Nonholonomic Tangent Spaces: Intrinsic Construction and Rigid Dimensions

A. Agrachev*, A. Marigo†

Abstract

A nonholonomic space is a smooth manifold equipped with a bracket generating family of vector fields. Its infinitesimal version is a homogeneous space of a nilpotent Lie group endowed with a dilation which measures the anisotropy of the space. We give the intrinsic construction of these infinitesimal objects and classify all rigid (i.e. not deformable) cases.

1 Introduction

Let M be a (C^∞-) smooth connected n-dimensional manifold and $\mathcal{F} \subset \mathrm{Vec}M$ be a set of smooth vector fields on M. Given $q \in M$ and an integer $l > 0$ we set

$$\Delta_q^l = span\{[f_1, [\cdots, [f_{i-1}, f_i] \cdots](q) : f_j \in \mathcal{F},\ 1 \leq j \leq i,\ i \leq l\} \subseteq T_qM.$$

Of course, $\Delta_q^l \subseteq \Delta_q^m$ for $l < m$. The set $\mathcal{F}$ is called bracket generating (or completely nonholonomic) at q if there exists m_q such that $\Delta_q^{m_q} = T_qM$. The minimal among these m_q is called the degree of nonholonomy of $\mathcal{F}$ at q. The set $\mathcal{F}$ is called bracket generating if it is bracket generating at every point.

We treat the pair $(M, \mathcal{F})$ as a "nonholonomic space", i.e. we assume that information propagates in M only along integral curves of the fields from $\mathcal{F}$ and concatinations of these curves. If $\mathcal{F}$ is bracket generating, then any point of M is "reachable" from any other, i.e. the points can be connected by

*SISSA, Trieste and Steklov Institute, Moscow; email: agrachev@ma.sissa.it
†IAC-CNR, Rome; email: marigo@iac.rm.cnr.it

an admissible curve. This is the statement of the Rashevski–Chow theorem [5, 7]. Actually, one can derive more from this theorem. Let $t \mapsto e^{tf}$ be the (local) one-parametric subgroup of diffeomorphisms generated by the vector field $f \in \mathcal{F}$ so that $t \mapsto e^{tf}(q)$, $q \in M$, are integral trajectories of the field f. The natural "nonholonomic topology" is the strongest topology on M such that the mappings

$$(t_1, \dots, t_k) \mapsto e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k}(q), \quad f_i \in \mathcal{F}\ q \in M,\ k = 1, 2, \dots$$

are continuous. It follows from the arguments of Rashevski and Chow that this nonholonomic topology coincides with the standard topology on M.

So the nonholonomic space $(M, \mathcal{F})$ is homeomeomorphic to the manifold M, but Analysis and Geometry on $(M, \mathcal{F})$ are very different from those on M. The main difference comes from the anisotropy of nonholonomic spaces: information propagates in all directions but with very different rates.

In this note we focus on tangent spaces to the nonholonomic space. Let $q \in M$; we denote $T_q^{\mathcal{F}}M$ the nonholonomic tangent space at q to be defined. We should make some remarks before we go to its construction, since this kind of spaces, burdened by extraneous structures, already appeared in various contexts. First of all, the nonholonomic tangent functor is a generalization of the usual one: if $\mathcal{F} = \mathrm{Vect}M$ then $T_q^{\mathcal{F}}M$ coincides with T_qM. Actually the tangent space $T_q^{\mathcal{F}}M$ is homeomorphic to $\mathbb{R}^n$ for any bracket generating $\mathcal{F}$. Moreover, $T_q^{\mathcal{F}}M$ is a homogeneous space of a nilpotent Lie group but it has not, in general, an intrinsic linear structure. On the other hand, it is intrinsically equipped with a canonical dilation which measures the anisotropy of the space.

Sub-Riemannian context. Assume that $\mathcal{F}$ is the space of sections of a vector subbundle of TM (a vector distribution) and that M is endowed with a Riemannian structure. The infumum of the lengths of admissible curves connecting two fixed points in M defines the Carnot–Caratheodory distance between the points. It is well-known that Carnot–Carateodory metric spaces have tangent metric spaces in the Gromov–Hausdorff sense. These tangent spaces are homeomorphic to $\mathbb{R}^n$ and their metrics are homogeneous with respect to certain dilations (see [3] for the detailed exposition). It happens that the space and the dilation are defined independently on any metric and just represent $T_q^{\mathcal{F}}M$.

Other contexts are mathematical control theory and hypoelliptic operators. Nonholonomic tangent functor appears here via nilpotent or graded approximations of the anisotropic objects to study (see [1, 2, 4, 6, 8]). The cited papers contain effective constructions and algorithms which allow to compute things explicitly. A weak point of these constructions is their heavy dependence on the choice of coordinates. Because of that, the approximations look like auxiliary technical tools rather than a fundamental functorial operation; the geometric insight and application of geometric machinery are highly impeded.

In this paper we introduce the nonholonomic tangent functor which performs the mentioned graded approximations in a natural coordinate free way and demonstrates the intrinsic meaning of the dilation. One more goal of the paper is to classify all "rigid" cases. What does it mean? Obviously, $T_q^{\mathcal{F}}M$ depends only on the germ of $\mathcal{F}$ at q. Moreover, it depends only on the generated by $\mathcal{F}$ module over the algebra of germs at q of smooth scalar functions. Assume that this module has d generators $f_1, \ldots, f_d$; then $T_q^{\mathcal{F}}M = T_q^{\{f_1 \ldots f_d\}}M$. We say that $T_q^{\{f_1 \ldots f_d\}}M$ is *rigid* if $T_q^{\{f_1 \ldots f_d\}}M$ is isomorphic to $T_q^{\{f_1' \ldots f_d'\}}M$ for all f_i' close enough to f_i in C^∞ topology. We call the pair (d, n) *rigid bi-dimension* if there exists at least one set of germs $\{f_1, \ldots, f_d\} \subset Vect_0\mathbb{R}^n$ such that $T_0^{\{f_1 \ldots f_d\}}\mathbb{R}^n$ is rigid. Given a rigid bi-dimension (d, n), it is not hard to show that generic germ $\{f_1, \ldots, f_d\}$ is rigid and there is only a finite number of mutually non isomorphic rigid $T_0^{\{f_1 \ldots f_d\}}\mathbb{R}^n$.

In this paper we completely characterize rigid bi-dimensions and indicate the number of different (up to an isomorphism) $T_0^{\{f_1 \ldots f_d\}}\mathbb{R}^n$ for any rigid bi-dimension. It happens that this number can be only 1, 2, or 3. The normal forms and the proofs will be given in the forthcoming long paper.

2 Tangent Functor

We call flow in M any one-parametric family of difeomorphisms $P_t \in \mathrm{Diff}M$ satisfying the condition $P_0 = id$. The flows form a group with respect to the point-wise composition: the product of P_t and $\hat{P}_t$ is the flow $t \mapsto P_t \circ \hat{P}_t$. Let $q \in M$. Tangent space T_qM consists of 1-jets at 0 of smooth curves $\gamma(t)$ in M such that $\gamma(0) = q$. To construct the nonholonomic tangent space $T_q^{\mathcal{F}}M$ we need jets of order m_q, where m_q is the degree of nonholonomy of $\mathcal{F}$ at q. The group of flows naturally acts on the space of smooth curves started from q: the flow P_t sends the curve $\gamma(t)$ to the curve $P_\cdot(\gamma) : t \mapsto P_t(\gamma(t))$. Of course, the l-jet at 0 of the curve $P_\cdot(\gamma)$ depends only on P_t and the l-jet of γ for any positive integer l. Let C^l_q be the space of l-jets of smooth curves in M started from q. We thus obtain the action of the group of flows on C^l_q. We keep notation q for the constant curve $\gamma(t) \equiv q$ and its jets; indeed, the trajectory $t \mapsto P_t(q)$ of the flow P_t is the image of the constant curve under the natural action.

The canonical dilation

$$\delta^l_\alpha : C^l_q \to C^l_q, \quad \delta^l_{\alpha_1\alpha_2} = \delta^l_{\alpha_1} \circ \delta^l_{\alpha_2}, \quad \alpha \in \mathbb{R}, \quad \delta_0(C^l_q) = q$$

is induced by the standard dilation δ_α on the space of smooth curves: $(\delta_\alpha \gamma)(t) = \gamma(\alpha t)$ for any curve γ.

We set $p^{l,k}(f) : C^l_q \to C^l_q$ the transformation induced by the flow $t \mapsto e^{t^k f}$, where $f \in \mathrm{Vec}M$. Let $\mathcal{P}^{l,\mathcal{F}}$ be the group of transformations generated by δ^l_α, $\alpha \neq 0$ and by $p^{l,k}(f)$, $f \in \mathcal{F}$, $k = 1, 2, 3, \ldots$. Let $\phi \in \mathcal{P}^{l,\mathcal{F}}$; we define $D(\phi) = \lim\limits_{s\to 0} \delta^l_s \circ \phi \circ \delta^l_{s^{-1}}$. Then $D : \mathcal{P}^{l,\mathcal{F}} \longrightarrow \{\delta_\alpha : \alpha \neq 0\}$ is the group homomorphism which projects $\mathcal{P}^{l,\mathcal{F}}$ on the dilation.

We denote by $\mathcal{O}^{l,\mathcal{F}} \subset C^l_q$ the orbit of the jet q under the action of the group $\mathcal{P}^{l,\mathcal{F}}$. Let us also consider the normal subgroup $\mathcal{P}^{l,\mathcal{F}}_\circ$ of $\mathcal{P}^{l,\mathcal{F}}$ generated by $p^{l,k}(f)$, $f \in \mathcal{F}$, $k = 2, 3, \ldots$ (the difference with $\mathcal{P}^{l,\mathcal{F}}$ is that $p^{l,1}(f)$ and δ^l_α are absent). Finally, we set:

- $T^{l,\mathcal{F}}_q M$ the quotient space of $\mathcal{O}^{l,\mathcal{F}}$ by the action of the group $\mathcal{P}^{l,\mathcal{F}}_\circ$;
- $T_q\mathcal{P}^{l,\mathcal{F}}$ the group of transformations of $T^{l,\mathcal{F}}_q M$ induced by the action of $\mathcal{P}^{l,\mathcal{F}}/\mathcal{P}^{l,\mathcal{F}}_\circ$.

We keep symbols δ^l_α and $p^{l,k}(f)$ for the induced transformations of $T^{l,\mathcal{F}}_q M$ and symbol D for the induced homomorphism of $T_q\mathcal{P}^{l,\mathcal{F}}$ on $\{\delta_\alpha : \alpha \neq 0\}$.

Proposition 1 *The codimension 1 normal subgroup $D^{-1}(id) \subset T_q\mathcal{P}^{l,\mathcal{F}}$ is a nilpotent Lie group generated by one-parametric subgroups $s \mapsto p^{l,1}(sf)$, $f \in \mathcal{F}$. The group $D^{-1}(id)$ acts transitively on $T^{l,\mathcal{F}}_q M$.*

Let $\pi^l : C^l_q \to C^{l-1}_q$ be the standard projection. Obviously, $\pi^l \circ p^{l,k}(f) = p^{l-1,k}(f) \circ \pi^l$, $\quad \pi^l \circ \delta^l_\alpha = \delta^{l-1}_\alpha \circ \pi^l$. Hence π^l sends the orbits of the groups $\mathcal{P}^{l,\mathcal{F}}$ and $\mathcal{P}^{l,\mathcal{F}}_\circ$ to the orbits of the groups $\mathcal{P}^{l-1,\mathcal{F}}$ and $\mathcal{P}^{l-1,\mathcal{F}}_\circ$ and induces a mapping of the quotient spaces. We keep symbol π^l for the induced mapping so that $\pi^l : T^{l,\mathcal{F}}_q M \to T^{l-1,\mathcal{F}}_q M$.

Proposition 2 *For any $l > 0$, $\pi^l : T^{l,\mathcal{F}}_q M \to T^{l-1,\mathcal{F}}_q M$ is a fiber bundle with the fiber $\Delta^l_q/\Delta^{l-1}_q$.*

In particular, $T_q^{l,\mathcal{F}}M$ is diffeomorphic to Δ_q^l and $T_q^{l,\mathcal{F}}M = T_q^{m_q,\mathcal{F}}M$ for $l \geq m_q$, where m_q is the degree of nonholonomy. Moreover, one can show that $T_q\mathcal{P}^{l,\mathcal{F}} = T_q\mathcal{P}^{m_q,\mathcal{F}}$ for $l \geq m_q$ as well.

Definition Let l be greater or equal to the degree of nonholonomy, i. e. $\Delta_q^l = T_qM$. The nonholonomic tangent space $T_q^{\mathcal{F}}M$ is the manifold $T_q^{l,\mathcal{F}}M$ equipped with the transitive action of the group $T_q\mathcal{P}^{\mathcal{F}} \stackrel{def}{=} T_q\mathcal{P}^{l,\mathcal{F}}$. For any $f \in \mathcal{F}$, the vector field $T_q^{\mathcal{F}}f \in \mathrm{Vec}T_q^{\mathcal{F}}M$ is the generator of the one-parametric group $s \mapsto p^{l,1}(sf)$; in other words, $e^{sT_q^{\mathcal{F}}f} \stackrel{def}{=} p^{l,1}(sf)$.

Obviously, $f \mapsto T_q^{\mathcal{F}}f$ is a homomorphism of the Lie algebras of vector fields; the group $T_q\mathcal{P}^{\mathcal{F}}$ is generated by the dilation and one-parametric subgroups $e^{sT_q^{\mathcal{F}}f}$, $s \in \mathbb{R}$, $f \in \mathcal{F}$. Moreover, just from the fact that the definition of $T_q^{\mathcal{F}}M$ is intrinsic it follows that any diffeomorphism $\Phi : M \to M$ automatically induces equivariant mappings $\Phi_*^{\mathcal{F}} : T_q^{\mathcal{F}}M \to T_{\Phi(q)}^{\Phi_*\mathcal{F}}M$ such that $(\Phi_1 \circ \Phi_2)_*^{\mathcal{F}} = \Phi_{1*}^{\Phi_{2*}\mathcal{F}} \circ \Phi_{2*}^{\mathcal{F}}$ for any pair of diffeomorphisms Φ_1, Φ_2. One more functorial property is as follows. Assume that $\mathcal{F} \subset \mathcal{G} \subset \mathrm{Vec}M$; the identical inclusion $\imath : \mathcal{F} \to \mathcal{G}$ induces a homomorphism $\imath_* : T_q\mathcal{P}^{\mathcal{F}} \to T_q\mathcal{P}^{\mathcal{G}}$ and an equivariant smooth mapping $\imath_* : T_q^{\mathcal{F}}M \to T_q^{\mathcal{G}}M$.

Proposition 3 *Let $\bar{\mathcal{F}} = \{\sum\limits_{j=1}^{k} a_j f_j \,:\, f_j \in \mathcal{F},\ a_j \in C^\infty(M),\ k > 0\}$ be the module over $C^\infty(M)$ generated by $\mathcal{F}$ and $\imath : \mathcal{F} \to \bar{\mathcal{F}}$ be the identical inclusion. Then $\imath_* : \left(T_q\mathcal{P}^{\mathcal{F}}, T_q^{\mathcal{F}}M\right) \longrightarrow \left(T_q\mathcal{P}^{\bar{\mathcal{F}}}, T_q^{\bar{\mathcal{F}}}M\right)$ is an isomorphism.*

Remark The last proposition states that nonholonomic tangent spaces depend on the submodule of $\mathrm{Vec}M$ generated by $\mathcal{F}$ rather than by $\mathcal{F}$ itself. In fact, from the very beginning of the paper, we could deal with submodules of $\mathrm{Vec}M$ instead of subsets. This approach would provide slightly more general functorial properties but the whole construction would become even more dry and abstract than it is now. Anyway, algebraically trained reader will easily recover missed functorial properties.

3 Coordinate Presentation

Given nonnegative integers $k_1, \ldots, k_l$, where $k_1 + \cdots + k_l = n$, we present $\mathbb{R}^n$ as a direct sum $\mathbb{R}^{k_1} \oplus \cdots \oplus \mathbb{R}^{k_l}$. Any vector $x \in \mathbb{R}^n$ takes the form

$$x = (x_1, \ldots, x_l), \quad x_i = (x_{i1}, \ldots, x_{ik_i}) \in \mathbb{R}^{k_i},\ i = 1, \ldots, l.$$

Differential operators on $\mathbb{R}^n$ with smooth coefficients have the form $\sum\limits_{\alpha} a_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}$, where $a_\alpha \in C^\infty(\mathbb{R}^n)$ and α is a multi-index: $\alpha = (\alpha_1, \dots, \alpha_l)$, $\alpha_i = (\alpha_{i1}, \dots, \alpha_{ik_i})$, $|\alpha_i| = \sum\limits_{j=1}^{k_i} \alpha_{ij}$, $i = 1, \dots, l$. The space of all differential operators with smooth coefficients forms an associative algebra with composition of operators as multiplication. The differential operators with polynomial coefficients form a subalgebra of this algebra with generators $1, x_{ij}, \frac{\partial}{\partial x_{ij}}$, $i = 1, \dots, l$, $j = 1, \dots, k_i$. We introduce a $\mathbb{Z}$-grading into this subalgebra by giving the weights ν to the generators by $\nu(1) = 0$, $\nu(x_{ij}) = i$, and $\nu(\frac{\partial}{\partial x_{ij}}) = -i$. Accordingly

$$\nu\left(x^\alpha \frac{\partial^{|\beta|}}{\partial x^\beta}\right) = \sum_{i=1}^{l} (|\alpha_i| - |\beta_i|)i,$$

where α and β are multi-indices.

A differential operator with polynomial coefficients is said to be *ν-homogeneous* of weight m if all the monomials occurring in it have weight m. It is easy to see that $\nu(A_1 \circ A_2) = \nu(A_1) + \nu(A_2)$ for any ν-homogeneous differential operators A_1 and A_2. The most important for us are differential operators of order 0 (functions) and of order 1 (vector fields). We have $\nu(ga) = \nu(g) + \nu(a)$, $\nu([g_1, g_2]) = \nu(g_1) + \nu(g_2)$ for any ν-homogeneous function a and vector fields g, g_1, g_2. A differential operator of order N has weight at least $-Nl$; in particular, the weight of nonzero vector fields is at least $-l$. Vector fields of nonnegative weights vanish at 0 while the values at 0 of the fields of weight $-i$ belong to the subspace $\mathbb{R}^{k_i}$, the i-th summand in the presentation $\mathbb{R}^n = \mathbb{R}^{k_1} \oplus \cdots \oplus \mathbb{R}^{k_l}$.

We introduce a dilation $\delta_t : \mathbb{R}^n \to \mathbb{R}^n$, $t > 0$, by the formula

$$\delta_t(x_1, x_2, \dots, x_l) = (tx_1, t^2 x_2, \dots, t^l x_l). \tag{1}$$

ν-homogeneity means a homogeneity with respect to this dilation. In particular, we have $a(\delta_t x) = t^{\nu(a)} a(x)$, $\delta_{t*} g = t^{-\nu(g)} g$ for ν-homogeneous function a and vector field g.

Now let $g = \sum\limits_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}}$ be an arbitrary smooth vector field. Expanding the coefficients a_{ij} in a Taylor series in powers of x_{ij} and grouping the terms with the same weights, we get an expansion $g \approx \sum\limits_{m=-l}^{+\infty} g^{(m)}$, where $g^{(m)}$ is a

ν-homogeneous field of weight m. This expansion enables us to introduce a decreasing filtration in the Lie algebra of smooth vector fields $\mathrm{Vec}\mathbb{R}^n$ by putting

$$\mathrm{Vec}^m(k_1,\dots,k_l)=\{X\in\mathrm{Vec}\mathbb{R}^n : X^{(i)}=0 \text{ for } i<m\},\quad -l\le m<+\infty.$$

It is easy to see that

$$[\mathrm{Vec}^{m_1}(k_1,\dots,k_l),\mathrm{Vec}^{m_2}(k_1,\dots,k_l)]\subseteq\mathrm{Vec}^{m_1+m_2}(k_1,\dots,k_l).$$

It happens that this class of filtrations is in a sense universal. The following theorem is a special case of general results proved in [2, 4].

Theorem 1 *Suppose that* $\dim(\Delta_q^i/\Delta_q^{i-1})=k_i,\ i=1,\dots,l.$ *Then there exists a neighborhood* O_q *of the point* q *in* M *and a coordinate mapping* $\chi : O_q\to\mathbb{R}^n$ *such that*

$$\chi(q)=0,\quad \chi_*(\Delta_q^i)=\mathbb{R}^{k_1}\oplus\cdots\oplus\mathbb{R}^{k_i},\ 1\le i\le l,$$

and $\chi_*(\mathcal{F})\subset \mathit{Vec}^{-1}(k_1,\dots,k_l)$.

The mapping $\chi : O_q\to\mathbb{R}^n$ which satisfies conditions of the theorem is called an *adapted coordinate map* for $\mathcal{F}$. All constructions of Section 2 have a simple explicit presentation in the adapted coordinates. Unfortunately, an adapted coordinate map for given $\mathcal{F}$ is far from being unique and it is not clear how to select a canonical one.

In any coordinates, l-jet of a curve is identified with its degree l Taylor polynomial. In particular, $\chi_*(C_q^l)=\{\sum\limits_{i=1}^{l}t^i\xi_i : \xi_i\in\mathbb{R}^n\}$

Proposition 4 *Let* χ *be an adapted coordinate map for* $\mathcal{F}$*; then*

$$\chi_*(\mathcal{O}^{l,\mathcal{F}})=\{\sum_{i=1}^{l}t^i\xi_i : \xi_i\in\mathbb{R}^{k_1}\oplus\cdots\oplus\mathbb{R}^{k_i},\ 1\le i\le l\}$$

and, for any $\gamma\in\mathcal{O}^{l,\mathcal{F}}$,

$$\chi_*(\mathcal{P}_\circ^{l,\mathcal{F}}(\gamma))=\chi_*(\gamma)+\{\sum_{i=1}^{l}t^i\xi_i : \xi_i\in\mathbb{R}^{k_1}\oplus\cdots\oplus\mathbb{R}^{k_{i-1}},\ 1\le i\le l\}.$$

Proof of this proposition is based on the expansions

$$\chi_*(f) \approx \chi_*(f)^{(-1)} + \sum_{j=0}^{\infty} \chi_*(f)^{(j)} \ \forall f \in \mathcal{F},$$

$$\chi(e^{\tau f}(q)) \approx \sum_{i=1}^{\infty} \frac{\tau^i}{i!} \underbrace{(\chi_* f) \circ \cdots \circ (\chi_* f)}_{i \ times} Id\,(0),$$

where $Id\,(x) \equiv x,\ x \in \mathbb{R}^n$, and on the fact that all ν-homogeneous functions of positive weight vanish at 0.

Proposition 3 implies the identification of $T_0^{\chi_*(\mathcal{F})}\mathbb{R}^n$ with the space $\{\sum_{i=1}^{l} t^i \xi_i : \xi_i \in \mathbb{R}^{k_i},\ 1 \leq i \leq l\}$. Similarly, the vector field $T_0^{\chi*(\mathcal{F})} f$ is identified with $t(\chi_*(f)^{(-1)}$. In particular, the fields $T_q^{\mathcal{F}} f,\ f \in \mathcal{F}$, are represented by ν-homogeneous vector fields of weight -1 in adapted coordinates.

4 Regularity and Rigidity

Definition(cf. [8]) We say that $\mathcal{F} \subset \mathrm{Vec}M$ is regular at $q_0 \in M$ if $\dim \Delta_q^i$ is constant in a neighborhood of q_0, $\forall i > 0$.

Let $\mathcal{F}$ be regular at q and $\dim \Delta_q^1 = d$. Take $f_1, \ldots, f_d \in \mathcal{F}$ such that vectors $f_1(q_0), \ldots, f_d(q_0)$ form a basis of $\Delta_{q_0}^1$. Then $f_1(q), \ldots, f_d(q)$ form a basis of Δ_q^1 for any q from a neighborhood of q_0. Hence, for any $f \in \mathcal{F}$ there exist smooth functions $a_1, \ldots, a_d$ such that $f(q) = \sum_{i=1}^{d} a_i(q) f_i(q)$ for any q from the same neighborhood. It follows that

$$\Delta_q^l = span\{[f_{i_1}, [\ldots, f_{i_l}] \cdots](q) : 1 \leq i_j \leq d\} + \Delta_q^{l-1}, \quad l = 1, 2, \ldots.$$

The regularity implies that one can select vector fields from the collection $\{[f_{i_1}, [\ldots, f_{i_l}] \cdots](q) : 1 \leq i_j \leq d\}$ in such a way that the values of the selected fields at q form a basis of $\Delta_q^l / \Delta_q^{l-1}$ for all q close enough to 0. With these bases in hands we easily obtain the following well-known fact:

Lemma 1 *Assume that $\mathcal{F} \subset Vec\,M$ is regular at q_0, $v_i, v_j \in Vec\,M$, $v_i(q) \in \Delta_q^i,\ v_j(q) \in \Delta_q^j\ \forall q$, and $v_i(q_0) = 0$. Then $[v_i, v_j](q_0) \in \Delta_{q_0}^{i+j-1}$.*

It follows immediately from this lemma that Lie brackets of the vector fields with values in Δ_q^i, $i = 1, 2, \ldots$, induce the structure of a graded Lie algebra on the space $\sum\limits_{i>0} \Delta_{q_0}^i / \Delta_{q_0}^{i-1}$. We denote this graded Lie algebra by $\mathrm{Lie}_{q_0}\mathcal{F}$. Obviously, $\mathrm{Lie}_{q_0}\mathcal{F}$ is generated by $\Delta_{q_0}^1$.

Proposition 5 *Let $\mathcal{F}$ be regular and bracket generating at $q \in M$. Then the mapping $f \mapsto T_q^{\mathcal{F}} f$, $f \in \mathcal{F}$, induces an isomorphism of Lie algebras $Lie_q\mathcal{F}$ and $Lie\{T_q^{\mathcal{F}} f : f \in \mathcal{F}\}$.*

Recall that $Lie\{T_q^{\mathcal{F}} f : f \in \mathcal{F}\}$ is the Lie algebra of a codimension 1 normal subgroup of $T_q\mathcal{P}^{\mathcal{F}}$, which acts transitively on $T_q^{\mathcal{F}} M$ (see Proposition 1). The quotient of $T_q\mathcal{P}^{\mathcal{F}}$ by this subgroup is the dilation. We thus have the transitive action of the n-dimensional nilpotent Lie group generated by the Lie algebra $\mathrm{Lie}_q\mathcal{F}$ on T_qM. Since $T_q^{\mathcal{F}} M$ is diffeomorphic to $\mathbb{R}^n$ and has the "origin" (the jet of the constant curve q), we obtain a canonical isomorphism of $T_q^{\mathcal{F}} M$ and the simply connected Lie group generated by $\mathrm{Lie}_q\mathcal{F}$.

We now turn to the generic case. Let $\mathcal{L}_d$ be the free Lie algebra with d generators (all algebras in this paper are over $\mathbb{R}$); in other words, $\mathcal{L}_d$ is the Lie algebra of commutator polynomials of d variables. We have $\mathcal{L}_d = \bigoplus\limits_{i=1}^{\infty} \mathcal{L}_d^i$, where $\mathcal{L}_d^i$ is the space of degree i homogeneous commutator polynomials. We set $\ell_d(i) = \dim \mathcal{L}_d^i$, $\ell_d^{(i)} = \sum\limits_{j=1}^{i} \ell_d(i)$. The classical recursion expression of $\ell_d(i)$ is: $\ell_d(i) = d^i - \sum\limits_{j|i} j\ell_d(j)$.

Below we deal with the space of germs at $q \in M$ of d-tuples of smooth vector fields $(f_1, \ldots, f_d)$ endowed with the standard C^∞-topology. The following statement is almost obvious.

Proposition 6 *For an open everywhere dense set of d-tuples $(f_1, \ldots, f_d)$, the set $\mathcal{F} = \{f_1, \ldots, f_d\}$ is regular and bracket generating at q with the degree of nonholonomy $m_q = \min\{i : \ell_d^{(i)} \geq n\}$ and $\dim(\Delta_q^i / \Delta_q^{i-1}) = \ell_d(i)$ for $i = 1, \ldots, m_q - 1$.*

Take $\mathcal{F}$ such that the module $\bar{\mathcal{F}}$ is generated by a generic d-tuple of vector fields. According to Propositions 5 and 6, the classification of $\left(T_q\mathcal{P}^{\mathcal{F}}, T_q^{\mathcal{F}} M\right)$ for such $\mathcal{F}$ is reduced to the classification of generic graded Lie algebras $\mathrm{Lie}_q\mathcal{F}$ or correspondent Lie groups.

Definitions of rigidity and of rigid bi-dimensions were done in the Introduction (isomorphism in this case means just the Lie groups isomorphism). In the next theorem we list all rigid bi-dimensions. It is convenient to give special names to some infinite series of bi-dimensions. For $d = 2, 3, 4, \ldots$, the bi-dimensions $\left(d, \ell_d^{(i)}\right)$, $i = 1, 2, 3, \ldots$, are called *free*; the bi-dimension $(d, d+1)$ is called *Darboux bi-dimension*, and the bi-dimension $(d, (d-1)(d+2)/2)$ is called *dual Darboux bi-dimension*.

Theorem 2 *All free, Darboux, and dual Darboux bi-dimensions are rigid, as well as bi-dimensions* $(2k-1, 2k+1), (2k-1, 2k^2-n-2)$, $k \geq 2$*; any of these bi-dimensions admits a unique up to an isomorphism rigid group. Besides that, there are 12 exceptional rigid bi-dimensions:*

$(2,4)_1,\ (2,6)_2,\ (2,7)_2,\ (4,6)_2,\ (4,7)_2,\ (4,8)_2,\ (5,8)_2,$
$(5,9)_3,\ (5,11)_3,\ (5,12)_2,\ (6,8)_2,\ (6,19)_2,$

where index j *in the expression* $(d,n)_j$ *indicates the number of isomorphism classes of rigid groups for given bi-dimension* (d,n).

All other bi-dimensions are not rigid.

5 Generalization

The construction of the functor $T_q^{\mathcal{F}}$ is easily generalized to the case of $\mathcal{F}$ with a fixed filtration (when various fields from $\mathcal{F}$ have various "weights"). More precisely, let

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_\nu \subseteq \cdots = \mathcal{F}$$

and $f_0(q) = 0$ for any $f_0 \in \mathcal{F}_0$. Then the (local) diffeomorphism e^{f_0} induces a transformation $p^{l,0}(f)$ of C_q^l. We now set:

- $\Delta_q^l = span\{[f_1, [\cdots, [f_{i-1}, f_i]\cdots](q) : f_j \in \mathcal{F}_{\nu_j},\ \nu_1 + \cdots + \nu_i \leq l,\ i > 0\}$;
- $\mathcal{P}^l(\mathcal{F})$ is the group of transformations generated by δ_α^l, $\alpha \in R \setminus 0$ and by $p^{l,k}(f)$, $f \in \mathcal{F}_k$, $k \geq 0$;
- $\mathcal{P}_\circ^l(\mathcal{F})$ is the normal subgroup of $\mathcal{P}^l(\mathcal{F})$ generated by $p^{l,k}(f)$, $f \in \mathcal{F}_{k-1}$, $k > 0$.

and repeat all the construction with these modified definitions. The presence of $\mathcal{F}_0$ brings some new phenomena. In particular, Lie algebra generated by

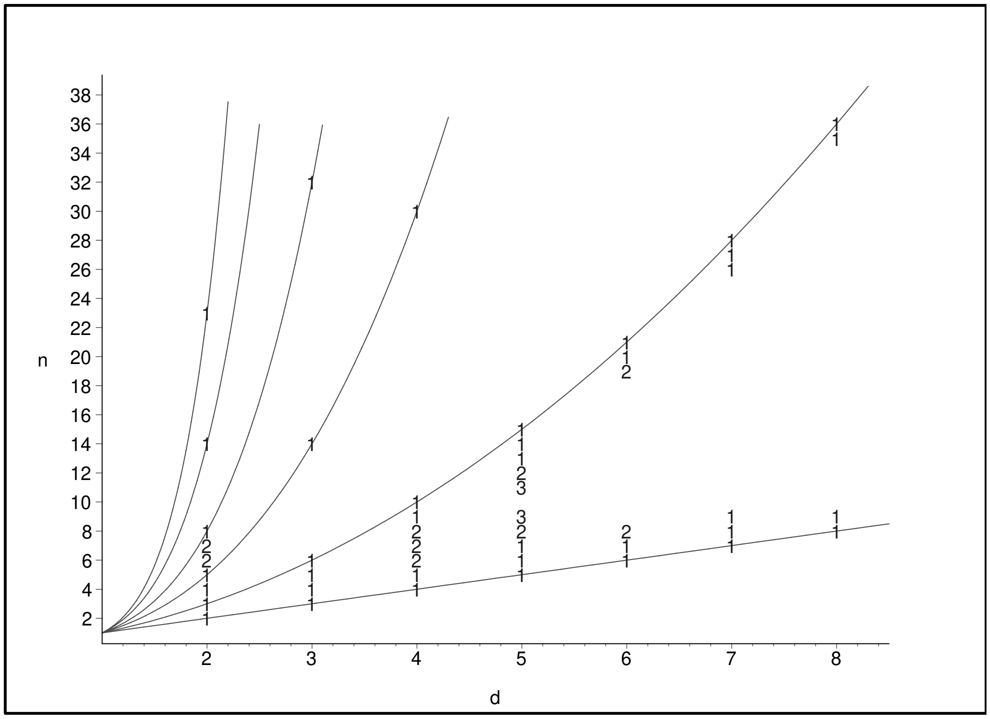

Figure 1: Rigid bi–dimensions with the indication of the number of isomorphism classes. The free bi–dimensions lie on the curves.

$T_q f$, $f \in \mathcal{F}$, is not, in general, nilpotent. Moreover, the substitution of $\mathcal{F}$ by the module $\bar{\mathcal{F}}$ with the induced filtration may enlarge this Lie algebra.

Typical examples are smooth nonlinear control systems with an equilibrium at $(q, 0)$:

$$\dot{x} = f(x, u), \quad x \in M,\ u \in \mathbb{R}^r, \quad f(q, 0) = 0. \tag{2}$$

We set $\mathcal{F}_\nu = \left\{ \frac{\partial^{|\alpha|}}{\partial u^\alpha} f(\cdot, 0) : |\alpha| \leq \nu \right\}$; the induced filtration of the module $\bar{\mathcal{F}}$ is feedback invariant. If the linearization of system (2) at $(q, 0)$ is controllable, then the tangent functor provides exactly the linearization and its module version admits a finite classification (Brunovsky normal forms). Number of isomorphism classes equals the number of partitions of r. It would be very interesting to study the tangent functor for some classes of systems with noncontrollable linearizations.

References

[1] A. A. Agrachev, A. V. Sarychev, Filtrations of a Lie algebra of vector fields and nilpotent approximation of control systems. Dokl. Akad. Nauk SSSR, t.295 (1987); English transl. in Soviet Math. Dokl., v.36 (1988), 104–108

[2] A. A. Agrachev, R. V. Gamkrelidze, A. V. Sarychev, Local invariants of smooth control systems. Acta Appl. Math., v.14 (1989), 191–237

[3] A. Bellaiche, The tangent space in sub-Riemannian geometry. In the book: Sub-Riemannian geometry, Birkhäuser, Progress in Math. 144 (1996), 1–78

[4] R. M. Bianchini, G. Stefani, Graded approximations and controllability along a trajectory. SIAM J. Control Optim., v.28 (1990), 903–924

[5] W-L. Chow, Über Systeme von linearen partiellen Differentialgleichungen ester Ordnung. Math. Ann., v.117 (1940/41), 98–105

[6] L. M. Rothschild, E. M. Stein, Hypoelliptic differential operators and nilpotent groups. Acta Math., v.137 (1976), 247–320

[7] P.K.Rashevsky, About connecting two points of a completely nonholonomic space by admissible curve. Uch. Zapiski Ped. Inst. Libknechta, No.2 (1938), 83–94

[8] A. M. Vershik, V. Ya. Gershkovich, Nonholonomic dynamic systems. Geometry of distributions and variational problems. Springer Verlag, EMS, v.16 (1987), 5–85