous with explicitly computed densities.
Let $\alpha : \Delta \to \mathbb{R}$ be an absolutely continuous function defined on an interval $\Delta$. We denote by $|d\alpha|$ a positive measure on $\Delta$ such that $|d\alpha|(S) = \int_S \left| \frac{d\alpha}{dt} \right| dt$, $S \subset \Delta$.

The operators $A_\omega$, $\omega \in W^*$, have purely imaginary eigenvalues. Let $0 \leq \alpha_1(\omega) \leq \cdots \leq \alpha_m(\omega)$ are such that $\pm i\alpha_j m, j = 1, \ldots, m$, are all eigenvalues of $A_\omega$ counted according the multiplicities.

Let $\bar{W}^* = (W \setminus 0)/(w \sim cw, \forall c \neq 0)$ be the projectivization of $W^*$, $\bar{W}^* = \mathbb{R}P^1$. 
Given \( w \in W \setminus 0 \), we take the line \( w^\perp \in W^* \) and consider the affine line

\[
\ell_w = \bar{W}^* \setminus \bar{w}^\perp \subset \bar{W}^*.
\]

Moreover, we define functions

\[
\lambda^w_j : \ell_w \to \mathbb{R}_+, \quad j = 1, \ldots, m, \quad \phi^w : \ell_w \to \mathbb{R}_+
\]

by the formulas:

\[
\lambda^w_j (\bar{\omega}) = \frac{\alpha_j (\omega)}{\langle \omega, w \rangle}, \quad \phi^w (\bar{\omega}) = \sum_{j=1}^{m} \lambda^w_j (\omega).
\]
Theorem 3. Assume that there exists $\omega \in W^*$ such that the matrix $A_\omega$ has simple spectrum. Then, for any $w \in W \setminus 0$, there exist the following limits in the weak topology of the space of positive measures on $\mathbb{R}_+$:

$$b_w = \lim_{s \to \infty} \mathcal{E}-\lim b(\bar{E}_w^s), \quad r_w = \lim_{s \to \infty} \mathcal{E}-\lim r(\bar{E}_w^s).$$

Moreover,

$$b_w = \phi^w_\ast \left( \sum_{j=1}^m |d\lambda^w_j| \right), \quad r_w = \chi[0,\min \phi^w] dt,$$

where $dt$ is the Euclidean measure.
General scheme.

The object to study is a Banach manifold \( \Omega \) equipped with a growing family of closed subsets \( \Omega^s, \ s \in \mathbb{R} \).

Auxiliary objects are a Banach space \( B \) and a submersion \( \Phi : U \to \Omega \), where \( U \subset B \) is a finite codimension submanifold of \( B \).

Moreover, \( U \) is equipped with an ordered by the inclusion directed and exhausting family \( \mathcal{V} \) of open bounded subsets and \( B \) is endowed by an ordered by the inclusion directed family \( \mathcal{E} \) of finite dimensional subspaces such that \( \bigcup_{E \in \mathcal{E}} E = B \).
Given $E \in \mathcal{E}$, $V \in \mathcal{V}$, $s \in \mathbb{R}$, we consider the relative homology groups:

$$G_i(E, V, s) \cong H_i\left(\Omega^s \cap \Phi(E \cap V), \Omega^{-s} \cap \Phi(E \cap V)\right).$$

Moreover, for $V, W \in \mathcal{V}$, $V \subset W$, we denote by

$$j^W_V : G_i(E, V, s) \to G_i(E, W, s)$$

the homology homomorphism induced by the inclusion $V \subset W$ and denote by $\mathcal{V}_V$ the directed subfamily:

$$\mathcal{V}_V \defeq \{W \in \mathcal{V} : W \supset V\}.$$
Finally, we select normalizing quantities \( r_i(E, s), \rho_i(E, s) \in \mathbb{R}_+ \) and build atomic measures:

\[
b(E, V, W, s) = \sum_{i \in \mathbb{Z}_+} \rho_i(E, s) \text{rank} \left( j_W^V G_i(E, V, s) \right) \delta_{r_i}(E, s)
\]

in such a way that their exist a limit:

\[
b = \lim_{s \to \infty} \mathcal{V} - \lim \mathcal{V} - \lim \mathcal{E} - \lim b(E, V, W, s).
\]