Differential geometry and optimal control problems*

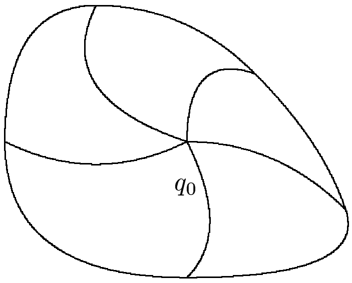
A.Agrachev

Consider a *control system*

$$\dot{q} = f(q, u) \quad q \in M, u \in U, \tag{1}$$

where M and U are smooth manifolds. An *admissible trajectory* is any trajectory of the differential equation (1) with fixed control function $u(\cdot), u(t) \in U$.

Fixing a point $q_0 \in M$ we define for $T \in \mathbb{R}$ an *attainable set* $\mathcal{A}_{q_0}(T)$ as a set of all points in M which could be reached from q_0 by an admissible trajectory in time $t \leqslant T$:



Also, we shall refer to the set $f(q_0, U) \subset T_{q_0}M$ as to an *infinitesimal attainable set*.

Our main goal is to characterize trajectories going to the boundary of the attainable set (*extremals*).

We shall make use of the fundamental notion of the so-called feedback equivalence of control systems. So, two systems of type (1) with the right hand sides $f, \tilde{f}$, and with u belonging to $U, \tilde{U}$ respectively, are feedback equivalent if there exists a map $\varphi : M \times U \to \tilde{U}$, being a diffeomorphism of U and $\tilde{U}$ for any fixed $q \in M$, such that $\tilde{f}(q, u) = f(q, \varphi(q, u))$.

Example Let M be a Riemannian surface, i.e. a 2-dimensional manifold with a Riemannian metric $\langle , \rangle_q$. Choose an orthonormal local frame $e_1(q), e_2(q) \in T_qM, \langle e_i, e_i \rangle \equiv 1, \langle e_1, e_2 \rangle \equiv 0$, and consider the following control system:

$$\dot{q} = u_1 e_1(q) + u_2 e_2(q), \quad u_1^2 + u_2^2 = 1.$$

Here the attainable set is a ball of radius T and trajectories going to the boundary are geodesics.

Note that the curvature, which is a basic invariant here, does not depend on changes of coordinates and of the frame (feedback invariant).

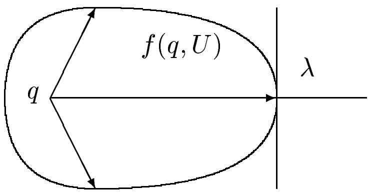
Now we make the following additional assumption:

$$f(q, U) = \overline{\mathrm{conv}\, f(q, U)}.$$

Define the *Hamiltonian* $H : T^*M \to \mathbb{R}$ of (1) as

$$H(\lambda) = \max_{u \in U} \langle \lambda, f(q, u) \rangle, \quad \lambda \in T_q^*M \tag{2}$$

(here $\langle , \rangle$ denotes the canonical pairing of the tangent and cotangent space). This function does not depend on u thus being also a feedback invariant.



*Lectures given on the school "Geometric control theory" which was held in the International Stefan Banach Center of Mathematical Sciences (Warsaw, September 15-20, 2002); written by A.Panasyuk

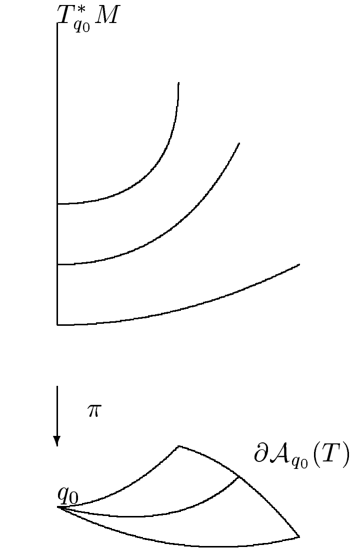
As usual, (assuming that H is smooth) we define the corresponding Hamiltonian vector field $\vec{H}(\lambda) \in T_\lambda(T^*M)$ by $d_\lambda H = \sigma(\cdot, \vec{H}(\lambda))$, where σ is the canonical symplectic form on T^*M, and the corresponding Hamiltonian system $\dot{\lambda} = \vec{H}(\lambda)$, or in local coordinates

$$\dot{q} = \frac{\partial H(p,q)}{\partial p} \qquad \dot{p} = -\frac{\partial H(p,q)}{\partial q}. \tag{3}$$

Recall a celebrated result called *Pontryagin's Maximum Principle*:

Theorem 1 *Assume $q(t), t \in [0,T]$, is an admissible trajectory of the system* (1) *such that $q(0) = q_0, q(T) \in \partial \mathcal{A}_{q_0}(T)$. Then there exists a "cotangent lift" of $q(t)$, i.e. a curve $\lambda : [0,T] \to T^*M$ with $\lambda(t) = (p(t), q(t))$, satisfying system* (3).

In other words, in order to describe $\mathcal{A}_{q_0}(T)$ it is sufficient to solve Hamiltonian system (3). We have to consider all trajectories of (3) starting from $T^*_{q_0}M$, up to time T, then to project them to M by the canonical projection $\pi : T^*M \to M$.



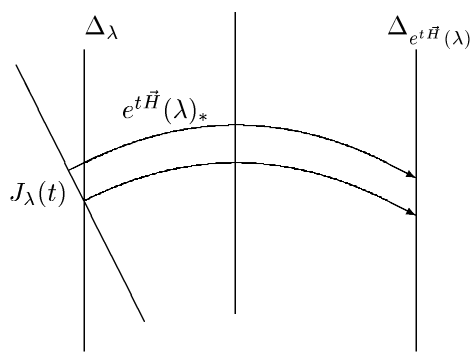
Putting $E_q := T^*_qM$, so that $T^*M = \bigcup_q E_q$, one can say that invariants (curvature e.t.c.) come from the pair $(E_q, \vec{H}(\lambda))$ (we can't rectify $\vec{H}(\lambda)$ saving fibers).

Now we are ready to give a basic definition of these lectures.

Definition 1 Let $e^{t\vec{H}} : T^*M \to T^*M$ denote the flow generated by $\vec{H}$, in particular $\frac{\partial}{\partial t} e^{t\vec{H}}(\lambda) = \vec{H}(e^{t\vec{H}}(\lambda)), e^{0\vec{H}}(\lambda) = \lambda$. Let $\Delta \subset T(T^*M)$ denote the "vertical" distribution: $\Delta_\lambda := T_\lambda E_q \subset T_\lambda(T^*M), q = \pi(\lambda)$. Put

$$J_\lambda(t) := e_*^{-t\vec{H}} \Delta_{e^{t\vec{H}}(\lambda)}.$$

For any λ we get a family $J_\lambda(t) \subset T_\lambda(T^*M)$ of n-dimensional subspaces in the 2n-dimensional space, i.e. a curve $t \mapsto J_\lambda(t)$ in the Grassmannian $G(T_\lambda(T^*M), n)$ which we will call a *Jacobi curve*.



We are going to study geometry of this curve. To do this it would be desirable to assume that for small $t \neq 0$

$$J_\lambda(t) \cap J_\lambda(0) = \{0\}. \tag{4}$$

This condition would allow us to study geometry of $J_\lambda(t)$ in terms of the projection $\pi_* : J_\lambda(t) \to T_{\pi(\lambda)}M$, which is one-to-one if (4) holds. However, this is never the case for Hamiltonians of the form (2).

Indeed, in our Hamiltonian $H(\lambda) = \max_{u \in U}\langle \lambda, f(q,u)\rangle = \max_{u\in U}\langle p, f(q,u)\rangle$ (we put $\lambda = (p,q)$) the ingredient $\langle p, f(q,u)\rangle$ is linear in p, hence H is homogeneous of degree 1 in fibers: $H(\alpha\lambda) = \alpha H(\lambda), \alpha > 0$. Consequently the Euler vector field $\theta = \sum p_i \partial_{p_i}$ is contained in $J_\lambda(t)$ for all t.

To avoid this obstruction we shall make a kind of symplectic reduction. Take the unit level of the Hamiltonian $H^{-1}(1) \subset T^*M$ and define the restricted vertical distribution as $\Delta^r_\lambda := T_\lambda H^{-1}(1) \cap T_\lambda(T^*_q M), \lambda \in H^{-1}(1), q = \pi(\lambda)$. Note that $\dim \Delta^r_\lambda = n-1$. Further on we define the restricted Jacobi curve

$$J^r_\lambda(t) = e_*^{-t\vec{H}} \Delta^r_{e^{t\vec{H}}(\lambda)} \subset T_\lambda H^{-1}(1) \tag{5}$$

(a curve in the Grassmannian $G(T_\lambda H^{-1}(1), n-1), \dim T_\lambda H^{-1}(1) = 2n-1$). The last step is the projection of $J^r_\lambda(t)$ with respect to the canonical projection $T_\lambda H^{-1}(1) \to \Sigma_\lambda := T_\lambda H^{-1}(1)/\mathbb{R}\vec{H}(\lambda)$. Note that the inclusion $\vec{H}(\lambda) \subset T_\lambda H^{-1}(1)$ and also (5) follow from the elementary symplectic geometry (the flow of the Hamiltonian vector field preserves the Hamiltonian). Moreover, Σ is endowed with a symplectic form which is a reduction of σ and which we will denote by the same letter. This fact will play a role later when we will consider a problem of conjugate points.

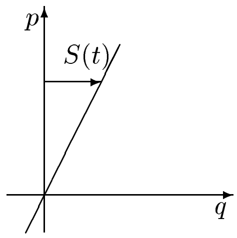
The projected Jacobi curve $J_\lambda(t) \subset \Sigma_\lambda$ (we use the same notation as for the initial curve) is free from the above disadvantage: the Euler vector field being transversal to the level sets of the Hamiltonian "disappears" after the restriction to $T_\lambda H^{-1}(1)$ and from now on we may have

$$J_\lambda(t) \cap J_\lambda(0) = \{0\}$$

for small $t \neq 0$.

Now we can realize our idea and the geometry of the Jacobi curve in terms of the projection operators. So consider a curve $J(t) \subset \Sigma$ of m-dimensional subspaces in a $2m$-dimensional space with the transversality condition (4).

Let us choose coordinates (p,q) in Σ in such a way that $J(0) = \{(p,0) \mid p \in \mathbb{R}^m\}$. Then $J(t) = \{(p, S(t)p) \mid p \in \mathbb{R}^m\}$, where $S(t)$ is a $m \times m$-matrix satisfying the conditions: $S(0) = 0$ and $t \neq 0$ then $\det S(t) \neq 0$ if $t \neq 0$ (cf. the figure).



The matrix

$$\pi_{t0} = \begin{bmatrix} I & -S^{-1}(t) \\ 0 & 0 \end{bmatrix}$$

represents in our coordinates the projector $\pi_{t0} : \Sigma \to J(0)$ onto $J(0)$ along $J(t)$. Note that all projectors onto $J(0)$ form an affine space, thus we get a curve $t \mapsto \pi_{t0}$ in the affine space.

So 0 is an isolated root of $\det S(t)$. Now we make an additional assumption that this is a root of a finite order. This implies that $S^{-1}(t)$ has a pole at 0: $-S^{-1}(t) = \sum_{i=-k}^{\infty} t^i S_i$ for some constant matrices S_i. On π_{t0} this decomposition reflects as

$$\pi_{t0} = \sum_{i=-k, i\neq 0}^{\infty} t^i \begin{bmatrix} 0 & S_i \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I & S_0 \\ 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} I & S_0 \\ 0 & 0 \end{bmatrix}$ can be regarded as a fixed point in the affine space of all projectors and $\begin{bmatrix} 0 & S_i \\ 0 & 0 \end{bmatrix}$ as points in the associated linear space. In a more compact form

$$\pi_{t0} = \sum_{i \neq 0} t^i \pi_i + \pi_0,$$

where $\pi_0 : \Sigma \to J(0)$ is a projection along some transversal to $J(0)$ subspace J°.

An analogous construction can be applied to any subspace $J(t_0)$ for $t_0 > 0$ small, instead of $J(0)$. As a result we get a new subspace $J^\circ(t_0)$ of dimension m, and moreover, a new curve $t \mapsto J^\circ(t)$ which will be called a *derivative curve.*

Exercise 1 1. Prove that

$$\pi_{J(\tau)J(t)} = \begin{bmatrix} S_{\tau t}^{-1} S(\tau) & -S_{\tau t}^{-1} \\ S(t) S_{\tau t}^{-1} S(\tau) & -S(t) S_{\tau t}^{-1} \end{bmatrix},$$

where $S_{\tau t} = S(\tau) - S(t)$.

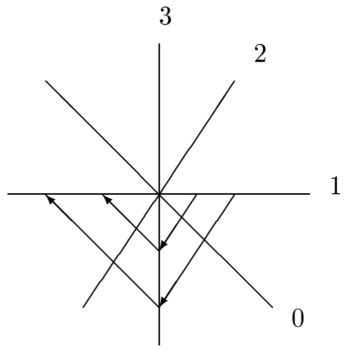
2. Show that if $S(0) = 0,\ \det(\dot S(0)) \neq 0$, then $S^{-1}(t)$ has a simple pole and $S^{-1}(t) = \frac{1}{t}\dot S^{-1}(0) - \frac{1}{2}\dot S^{-1}(0)\ddot S(0)\dot S^{-1}(0) + O(t)$. Derive from here and from 1. the formula

$$J^\circ(t) = \left\{ \left(-\frac{1}{2}\dot S^{-1}(0)\ddot S(0)\dot S^{-1}(0)q, q \right) : q \in \mathbb{R}^m \right\}.$$

Note that for small different t_0, t_1 the subspaces $J(t_0), J^\circ(t_0), J(t_1), J^\circ(t_1)$ are in general position and this allows us to apply a multidimensional version of the classical notion of the cross-ratio, which we will recall now.

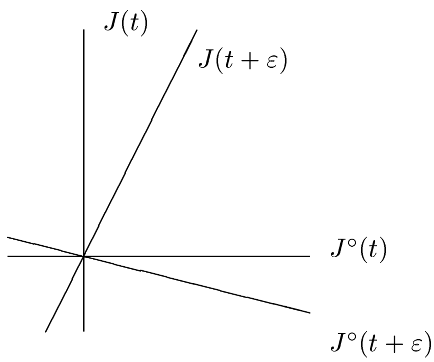
Definition 2 Given four m-dimensional subspaces $J_0, \dots, J_3$ in general position in a $2m$-dimensional space Σ, let π_{ij} denote a projector onto J_j along J_i, $i, j = 0, \dots, 3$. The *cross-ratio* of the subspaces $J_0, \dots, J_3$ is the operator

$$[J_0, J_1, J_2, J_3] := (\pi_{01}\pi_{23}|_{J_1} : J_1 \longrightarrow J_1).$$



Note that $\pi_{ij} + \pi_{ji} = I,\ \pi_{ik}\pi_{jk} = \pi_{jk},\ \pi_{ij}\pi_{ik} = \pi_{ij}$.

Now, the idea to define curvature is as follows: take $J(t), J(t+\varepsilon), J^\circ(t), J^\circ(t+\varepsilon)$ and compute the first term of the cross-ratio as ε tends to 0.



Definition 3 A *curvature* (of a curve $t \mapsto J(t)$) is an operator $R(t) : J(t) \to J(t)$ given by the formula

$$R(t) = [\frac{\partial \pi_{J^\circ(t)J(\tau)}}{\partial t} \frac{\partial \pi_{J^\circ(t)J(\tau)}}{\partial \tau}|_{\tau=t}]|_{J(t)} = -[\pi_{J^\circ(t)J(t)} \frac{\partial^2 \pi_{J^\circ(t)J(\tau)}}{\partial t \partial \tau}|_{\tau=t}]|_{J(t)} = -[\frac{\partial^2 \pi_{J^\circ(t)J(\tau)}}{\partial t \partial \tau}|_{\tau=t} \pi_{J^\circ(t)J(t)}]|_{J(t)},$$

where the last two equalities are obtained by the differentiation of the identities $\pi_{J^\circ(t)J(\theta)}\pi_{J^\circ(t)J(\tau)} = \pi_{J^\circ(t)J(\theta)}$ and $\pi_{J^\circ(t)J(\tau)}\pi_{J^\circ(\theta)J(\tau)} = \pi_{J^\circ(\theta)J(\tau)}$ with respect to t and τ.

Exercise 2 Compute $R(t)$ of a curve $J(t) = \{(p, S(t)p) \mid p \in \mathbb{R}^m\}$ under an assumption $\det \dot S(t) \neq 0$.
Answer: $R(t) = ((2\dot S)^{-1}\ddot S)' - ((2\dot S)^{-1}\ddot S)^2 = (1/2)\dot S^{-1}S''' - (3/4)(\dot S^{-1}\ddot S)^2$ (matrix version of Schwartzian derivative coinciding with the classical one if S is scalar).

Now we come back to the setting of control theory and Jacobi curves. Our next aim is to characterize the so-called conjugate points in terms of the Jacobi curves and discuss their relations with curvature.

Definition 4 We say that $e^{t_* \vec{H}}(\lambda)$ is *conjugate* to λ (or t_* is conjugate to 0) if

$$e_*^{t_* \vec{H}} \Delta_\lambda^r \cap \Delta_{e^{t_* \vec{H}}}^r(\lambda) \neq 0.$$

Obviously, this condition is equivalent to $\Delta_\lambda^r \cap e_*^{-t_* \vec{H}} \Delta_{e^{t_* \vec{H}}}^r(\lambda) \neq 0$. The first term in the left hand side is $J_\lambda(0)$ while the second one is $J_\lambda(t_*)$. This allows us to say that t_* is conjugate to t_0 if

$$J(t_0) \cap J(t_*) \neq 0.$$

Now recall that the space Σ where our Jacobi curve $t \mapsto J(t)$ takes values has the symplectic form σ (the reduction of the canonical symplectic form on T^*M) and that a subspace $J \subset \Sigma$ of dimension $(1/2) \dim \Sigma$ is called Lagrangian if $\sigma|_J = 0$.

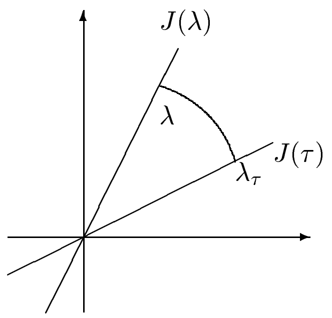
Exercise 3

1. Show that if $J(t)$ is a Jacobi curve (constructed from some control system) $J(t)$ is Lagrangian for any t.
2. Let $\Sigma = \mathbb{R}^{2m}$ with the standard symplectic form $dp \wedge dq$. Show that a subspace $\Lambda = \{(p, Sp) \mid p \in \mathbb{R}^m\}$ is Lagrnagian if and only if S is symmetric. Deduce from this that dimension of the space $L(\Sigma)$ of all Lagrangian subspaces (Lagrangian Grassmannian) equals $m(m+1)/2$.

Given any Jacobi curve $J(t) = \{(p, S(t)p) \mid p \in \mathbb{R}^m\}, S(0) = 0$, one can intrinsically identify its velocity $\dot{J}(t)$ with a quadratic form on $J(t)$:

$$\dot{J}(t) : \lambda \mapsto \sigma(\lambda, \frac{\partial}{\partial \tau} \lambda_\tau|_{\tau=t}) : J(t) \to \mathbb{R}$$

(here λ_τ is any smooth curve belonging to $J(\tau)$).



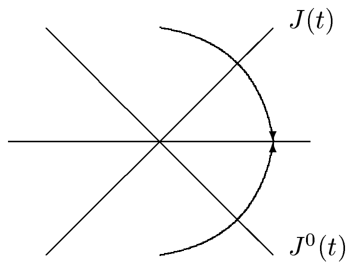
In particular $\dot{J}(0) : p \mapsto \langle p, \dot{S}(0)p \rangle$. We say that $J(t)$ is *monotonic* if $\dot{J}(t) \geqslant 0$ ($\dot{J}(t) \leqslant 0$).

Exercise 4

1. Prove that $\dot{S}(0) = -\frac{\partial^2 H}{\partial p^2}$ and $\dot{J}_\lambda(t) \sim -\frac{\partial^2 H}{\partial p^2}|_{e^{t\vec{H}}(\lambda)}$ ($\sim$ means equivalence of quadratic forms with respect to linear changes of variables). Conclusion: if H is convex then $J(t)$ is monotonic.
2. If $\dot{J}(t)$ is an Euclidean structure then the curvature operator $R(t)$ is symmetric in it (and we also can distinguish the definite cases: $R(t) \gtrless 0$).

Theorem 2 *Assume we are in the regular situation:* $\dot{J}(t) > 0$. *Then* $R(t) \leqslant 0$ *implies nonexistence of conjugate points.*

In particular, for $m = 1$ (Σ is a plane) there exist some limiting lines to which tend $J(t)$ and $J^0(t)$:



Exercise 5 Prove that the condition $R(t) \leqslant 0$ is equivalent to $\dot{J}^0(t) \leqslant 0$.

Hint: We have $\Sigma = J(t) \oplus J^\circ(t)$ and symplectic form on Σ defines a non degenerate pairing of $J(t)$ and $J^\circ(t)$ so that $J^circ(t) \cong J(t)^*$, $J(t) \cong J^circ(t)^*$. Velocity $\dot{J}(t)$ is identified with a quadratic form on the space $J(t)$ or, in other words, with a self-adjoint linear mapping from $J(t)$ to $J(t)^*$. We obtain:

$$\dot{J}(t) : J(t) \to J(t)^* \cong J^\circ(t) \qquad \dot{J}(t) : J(t) \to J^\circ(t)$$
$$\dot{J}^\circ(t) : J^\circ(t) \to J(t) \qquad R(t) = \dot{J}^\circ(t)\dot{J}(t) : \quad J(t) \to J(t).$$

Theorem 3 *(Comparison Theorem)*

1. *If $R(t) \leqslant C\,\mathrm{Id}$ (all eigenvalues are less or equal to C) and points t_0, t_1 are conjugate, then $|t_1 - t_0| \geqslant \pi/\sqrt{C}$.*
2. *If $(1/m)\,\mathrm{tr}\,R(t) \geqslant C$, then for any t_0 and for any $t \geqslant t_0$ the segment $[t, t + \pi/\sqrt{C}]$ contains a point t_1 conjugate to t_0.*

Now we will discuss the special case of constant curvature:

$$R(t) = C\,\mathrm{Id}\,. \tag{6}$$

If $C = 0$, i.e. $R(t) = 0$, then $J^\circ(t) = const$. This corresponds to the case of "straight lines": $J(t) = \{(p, tp) \mid p \in \mathbb{R}^m\}$.

If $C \neq 0$ (note that $J^\circ(t) \cap J(t) = \{0\}$, $J^{\circ\circ}(t) \cap J^\circ(t) = \{0\}$) condition (6) implies $J^{\circ\circ}(t) = J(t)$. In general, solutions of the equation $J^{\circ\circ}(t) = J(t)$ have normal forms including the case (6). In Riemannian geometry $J^{\circ\circ}(t) = J(t)$ corresponds to symmetric spaces.

Exercise 6 If $J(t) = \{(p, S(t)p) \mid p \in \mathbb{R}^m\}$, $S(0) = 0$, $\dot{S}(0) = I$ then the condition $J^{\circ\circ}(t) \equiv J(t)$ implies $S(t) = (2R)^{-1/2}\tan(t(2R)^{1/2})$ ($S(t)$ is symmetric, hence $\sqrt{\cdot}$ is well-defined) and the comparison theorem is sharp for the constant curvature.

We conclude with the discussion how to construct a Jacobi curve in the degenerate case. Come back to a control system (1) and assume that

$$f(q,u) = f_0(q) + \sum_i u_i f_i(q), \quad U = \mathbb{R}^n.$$

Recall that an "optimal" trajectory is one going to the boundary of the attainable set. Put $F_t : u(\cdot) \mapsto q(q_0, u(\cdot); t)$, where $q(q_0, u(\cdot); t)$ is the trajectory with the initial condition q_0 and control u evaluated at time t, and call it *input-state* mapping. So the attainable set is the image of F_t and if $u(\cdot)$ is an "optimal" control then it is a critical point of F_t, i.e. $im D_{u(\cdot)} F_t \neq T_{q(t)}M$, or equivalently, there exists $\lambda_t \in T^*_{q(t)}M, \lambda_t \neq 0$, such that $\lambda_t D_{u(\cdot)} F_t = 0$. In turn, if $u(t)$ is critical for F_t then for any $\tau \leqslant t$, $u|_{[0,\tau]}$ is critical for F_τ, or there exists a nontrivial $\lambda_\tau \in T^*_{q(\tau)}M$ such that $\lambda_\tau D_{u|_{[0,\tau]}} F_\tau = 0$.

This can be expressed as the following system of equations:

$$\dot{\lambda}_\tau = \vec{h}(\lambda, u(\tau)) \qquad \frac{\partial h}{\partial u}(\lambda, u(\tau)) = 0,$$

where $h(\lambda, u) = \langle \lambda, f(q,u) \rangle$. In coordinates we have $\lambda = (p,q)$, $h(\lambda,u) = pf(q,u)$ and

$$\dot{p} = -\frac{\partial h}{\partial q}(p,q,u) \qquad \dot{q} = \frac{\partial h}{\partial p}(p,q,u) \qquad \frac{\partial h}{\partial u}(p,q,u) = 0.$$

If $\det \frac{\partial^2 h}{\partial u^2} \neq 0$ then the last equation can be solved at least locally, i.e. there exists $u(\lambda)$ such that $\frac{\partial h}{\partial u}(\lambda, u(\lambda)) = 0$. If, in addition, $\frac{\partial^2 h}{\partial u^2} < 0$ this u maximizes $h(\lambda, \cdot)$: $h(\lambda, u(\lambda)) = H(\lambda)$, so we come to the situation described earlier (see (2)). In general non degenerate case we can substitute $u(\lambda)$ into first two equations of the system and proceed as we did for H in order to construct the Jacobi curve.

Now, if $\frac{\partial^2 h}{\partial u^2}$ is degenerate (it is identical 0 for the above affine in control system) we need to linearize the system first and then to construct the Jacobi curve. This way we come to a notion of an *$\mathcal{L}$-derivative*.

Let W be a smooth (possibly infinite-dimensional) manifold, $F : W \to M$ be a smooth map. Consider the system

$$\lambda D_w F = 0 \qquad \lambda \in T^*_{F(w)}M, \tag{7}$$

which in coordinates ($\lambda = (p,q)$) can be rewritten as

$$p\frac{\partial F}{\partial w} = 0 \qquad q = F(w).$$

Let us try to linearize it:

$$p'\frac{\partial F}{\partial w} + p\frac{\partial^2 F}{\partial w^2}w' = 0 \qquad q' = \frac{\partial F}{\partial w}w'.$$

Define $\mathcal{L}_{(w,\lambda)}F \subset T_\lambda(T^*M)$ as

$$\mathcal{L}_{(w,\lambda)}F = \left\{ \left[\begin{array}{c} p' \\ q' \end{array} \right] : \exists w' \text{ s.t. } p'\frac{\partial F}{\partial w} + p\frac{\partial^2 F}{\partial w^2}w' = 0, \;\; q' = \frac{\partial F}{\partial w}w' \right\}.$$

Proposition *If* $\dim W < \infty$*, then* $\mathcal{L}_{(w,\lambda)}F$ *is a Lagrangian subspace of* $T_\lambda(T^*M)$*.*

In the infinite-dimensional case $\frac{\partial^2 F}{\partial w^2}$ may have a non-closed image; more precisely, the image is closed if and only if we are in the regular situation. So we need to change the definition.

By the proposition above for any finite-dimensional submanifold $V \subset W$ and any $w \in V$ the subspace $\Lambda_V := \mathcal{L}_{(w,\lambda)}(F|_V)$ is Lagrangian in $T_\lambda(T^*_{F(w)}M)$. The set of all finite-dimensional submanifolds of W is partially ordered by $\subset$ and Λ_V form a generalized sequence; we can try to pass to the limit.

Theorem 4 *The limit* $\lim\limits_{V} \Lambda_V =: \mathcal{L}_{(w,\lambda)}F$ *exists if and only if the Hessian* $\mathrm{Hess}_w\, \lambda F$ *has finite positive or negative inertia index.*

To compute the $\mathcal{L}$-derivative one should approximate W by finite-dimensional manifolds.

Now return to a control system (1) and define a map $G_\tau : v(\cdot) \mapsto \hat{q}(0)$, where $v(\cdot)$ is in a space of controls and $\frac{d\hat{q}}{d\theta} = f(\hat{q}, v(\theta)), \hat{q}(\tau) = q(\tau)$. Then for any $\tau \in [0,t]$ we can write a variant of system (7) for G_τ

$$\lambda(0)D_{u(\cdot)|_{[0,\tau]}}G_\tau = 0$$

and then linearize it in order to construct an $\mathcal{L}$-derivative.

Finally, the Jacobi curve we are looking for is defined as $J_{\lambda(0)}(\tau) = \mathcal{L}_{(u(\cdot),\lambda(0))}G_\tau$ ($J(\tau)$ is a Lagrangian subspace of $T_{\lambda(0)}(T^*M)$).

References: Journal of Dynamical and Control Systems, **3**(1997), 343–389; **4**(1998), 583–604; **8**(2002), 93–140, 167–215.