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Local Controllability and Semigroups of Diffeomorphisms

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Abstract. The local structure of orbits of semigroups, generated by families of diffeomorphisms, is studied by Lie theory methods. New sufficient conditions for local controllability are obtained which take into account ordinary, as well as fast-switching variations.

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1. Introduction

1. Let M be a real-analytic manifold, $\operatorname{Vect} M$ the Lie algebra of analytic vector fields on M. We consider a vector field $f \in \operatorname{Vect} M$ as a differential operator of first order on the algebra of smooth functions $C^\infty(M)$ which satisfies the differentiation rule

$$f(\varphi_1\varphi_2) = (f\varphi_1)\varphi_2 + \varphi_1(f\varphi_2) \qquad \forall \varphi_1, \varphi_2 \in C^\infty(M).$$

A point $x \in M$ will be identified with the corresponding homomorphism

$$x\colon C^\infty(M) \longrightarrow \mathbb{R}, \qquad \varphi \mapsto \varphi(x).$$

A tangent vector $\xi \in T_xM$ is a linear functional $\xi\colon C^\infty(M) \to \mathbb{R}$ which satisfies condition

$$\xi(\varphi_1\varphi_2) = (\xi\varphi_1)\varphi(x) + \varphi_1(x)(\xi\varphi_2),$$

so that the value of the field at the point x is considered as a composition:

$$x \circ f\colon C^\infty(M) \xrightarrow{f} C^\infty(M) \xrightarrow{x} \mathbb{R}.$$

The Lie bracket of the fields f_1, f_2 has the usual meaning as the commutator of operators $[f_1, f_2] = f_1 \circ f_2 - f_2 \circ f_1$. The symbol e^{tf} denotes a series in degrees of t with operator coefficients of the form

$$\mathrm{e}^{tf} = \mathrm{id} + \sum_{k=1}^{\infty} \frac{t^n}{n!} f^n, \quad f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}.$$

Let $x_0 \in M$. If the function φ is analytic in some neighborhood of x_0, then the power series with real coefficients $x \circ e^{tf} \varphi$ is convergent for small values of t and all x sufficiently close to x_0. In addition, we have $x(0) \circ e^{tf} \varphi = \varphi(x(t))$, where $t \mapsto x(t)$ is a solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x = x \circ f. \tag{1}$$

Therefore, in the sequel we use the notation $x(t) = x(0) \circ e^{tf}$. For all future constructions, we fix a point $x_0 \in M$ and suppose that $\mathcal{F} \subset \operatorname{Vect} M$ is an arbitrary set of vector fields, which is uniformly bounded in the C^1-topology in some neighborhood of x_0. Hence, there exists such a neighborhood O of $(x_0, 0) \in M \times \mathbb{R}$ that $x(t) = x(0) \circ e^{tf}$ is defined for all $(x(0), t) \in O$ and for all $f \in \mathcal{F}$. All our considerations will be local with regard to x, as well as with regard to t. In order to elucidate the exposition, we shall always suppose that the expression $x \circ e^{tf}$ is defined for all values x, t, under consideration, which certainly will not restrict the generality or the rigor of reasoning.

2. The set of differential equations

$$\dot{x} = x \circ f, \quad f \in \mathcal{F}, \quad x \in M \tag{2}$$

with the fixed initial condition $x(0) = x_0$ defining a *control system.* We call a *control (function)* an arbitrary piecewise constant mapping of $\mathbb{R}_+ = \{t \mid t \geqslant 0\}$ into $\mathcal{F}$. The points of discontinuity of this mapping we call the *switching points* of the control. The trajectory of the control system, which corresponds to the control $f_\tau, \tau \geqslant 0$, is the solution of the differential equation

$$\dot{x} = x \circ f_\tau, \quad x(0) = x_0.$$

Denote

$$f_\tau = f_i \quad \text{for} \quad t_{i-1} < \tau < t_i, \quad 0 = t_0 < t_1 < \ldots < t_k \leqslant t,$$

then,

$$x(t) = x_0 \circ e^{t_1 f_1} \circ e^{(t_2 - t_1) f_2} \circ \cdots \circ e^{(t - t_k) f_k}.$$

DEFINITION. The set

$$\mathcal{A}_t(\mathcal{F}) = \left\{ x_0 \circ e^{s_1 f_1} \circ \ldots \circ e^{s_k f_k} \mid f_i \in \mathcal{F},\ s_i \geqslant 0,\ i = 1, \ldots, k; \right.$$
$$\left. \sum_{i=1}^{k} s_i < t,\ k > 0 \right\},$$

is called the *attainable set for the system* (2) *for times less than* t.

One of the basic problems in control theory is to investigate the directions in which the controlled trajectory of (2) moves starting from x_0, or, how do the attainable sets $\mathcal{A}_t$ behave for small $t > 0$. For example, if $\alpha, \beta \geqslant 0$, $f_1, f_2 \in \mathcal{F}$, then, for every given $t > 0$ the point $x_0 \circ e^{\varepsilon\alpha f_1} \circ e^{\varepsilon\beta f_2}$ belongs to $\mathcal{A}_t$ for sufficiently small $\varepsilon > 0$ and is situated 'almost' in the direction of $x_0 \circ (\alpha f_1 + \beta f_2)$ from the point x_0. If $f_i, -f_i \in \mathcal{F}$, $i = 1, 2$, then the point $x_0 \circ e^{\varepsilon f_1} \circ e^{\varepsilon f_2} \circ e^{-\varepsilon f_1} \circ e^{-\varepsilon f_2}$ belongs to $\mathcal{A}_t$ for small ε and is situated 'almost' in the direction of $x_0 \circ [f_1, f_2]$ from the point x_0, etc.

DEFINITION. The system (2) is called *locally controllable at x_0 for small time* if $x_0 \in \operatorname{int} \mathcal{A}_t(\mathcal{F})\ \forall t > 0$.

If we remove in the definition of attainable sets all restrictions on s_i, we obtain an upper estimate for $\mathcal{A}_t(\mathcal{F})$ which is relatively easy to compute. For this purpose, we introduce the following definition.

DEFINITION. The set

$$\mathcal{O}(\mathcal{F}) = \{x_0 \circ e^{s_1 f_1} \circ \ldots \circ e^{s_k f_k} \mid f_i \in \mathcal{F},\ s_i \in \mathbb{R},\ i = 1, \ldots, k > 0\}$$

is called the *orbit of the system* (2) (*through* x_0).

Denote by $\operatorname{Lie} \mathcal{F}$ the Lie subalgebra in $\operatorname{Vect} M$ generated by the set $\mathcal{F}$ and let

$$\operatorname{Lie}_x \mathcal{F} = \{x \circ f \mid f \in \operatorname{Lie} \mathcal{F}\} \quad x \in M.$$

The following well-known theorem holds

THEOREM 1 (Nagano–Sussmann). *The orbit $\mathcal{O}(\mathcal{F})$ is an immersed analytic submanifold in M and*

$$T_x \mathcal{O}(\mathcal{F}) = \operatorname{Lie}_x \mathcal{F} \quad \forall x \in \mathcal{O}(\mathcal{F}).$$

Since $\mathcal{A}_t(\mathcal{F}) \subset \mathcal{O}(\mathcal{F})$ for all $t \geqslant 0$, an evident necessary condition for local controllability is the equality

$$\operatorname{Lie}_{x_0} \mathcal{F} = T_{x_0} M.$$

This is the so-called rank condition for controllability which we suppose to be fulfilled in the sequel without further mention.

The following relation is well known:

$$\mathcal{A}_t(\mathcal{F}) \subset \overline{\operatorname{int} \mathcal{A}_t(\mathcal{F})}.$$

Thus, $\mathcal{A}_t(\mathcal{F})$ contains interior points arbitrarily close to x_0 (provided the rank condition is fulfilled). The main problem is to find the conditions under which the initial point x_0 is an interior point.

3. To fix the system of notions in which the answer should be given we shall describe here a complete system of invariants of a family of analytic vector fields. Let

$$\mathcal{F} = \{f_\nu \mid \nu \in \mathcal{N}\} \qquad \text{and} \qquad \mathcal{G} = \{g_\nu \mid \nu \in \mathcal{N}\}$$

be two families of analytic vector fields with identical index sets, and suppose that $\operatorname{Lie}_{x_0} \mathcal{F} = T_{x_0}M$. Furthermore, suppose that $\operatorname{Lie}\mathcal{N}$ is a real free Lie algebra with the set of generators $\mathcal{N}$. The mappings $\nu \mapsto f_\nu$ and $\nu \mapsto g_\nu$ are uniquely extended to the homomorphisms

$$\bar{f}\colon \operatorname{Lie}\mathcal{N} \longrightarrow \operatorname{Vect} M, \qquad \bar{g}\colon \operatorname{Lie}\mathcal{N} \longrightarrow \operatorname{Vect} M,$$

respectively. The system of invariants is described by the following proposition, which is a consequence of Theorem 1.

PROPOSITION 1. *The following two assertions are equivalent.*

(*1*) *There exists a diffeomorphism* $\Phi\colon O_{x_0} \to O'_{x_0}$ *defined on some neighborhood* O_{x_0} *of* $x_0 \in M$, *such that* $\Phi(x_0) = x_0$ *and* $\Phi_* f_\nu = g_\nu|_{O'_{x_0}}$ $\forall \nu \in \mathcal{N}$.

(*2*)

$$\{\lambda \in \operatorname{Lie}\mathcal{N} \mid x_0 \circ \bar{f}(\lambda) = 0\} = \{\lambda \in \operatorname{Lie}\mathcal{N} \mid x_0 \circ \bar{g}(\lambda) = 0\}.$$

Thus, the local behavior of a control system is completely defined by the values at x_0 of the commutator polynomials of the initial vector fields. We start with the formulation of results which could be obtained without considering the commutators, i.e., using only 'commutator polynomials' of vector fields of the first degree — linear combinations of vector fields. Denote by $\operatorname{conv}\mathcal{F}$ the convex hull of the subset $\mathcal{F} \subset \operatorname{Vect} M$. A result from the sliding state theory implies

$$\operatorname{int}\mathcal{A}_t(\mathcal{F}) \supset \operatorname{int}\mathcal{A}_\tau(\operatorname{conv}\mathcal{F}) \qquad \forall t > \tau > 0.$$

Using this relation and the uniform boundedness of $\mathcal{F}$ around x_0, and denoting by $\operatorname{rel\,int} S$ the relative interior (i.e., the interior relative to the carrier) of the convex set S, we come to the assertions

$$0 \in x_0 \circ (\operatorname{rel\,int}\operatorname{conv}\mathcal{F}) \Longrightarrow x_0 \in \operatorname{int}\mathcal{A}_t(\mathcal{F}) \quad \forall t > 0 \Longrightarrow 0 \in \overline{x_0 \circ (\operatorname{conv}\mathcal{F})};$$

$$x_0 \in \operatorname{int}\mathcal{A}_t(\mathcal{F}) \quad \forall t > 0 \iff x_0 \in \operatorname{int}\mathcal{A}_t(-\mathcal{F}) \quad \forall t > 0.$$

The last relation asserts that the local controllability for small time is equivalent to the possibility of reaching the point x_0 for small time from every point sufficiently close to x_0 along the trajectories of system (2). This property indicates some superficial analogies between the problem of local controllability and the problem of asymptotic stability for dynamical systems (without controls).

4. In order to obtain deeper controllability conditions, we must use the commutator polynomials of vector fields from $\mathcal{F}$. We shall mention here some illuminating results for the most descriptive (though, by far not easy) case of two vector fields $\mathcal{F} = \{X, Y\}$, to give some motivation for the general considerations in further sections.

The permutation $X \leftrightarrow Y$ defines an involution on $\operatorname{span}\mathcal{F}$. Put $f = \frac{1}{2}(X+Y)$, which is a fixed element for this involution, and $g = \frac{1}{2}(X-Y)$ is an eigenvector with the eigenvalue (-1). Furthermore,

$$\operatorname{conv}\mathcal{F} = \{f + ug \mid |u| \leqslant 1\}.$$

The earlier controllability conditions used the filtration of the algebra $\operatorname{Lie}\{f, g\}$ according to the powers of g, which corresponds to the asymptotic development of the solutions of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f + u(t)g, \quad |u| \leqslant 1, \tag{3}$$

according to the powers of u. Suppose π is a commutator monomial of several variables. Denote by $\deg_i \pi$ the degree of π with respect to the ith variable, and by $\deg \pi$ the total degree over all variables. The set of all commutator monomials of two variables we denote by $\mathfrak{M}_2$. Put

$$L^n(g, f) = \operatorname{span}\{\pi(g, f) \mid \deg_1 \pi \leqslant n, \ \pi \in \mathfrak{M}_2\}.$$

In particular,

$$L^0(g, f) = \{f\}, \qquad L^1(g, f) = \operatorname{span}\{(\operatorname{ad} f)^i g \mid i = 0, 1, \dots\}, \ \text{ etc.},$$

where, as usual,

$$(\operatorname{ad} f)v = [f, v], \qquad (\operatorname{ad} f)^{i+1}v = [f, (\operatorname{ad} f)^i v], \quad \forall v \in \operatorname{Vect} M, \ i = 1, 2, \dots.$$

The obvious necessary condition of local controllability for small time, which will be always supposed to be satisfied, is $x_0 \circ f = 0$. The simplest sufficient condition — controllability by linear approximation:

$$x_0 \circ L^1(g, f) = T_{x_0}M.$$

A more sophisticated necessary condition — the generalized Legendre–Klebsch condition:

$$x_0 \circ (\operatorname{ad} g)^2 f \in x_0 \circ L^1(g, f).$$

Already, these lower order conditions show that the monomials of even and odd degrees in g play different roles. The reason for such a behavior can be explained if we observe that the involution $g \mapsto -g$ changes the sign of the monomials of odd degree in g and leaves unchanged the monomials of even degree. The following theorem of Hermes and Sussmann, cf. [22], contains everything that could be extracted, if we restrict our considerations to the filtration $L^n(g, f)$ and the involution $g \to -g$.

THEOREM 2. *If*

$$x_0 \circ L^{2k}(g,f) \subset x_0 \circ L^{2k-1}(g,f), \quad \forall k \geqslant 0,$$

then the system is locally controllable for small time.

The following necessary condition, generalizing the Legendre–Klebsch conditions, is due to Stefani [19]:

PROPOSITION 2. *If the system* (3) *is locally controllable for small time, then*

$$x_0 \circ (\operatorname{ad} g)^{2k} f \in x_0 \circ L^{2k-1}(g,f), \quad \forall k \geqslant 0.$$

But still there is a deep gap between the necessary and sufficient conditions of local controllability. It could be somewhat narrowed if some other filtrations and additional symmetry considerations are used. First of all, we can make in (3) the change of time variable $t = \int_0^\tau w(\theta)\,\mathrm{d}\theta$, which leads to the system

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = wf + wug, \quad w \geqslant 0, \quad |u| \leqslant 1. \tag{4}$$

Asymptotic expansion of solutions according to the powers of u or w leads us to a family of filtrations of the Lie algebra, generated by the fields g and f: if the field f has a 'zero weight' in the filtration $L^n(g,f)$, then we can assign to the field f an arbitrary nonnegative weight, not exceeding the weight of g. The following theorem due to Sussmann [23] is based on such filtrations and special symmetries of concrete problems (they will appear in a more general context below), which allow us to distinguish the polynomials of even and odd degrees not only with respect to g, but also with respect to f, as well.

THEOREM 3. *Suppose an $\alpha \in [0,1]$ exists, such that for every $\pi \in \mathfrak{M}_2$ which is of even degree in the first variable and odd degree in the second, the following relation holds*:

$$x_0 \circ \pi(g,f) \in \operatorname{span}\{x_0 \circ \pi'(g,f) \mid \deg_1 \pi' + \alpha \deg_2 \pi' < \deg_1 \pi + \alpha \deg_2 \pi,\ \pi' \in \mathfrak{M}_2\}.$$

Then the system (3) *is locally controllable for small time.*

Some additional necessary and sufficient conditions based on the same filtrations can be found in [6, 16]. It turns out, however, that the described filtrations are insufficient for obtaining satisfactory conditions of local controllability, even for simple polynomial systems of small dimensions. This is well demonstrated by the following remarkable example due to Kawski [15] which strongly influenced the content of this paper.

EXAMPLE. Let

$$M = R^4, \qquad x_0 = 0, \qquad x = (x^1, \dots, x^4) \in \mathbb{R}^4,$$

$$\partial_i = \frac{\partial}{\partial x^i} \in \operatorname{Vect} \mathbb{R}^4, \quad i = 1, \dots, 4.$$

Suppose

$$g = \partial_1, \qquad f = x^1 \partial_2 + (x^1)^3 \partial_3 + \big((x^3)^2 - (x^2)^7\big)\partial_4. \tag{5}$$

At the origin, only linear combinations of the following commutator monomials of f, g differ from zero:

$$g = \partial_1, \qquad 0 \circ [g, f] = \partial_2, \qquad (\operatorname{ad} g)^3 f = 6\partial_3,$$

$$\pi_1(g, f) = \big(\operatorname{ad}(\operatorname{ad} g)^3 f\big)^2 f = 72\partial_4, \qquad \pi_2(g, f) = \big(\operatorname{ad}[g, f]\big)^7 f = -7!\partial_4.$$

We have

$$\deg_1 \pi_1 = 6, \qquad \deg_2 \pi_1 = 3; \qquad \deg_1 \pi_2 = 7, \qquad \deg_2 \pi_2 = 8.$$

It is clear, that the fields g, f violate the above formulated sufficient condition of local controllability. One can even prove, that for all $N > 0$ there exists $t > 0$ such that, for every

$$x = (x^1, \dots, x^4) = 0 \circ e^{t_1(f + u_1 g)} \circ \dots \circ e^{t_N(f + u_N g)}, \quad |u_i| \leqslant 1, \quad \sum_{i=1}^N t_i \leqslant t,$$

the condition $x^1 = x^2 = x^3 = 0$ implies $x^4 \geqslant 0$.

Nevertheless, as was shown by Kawski, the system (3) with the fields (5) is locally controllable for small time. All previous sufficient conditions failed for this system, since for $t \to 0$ the number of switchings, which is necessary to attain all points of a neighborhood of 0, increases to the infinity.

Hence, some new methods should be introduced to handle this 'fast switching' phenomenon. The development of such methods is one of the purposes of this paper. But, as it frequently happens, to fully understand this particular phenomenon, we have to look at it from a much more general point of view.

5. We start with a remark that the definition of attainable sets employs not the vector fields but the mappings of the form

$$(t, x) \longmapsto x \circ e^{tf}. \tag{6}$$

Therefore, nothing prevents us to consider from the beginning not a family of vector fields $\mathcal{F}$ but an arbitrary family $\mathcal{P}$ of analytic mappings

$$(t, x) \longmapsto p(t, x), \tag{7}$$

defined on a region in $\mathbb{R} \times M$, with values in M and the initial condition $p(0, x) = x$. We shall adopt this approach. Furthermore, the parameter t does not necessarily plays the role of time, and the corresponding notion of local controllability is applicable not only to systems with continuous time, but to a broad class of systems with discrete time. Moreover, only in this more general situation can we achieve sufficiently flexible means for an adequate investigation of the original problem (2). Indeed, already the mapping $(t, x) \mapsto x \circ e^{tf_1} \circ e^{tf_2}$ could not be represented in the form (6) if $[f_1, f_2] \neq 0$.

The germs of mappings of the form (7) at $(0, x)$ constitute a semigroup: the product of the germs, represented by the mappings p_1, p_2, is the germ of the mapping $(t, x) \mapsto p_2(t, p_1(t, x))$. In fact, all germs are invertible and thus constitute a group. The first part of Section 2 contains a version of Lie theory of subgroups of this infinite-dimensional group.

Let $t \mapsto \gamma(t)$, $\gamma(0) = x_0$, be a smooth curve in M, hence $t \mapsto p(t, \gamma(t))$ is also a smooth curve starting at x_0. The transition from $\gamma(t)$ to $p(t, \gamma(t))$ defines an action of the group of germs of mappings of the form (7) on the set of germs of curves at x_0. This action induces, in turn, an action of the same group on the space of n-jets of curves at x_0 $\forall n > 0$. Note that the space of 1-jets of curves at x_0 is, by definition, $T_{x_0}M$, and the corresponding action depends only on the linearization of the mappings of the form (7) at $(0, x_0)$. For $n > 1$, higher polynomial approximations of these mappings enter into play.

The action of the group of germs of mappings of the form (7) on the spaces of jets of curves, thus introduced, play a central role in our investigation of the problem. It seems reasonable to suppose that similar considerations might render quite useful in other problems of geometric control and in the geometry of distributions. For example, the controllability problem is a special case of the problem of investigating attainable sets in the space of jets of curves. The results obtained can be easily applied to popular problems of curve interpolation.

The second half of Section 2 is devoted to a detailed study of the orbits, in the space of jets of curves, of the group generated by a given family $\mathcal{P}$ of mappings of the form (7). In Section 3, we study the action of a semigroup, generated by the family $\mathcal{P}$, in the same spaces: we investigate the situation of orbits of semigroups and of stable subsemigroups in the group orbits and, correspondingly, in the stable subgroups.

The basic results of the paper are formulated in Section 4. Theorem 4.1 formulates conditions which are sufficient for the orbit of a semigroup in the space of n-jets of curves to coincide with the orbit of group generated by this semigroup.* This theorem immediately implies an effective sufficient condition for controllability, which includes as a particular case Theorem 3 formulated above. But much more is achieved if we extend the semigroups with the aid of the so-called 'fast switching

* Formulas, theorems, propositions and lemmas are numbered in each section separately. The references for the same items from a different section have a double numbering, the firs number indicating the number of the corresponding section.

variations'. Namely, the elements of the semigroup are products of finite numbers of elements of the family $\mathcal{P}$, and the fast-switching variations make it possible to indefinitely increase the number of factors as t tends to zero. As a final outcome of the combination of all possible variations is our Theorem 4.2 — the central result of this paper. The final Section 5 is devoted to the technique of handling of fast-switching variations and to the proof of Theorem 4.2.

Fast-switching variations can be considered to be a further development of the idea of sliding states. Such fast-switching controls are successfully used in applications where one should not always consider the parameter t as a time-parameter. A good example is given by multi-layered materials: to attain the desired physical properties of a physical surface it is sometimes useful to combine a large number of very thin layers of different materials. Here t plays the role of the thickness of the layer.

6. We shall not try to formulate all main results of the paper in this introductory section, but still we can give here some special consequences pertaining to the system (3). The following assertion uses the notion of a reachable subspace in $T_{x_0}M$. The exact definition is given in Section 4, here we can only mention that the sum of two reachable subspaces is a reachable subspace and that the reachability of the space $T_{x_0}M$ implies the local controllability.

THEOREM 4 (cf. Proposition 4.9). *Let $\nu \in [0,1]$, $r \geqslant 0$, and let Π^1 be a set of bihomogeneous commutator polynomials in two variables such that they generate a Lie algebra* $\operatorname{Lie}\Pi^1$ *with the following properties*:

(a) Π^1 *is a set of free generators of the algebra* $\operatorname{Lie}\Pi^1$;

(b) $\operatorname{Lie}\Pi^1$ *contains all commutator monomials which are of even degree in the first argument and of odd degree in the second.*

Put

$$\Pi^{k+1} = [\Pi, \Pi^k], \qquad d_\nu(\pi) = \deg\pi - \nu k \quad \forall \pi \in \Pi^k,\ k = 1, 2, \ldots;$$

$$\Pi = \bigcup_{k=1}^{\infty} \Pi^k.$$

Suppose that for all $l \geqslant 0$ and every $\pi \in \Pi^{2l+1}$, which is of even degree in the first argument and of odd degree in the second and satisfies the relation $d_\nu(\pi) \leqslant r$, the inclusion

$$x_0 \circ \pi(g, f) \in \operatorname{span}\{x_0 \circ \pi'(g, f) \mid \pi' \in \Pi,\ d_\nu(\pi') < d_\nu(\pi)\}$$

holds. Then the subspace

$$\operatorname{span}\{x_0 \circ \pi(g, f) \mid \pi \in \Pi,\ d_\nu(\pi) \leqslant r\} \subset T_{x_0}M$$

is reachable for the system (3).

Let Lie $\{y, z\}$ be the Lie algebra of all commutator polynomials in y, z, i.e., a free Lie algebra with two free generators y, z. Every subalgebra of a free Lie algebra is free. Therefore, to apply the formulated theorem, we can first choose a bigraduated Lie subalgebra in Lie $\{y, z\}$, which contains all monomials of even degree in the first argument and of odd degree in the second, and only then choose some free generators in this subalgebra. For this purpose the so-called elimination theorem is quite useful, cf. [8]: Let S be an arbitrary set, $s_0 \in S$, then the linear hull of all commutator monomials in the elements of S, except the first degree monomial s_0, is a Lie algebra freely generated by the set

$$\{(\operatorname{ad} s_0)^i s \mid s \in S \setminus \{s_0\},\ i = 0, 1, 2, \dots\}.$$

The only Lie subalgebra of codimension 1 in Lie $\{y, z\}$, which satisfies the conditions of Theorem 4, is the linear hull of all commutator polynomials except y. The elimination theorem implies that this subalgebra is freely generated by the set $\{(\operatorname{ad} y)^i z \mid i = 0, 1, 2, \dots\}$, i.e., by the set of all monomials of degree 1 in the second variable. Thus, if we put $\Pi^1 = \{(\operatorname{ad} y)^i z \mid i = 0, 1, 2, \dots\}$, we obtain $d_\nu(\pi) = \deg_1 \pi + (1 - \nu) \deg_2 \pi\ \forall \pi \in \Pi$.

Put $f_i = (\operatorname{ad} g)^i f$, $i = 0, 1, 2, \dots$. Since $x_0 \circ g \neq 0$, the field g can be represented in some local coordinates in a neighborhood of x_0 in the form $g = \partial/\partial x^1 = \partial_1$. Substituting, if necessary, the field f by the field $f + \varphi g$, where $\varphi \in C^\omega(M)$, $\varphi(x_0) = 0$, i.e., after executing an elementary feedback transformation which does not influence the local controllability properties, we can represent the field f as $f = \sum_{j \neq 1} \varphi_j \partial_j$; then $f_i = \sum_{j \neq 1} (\partial_1^i \varphi_j) \partial_j$. Thus, without a loss of generality, we can assume that

$$x_0 \circ g \notin \operatorname{Lie}_{x_0}\{f_i \mid i = 0, 1, \dots\}.$$

Summing up, we can assert that Theorem 3 is a Corollary of Theorem 4 for

$$\Pi^1 = \{(\operatorname{ad} y)^i z \mid i = 0, 1, \dots\}.$$

Now we shall examine the case when Π^1 generates a subalgebra of codimension 2 in Lie $\{y, z\}$. It is easy to indicate all such subalgebras satisfying the conditions of Theorem 4. They depend on one integer-valued parameter which takes only odd values. The subalgebra corresponding to n is a linear hull of all monomials except y and $(\operatorname{ad} y)^n z$. According to the elimination theorem, it is freely generated by the set $\{(\operatorname{ad}(\operatorname{ad} y)^n z)^j (\operatorname{ad} y)^i z \mid i, j \geqslant 0,\ i \neq n\}$. Put

$$h_{ij}^n = (\operatorname{ad} f_n)^j f_i = \left(\operatorname{ad}(\operatorname{ad} g)^n f\right)^j (\operatorname{ad} g)^i f,$$

$$H_{ij}^k(n) = \operatorname{span}\left\{ [h_{i_k j_k}^n, [\dots, [h_{i_2 j_2}^n, h_{i_1 j_1}^n] \dots] \mid \sum_{\iota=1}^k i_\iota = i,\ \sum_{\iota=1}^k j_\iota = j,\ i_\iota \neq n \right\}.$$

The space $H_{ij}^k(n)$ consists of commutator polynomials in the fields g, f, of degree $nj + i$ in g and $j + k$ in f. As a Corollary of Theorem 4, we obtain

PROPOSITION 3. *Let $\mu \in [0,1]$, $r \geqslant 0$, and let n be a positive odd number. Suppose that for all even i, j and odd k, which satisfy the relation $(n+1)j + i + \mu k \leqslant r$, the inclusion*

$$x_0 \circ H_{ij}^k(n) \subset \operatorname{span}\{x_0 \circ H_{i'j'}^{k'}(n) \mid (n+1)j' + i' + \mu k' < (n+1)j + i + \mu k\}$$

holds. Then the space

$$x_0 \circ \operatorname{span}\{x \circ H_{ij}^k(n) \mid (n+1)j + i + \mu k \leqslant r\}$$

is reachable for the system (3).

Thus, if all elements of the spaces $x_0 \circ H_{ij}^k(n)$ for even i, j and odd k are 'neutralized by the brackets of lesser weight', then the system (3) is locally controllable. If i, j are even and k is odd, then the polynomials in g, f, constituting the space $H_{ij}^k(n)$, are of even degree in g and of odd degree in f. But the same is true if i, j are odd and k is even. Thus, contrary to Theorem 3, it is sufficient to neutralize only part of all brackets (roughly speaking, half of them), which are even in g and odd in f. For such neutralization we can use all brackets of 'lesser weight' except $x_0 \circ (\operatorname{ad} g)^n f$. Certainly, the set of the brackets which are to be neutralized, as well as the system of weights, depend on n.

Let us consider in more detail what happens with the brackets of fourth degree in g. For simplicity, we take the case $\mu = 0$, then the 'weights' do not depend on k. Put $H_{ij}(n) = \sum_{k=1}^{\infty} H_{ij}^k(n)$. For the fourth-order brackets, the interesting cases could be only for $n = 1, 3$. The spaces $x_0 \circ H_{3,1}(1)$ and $x_0 \circ H_{1,1}(3)$ consist completely of brackets of fourth degree in g which do not need neutralization. A simple calculation using the Jacobi identity leads us to the relations

$$H_{3,1}(1) = H_{1,1}(3) = \operatorname{span}\{[(\operatorname{ad} f)^i g,\ (\operatorname{ad} f)^j (\operatorname{ad} g)^3 f],\ i \geqslant 1,\ j \geqslant 0\}.$$

All remainder brackets of fourth degree in g are contained in

$$H_{4,0}(3) = H_{4,0}(1) + H_{2,2}(1) + H_{0,4}(1).$$

For $n = 3$ we obtain the following

COROLLARY. *If*

$$x_0 \circ L^2(g, f) \subset x_0 \circ L^1(g, f),$$

$$x_0 \circ H_{4,0}^k(3) \subset \left(x_0 \circ L^2(g, f) + x_0 \circ H_{3,0}(3)\right), \quad k = 1, 3, 5, \ldots,$$

then the space $x_0 \circ L^4(g, f) + x_0 \circ H_{5,0}(3)$ is reachable for the system (3).

(We used here the equality $H_{2,0}(3) = L^2(g, f)$).

For $n = 1$, the fourth degree brackets are obtained in a somewhat different way.

COROLLARY. *If*

$$x_0 \circ H_{2,0}(1) = 0, \quad x_0 \circ H_{0,2}(1) \subset x_0 \circ H_{0,1}(1),$$

$$x_0 \circ H^k_{4-j,j}(1) \subset \sum_{i+2\iota<4+j} x_0 \circ H_{i,\iota}(1)$$

for $j = 0, 2, 4$, $k = 1, 3, 5, \ldots$, *then the space* $x_0 \circ L^4(g, f) + x_0 \circ H_{5,0}(1)$ *is reachable for the system* (3).

It is interesting to notice that, according to the last corollary, the bracket $x_0 \circ (\mathrm{ad}[g, f])^4 f$, which belongs to $x_0 \circ H^1_{0,4}(1)$, can be neutralized by an arbitrary element from the space $x_0 \circ H_{7,0}(1)$, which contains brackets of degree 7 in g and of arbitrarily high degree in f. More generally, for arbitrary odd n and even j, the bracket $x_0 \circ (\mathrm{ad}\,(\mathrm{ad}\,g)^n f)^j f = x_0 \circ h^n_{0j}$ can be neutralized by the elements of the spaces $H_{i,0}(n)$ for $i < (n+1)j$. In the Kawski example, the bracket $x_0 \circ h^3_{02}$ is neutralized by the bracket $x_0 \circ (\mathrm{ad}\,[g, f])^7 f \in x_0 \circ H_{7,0}(3)$.

7. In this paper, we are considering only analytic systems, but all results of Section 4 and everything pertaining to jets is valid for the C^∞ case, and the proofs are similar to those given here. But the technique developed in this paper does not permit us to obtain the necessary conditions of controllability. At the same time, it seems quite promising to use this technique for obtaining sufficient conditions of 'controllability along the trajectory' and of the necessary conditions of optimality.

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2. Orbits of Groups of Diffeomorphisms and Groups of Flows

1. Let $\mathcal{P}$ be a set of germs at $(0, x_0)$ of analytic mappings $(s, x) \mapsto P(s, x)$ from $\mathbb{R} \times M$ into M, satisfying the condition $P(0, x) \equiv x$. All subsequent considerations are local in both variables s and x, therefore we shall always consider instead of germs their representatives. Without restricting generality, we can also suppose that for every fixed s the mapping $P(s, \cdot)$: $x \mapsto P(s, x)$ is a diffeomorphism of M, since for $s = 0$ we have the identity mapping of M and all considerations are local in s and x.

Vector fields on M are considered in the standard way as linear operators (differentiations) on the algebra of smooth functions $C^\infty(M)$. Diffeomorphisms of M will also be identified with linear operators on $C^\infty(M)$ — with automorphisms of the algebra $C^\infty(M)$. The image of a function $\varphi \in C^\infty(M)$ under the diffeomorphism p is defined according to the relation $(p\varphi)(x) = \varphi(p(x))$. The points of M are identified in the usual way with linear multiplicative functionals on $C^\infty(M)$ — the algebra of homomorphisms $C^\infty(M) \to \mathbb{R}$, acting according to the relation x: $\varphi \mapsto x\varphi \stackrel{\mathrm{def}}{=} \varphi(x)$. Therefore, the point $p(x)$ is identified with the composition

$$x \circ p\colon\ \varphi \longmapsto x \circ p(\varphi) = (p\varphi)(x) = \varphi(p(x)).$$

The operator notations are very convenient while dealing with asymptotic expansions.

After these introductory remarks we can consider $\mathcal{P}$ as a set of analytic curves $s \mapsto p(s)$ in the group of diffeomorphisms: $p(s) \in \operatorname{Diff} M\ \forall s,\ p(0) = \mathrm{id}$. Such curves in $\operatorname{Diff} M$ starting at id are called (nonstationary) flows on M. Standard examples of such curves are given by one parameter subgroups $s \mapsto e^{sf},\ f \in \operatorname{Vect} M$, but, certainly, there are numerous other examples.

DEFINITION. The sets

$$\mathcal{A}_t(\mathcal{P}) = \Bigg\{ x_0 \circ p_1(s_1) \circ \cdots \circ p_k(s_k) \mid p_i(\cdot) \in \mathcal{P}, \\ s_i \geqslant 0,\ i = 1, \ldots, k,\ \sum_{i=1}^{k} s_i < t; k > 0 \Bigg\}$$

are called *attainable sets* for the family $\mathcal{P}$. We say that $\mathcal{P}$ is *locally controllable at* x_0 if

$$x_0 \in \operatorname{int} \mathcal{A}_t(\mathcal{P}) \qquad \forall t > 0.$$

More generally, let $p \in \mathcal{P}$ be given; we say that $\mathcal{P}$ is *locally controllable along the curve* $s \mapsto x_0 \circ p(s)$, if $x_0 \circ p(t) \in \operatorname{int} \mathcal{A}_t(\mathcal{P})\ \forall t > 0$.

If $\mathcal{P}$ consists of one-parameter subgroups only, the given definitions coincide with the usual definitions of local controllability for small time and local controllability along a trajectory for the corresponding control system, cf. Section 1. The general case also includes the local controllability for arbitrary time (with small controls in the integral norm) for a vast class of systems with continuous, as well as discrete time.

2. Attainable sets $\mathcal{A}_t(\mathcal{P})$ are contained in the orbit through x_0 of the group of diffeomorphisms, generated by the diffeomorphisms $p(s),\ p \in \mathcal{P},\ s \in \mathbb{R}$. It is desirable to give a maximally explicit description of this orbit, as in Section 1, which leads to a simple 'upper bound' for the attainable sets.

DEFINITION. The set

$$\mathcal{O}(\mathcal{P}) = \{x_0 \circ q_1(s_1) \circ \cdots \circ q_k(s_k) \mid q_i \in \mathcal{P} \cup \mathcal{P}^{-1},\ s_i \in \mathbb{R}, i = 1, \ldots, k;\ k > 0\},$$

where $\mathcal{P}^{-1} = \{s \mapsto p(s)^{-1} \mid p(\cdot) \in \mathcal{P}\}$, is called the *orbit of the family* $\mathcal{P}$ through the point x_0.

We shall consider the group generated by diffeomorphisms $p(s), p \in \mathcal{P},\ s \in \mathbb{R}$, as well as the group of flows generated by flows which are obtained from flows in $\mathcal{P}$ by arbitrary polynomial substitutions of the parameter.

Let A be the space of all real polynomials without free coefficients: $a(0) = 0$ $\forall a \in A$. Put

$$\begin{aligned}\operatorname{Gr}(\mathcal{P}) = \bigl\{s \mapsto q_1\bigl(a_1(s)\bigr) \circ \cdots \circ q_k(a_k(s)) \mid q_i \in \mathcal{P} \cup \mathcal{P}^{-1}, \\ a_i \in A,\ i = 1, \dots, k;\ k > 0\bigr\}.\end{aligned} \tag{1}$$

Evidently,

$$\mathcal{O}(\mathcal{P}) = \bigl\{x_0 \circ q(s) \mid q \in \operatorname{Gr}(\mathcal{P})\bigr\} \quad \forall s \in \mathbb{R}.$$

3. Since the diffeomorphisms and vector fields are linear operators in $C^\infty(M)$, they can be added and multiplied, as well as multiplied with smooth functions. As a result, we again obtain linear operators, which are in general neither diffeomorphisms nor vector fields. We shall also differentiate and integrate with respect to the scalar parameter $s \in \mathbb{R}$ one-parameter families of linear operators $D(s)$ in $C^\infty(M)$. We say that $D(s)$ tends to D for $n \to \infty$ if all derivatives of $D(s_n)\varphi$ tend uniformly to the corresponding derivatives of $D\varphi$ in a neighborhood of x_0 for $\forall \varphi \in C^\infty M$. The derivative and the integral of $D(s)$ are always considered in the week operator sense, for example

$$x \circ \left(\frac{\mathrm{d}}{\mathrm{d}s} D(s)\right) \varphi \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}s}(x \circ D(s)\varphi) \quad \forall \varphi \in C^\infty(M), \quad x \in M.$$

The validity of the standard rules for differentiation of products, integration by parts, etc. in situations in which they are used in this article, should not cause any doubts. They are discussed in detail in [1].

Let $s \mapsto q(s)$ be a flow, i.e., a smooth curve in $\operatorname{Diff} M$ with the initial condition $q(0) = \mathrm{id}$. Put

$$\operatorname{ord} q = \min\left\{k > 0 \,\Big|\, \frac{\mathrm{d}^k}{\mathrm{d}s^k} q(s)\Big|_{s=0} \neq 0\right\},$$

and call the number $\operatorname{ord} q$ the order of the curve q. If $\operatorname{ord} q = n$ the operator

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n} q(s)\Big|_{s=0}$$

is a vector field. Indeed,

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}s^n} q(s)\Big|_{s=0}(\varphi_1\varphi_2) &= \frac{\mathrm{d}^n}{\mathrm{d}s^n}\bigl((q(s)\varphi_1)\,(q(s)\varphi_2)\bigr)\Big|_{s=0} \\ &= \left(\frac{\mathrm{d}^n}{\mathrm{d}s^n} q(s)\Big|_{s=0}\varphi_1\right)\varphi_2 + \varphi_1\left(\frac{\mathrm{d}^n}{\mathrm{d}s^n} q(s)\Big|_{s=0}\varphi_2\right).\end{aligned}$$

The operator

$$T_0 q \overset{\text{def}}{=} \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\, q(s)\Big|_{s=0}$$

can be considered as the 'tangent field to the flow at $q(0) = \mathrm{id}$'. Finally, put

$$\mathrm{Gr}\,(\mathcal{P})_n = \{q \in \mathrm{Gr}\,(\mathcal{P}) \mid \mathrm{ord}\, q = n\}, \qquad \mathrm{Gr}\,(\mathcal{P}) = \bigcup_{n=1}^{\infty} \mathrm{Gr}\,(\mathcal{P})_n.$$

Our nearest goal is to describe all fields, tangent at id to the flows in $\mathrm{Gr}\,(\mathcal{P})_n$.

4. To each $p \in \mathcal{P}$ we correspond an analytic curve $s \mapsto \omega_p(s)$ in $\mathrm{Vect}\, M$ defined by the relation

$$\omega_p(s) = p(s)^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}s}\, p(s). \tag{2}$$

It is easily seen that if $\omega_p(s) \not\equiv 0$, then the operator $\omega_p(s)$ is the tangent field to the flow $t \mapsto p(s)^{-1} \circ p(s+t)$. The relation (2) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s} x \circ p(s) = x \circ p(s) \circ \omega(s) \qquad \forall x \in M.$$

In standard notations, this is equivalent to a differential equation on M defined by the nonstationary field $(x, s) \mapsto (x \circ \omega_p(s)) \in T_x M$. Thus, $s \mapsto p(s)$ is a flow on M generated by the nonstationary vector field ω_p. The operator notations directly lead to the asymptotic representation of $p(s)$ as a Volterra series

$$\begin{aligned} p(s) &= \mathrm{id} + \int_0^s p(\tau) \circ \omega_p(\tau)\,\mathrm{d}\tau = \mathrm{id} + \int_0^s \omega_p(\tau)\,\mathrm{d}\tau + \\ &\quad + \iint_{0 \leqslant \tau_2 \leqslant \tau_1 \leqslant s} p(\tau_2) \circ \omega(\tau_2) \circ \omega(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 = \cdots \\ &\approx \mathrm{id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_s^n} \omega_p(\tau_n) \circ \cdots \circ \omega_p(\tau_1)\,\mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n, \end{aligned} \tag{3}$$

where

$$\Delta_s^n = \{(\tau_1, \ldots, \tau_n) \mid 0 \leqslant \tau_n \leqslant \cdots \leqslant \tau_1 \leqslant s\}. \tag{4}$$

The operator series (3) is divergent, but $\forall x_0 \in M$ and every real analytic function φ in the neighborhood of x_0, the series

$$x\varphi + \sum_{n=1}^{\infty} x \circ \int \cdots \int_{\Delta_s^n} \omega_p(\tau_n) \circ \cdots \circ \omega_p(\tau_1)\,\mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \varphi$$

converges to $x \circ p(s)\varphi$ for (x, s) close to $(x_0, 0)$. We call the flow p which satisfies (3), the *right chronological exponent of* ω_p and denote it by

$$p(s) = \overrightarrow{\exp} \int_0^s \omega_p(\tau)\, \mathrm{d}\tau.$$

Formal properties of chronological exponents are studied in [1, 2].

5. Let

$$\omega_p(s) = \sum_{n=1}^{\infty} s^{n-1}\omega_p^n, \quad \omega_p^n \in \operatorname{Vect} M,$$

be a Taylor series, expansion of the analytic curve $s \mapsto \omega_p(s)$. Put

$$\Omega_n = \operatorname{span}\left\{ [\omega_{p_k}^{i_k}, [\ldots, [\omega_{p_1}^{i_1}, \omega_{p_0}^{i_0}] \ldots] \mid \sum_{j=0}^{k} i_j \leqslant n,\ p_j \in \mathcal{P},\ 0 \leqslant k < n \right\},$$

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

It is easily seen that $\Omega = \operatorname{Lie}\{\omega_p^n \mid p \in \mathcal{P},\ n = 1, 2, \ldots\}$ is a Lie subalgebra in $\operatorname{Vect} M$ generated by the fields ω_p^n, and the subspaces Ω_n constitute an increasing filtration of this subalgebra.

THEOREM 1. *For every $n > 0$, the following relation holds*

$$\Omega_n = \left\{ T_0 q \mid q \in \operatorname{Gr}(\mathcal{P})_n \right\}.$$

Proof. Let $s \mapsto q(s)$ be an analytic flow. Put

$$\overrightarrow{\ln}\, q(s) = q(s)^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}s}\, q(s).$$

It is easily seen that

$$\int_0^s \overrightarrow{\ln}\, q(\tau)\, \mathrm{d}\tau = s^{\operatorname{ord} q} T_0 q + O(s^{\operatorname{ord} q + 1}). \tag{5}$$

Furthermore, for arbitrary flows q_1, q_2 the following relation holds which is a direct consequence of the variation formula, cf. [1, 2]:

$$\overrightarrow{\ln}\big((q_1 \circ q_2^{-1})(s)\big) = \overrightarrow{\exp} \int_0^s \operatorname{ad} \overrightarrow{\ln}\, q_2(\tau)\, \mathrm{d}\tau \big(\overrightarrow{\ln}\, q_1(s) - \overrightarrow{\ln}\, q_2(s)\big), \tag{6}$$

where by definition

$$\overrightarrow{\exp} \int_0^s \operatorname{ad} v(\tau)\, d\tau w(s) \approx w(s) + \\ + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_s^n} \left[v(\tau_n), [\ldots [v(\tau_1), w(s)] \ldots \right] d\tau_1 \cdots d\tau_n$$

for arbitrary curves v, w in $\operatorname{Vect} M$. Hence, the relations (5), (6) imply the inclusion

$$\Omega_n \supset \{T_0 q \mid q \in \operatorname{Gr}(\mathcal{P})_n\}.$$

To prove the inverse inclusion, we need some preparatory work. The following three statements are proved by direct calculation.

LEMMA 1. *For every flow q and every real polynomial $a(s) = \sum_{i=1}^k \alpha_i s^i$, $\alpha_1 \neq 0$, the following equalities hold*

$$\operatorname{ord} q = \operatorname{ord} q^{-1} = \operatorname{ord} q\big(a(\cdot)\big); \qquad T_0 q^{-1} = -T_0 q, \quad T_0 q\big(a(\cdot)\big) = \alpha_1^{\operatorname{ord} q} T_0 q.$$

LEMMA 2. *Let q_1, q_2 be flows such that* $\operatorname{ord} q_1 = \operatorname{ord} q_2 = n$, $T_0 q_1 + T_0 q_2 \neq 0$. *Then*

$$\operatorname{ord}(q_1 \circ q_2) = n, \qquad T_0(q_1 \circ q_2) = T_0 q_1 + T_0 q_2.$$

LEMMA 3. *For arbitrary flows q_1, q_2 the equalities*

$$\operatorname{ord}(q_1 \circ q_2 \circ q_1^{-1} \circ q_2^{-1}) = \operatorname{ord} q_1 + \operatorname{ord} q_2, \qquad T_0(q_1 \circ q_2 \circ q_1^{-1} \circ q_2^{-1}) = [T_0 q_1, T_0 q_2]$$

hold.

Lemmas 1–3 imply that the inclusion $\Omega_n \subset \{T_0 q \mid q \in \operatorname{Gr}(\mathcal{P})_n\}$ holds if we prove the inclusions

$$\omega_p^k \in \{T_0 q \mid q \in \operatorname{Gr}(\mathcal{P})_n\}, \quad k = 1, \ldots, n, \quad p \in \mathcal{P}. \tag{7}$$

We first prove by induction over n that $\omega_p^n = T_0 q$ for some $q \in \operatorname{Gr}(\mathcal{P})$. For $n = 1$, the statement is evident. Applying Lemmas 1–3 and the inductive assumption, we come to the conclusion that it is sufficient to find $q \in \operatorname{Gr}(\mathcal{P})_n$, for which

$$T_0 q = \alpha \omega_p^n + \pi\big(\omega_p^1, \ldots, \omega_p^{n-1}\big), \quad \alpha \neq 0, \tag{8}$$

where π is some commutator polynomial of $\omega_p^1, \ldots, \omega_p^{n-1}$ of weight n, if we agree that the weight of the variable ω_p^i is equal to i, $\forall i > 0$. We shall deduce (8) from a slightly more general assertion. Namely, we prove by induction over n the existence of a $q \in \operatorname{Gr}(\mathcal{P})$, such that

$$\overrightarrow{\ln}\, q(s) = \sum_{k=n}^{\infty} s^{k-1}\big(\alpha_k \omega_p^k + \pi_k(\omega_p^1, \ldots, \omega_p^{k-1})\big),$$

where $\alpha_k \neq 0$, and π_k is a commutator polynomial of weight k, $k = n, n+1, \ldots$. The induction step:

Let $q \in \mathrm{Gr}(\mathcal{P})$ be such that $\overrightarrow{\ln} q(s) = \sum_{k=n-1}^{\infty} s^{k-1} v_k$. Put

$$\widehat{q}(s) = q\left(2^{-\frac{1}{n-1}} s\right) \circ q(s)^{-1} \circ q\left(2^{-\frac{1}{n-1}} s\right).$$

Relation (6) implies

$$\overrightarrow{\ln} \widehat{q}(s) = \sum_{k=n}^{\infty} s^{k-1} \left(\left(1 - 2^{\frac{n-k-1}{n-1}}\right) v_k + \widehat{\pi}_k(v_1, \ldots, v_{k-1}) \right),$$

where $\widehat{\pi}_k$ is a commutator polynomial of weight k of the variables $v_1, \ldots, v_{k-1}$, if we assume that the weight of v_i is equal to i, $i > 0$. Thus, the induction is complete and the inclusion $\omega_p^n \in \{T_0 q \mid q \in \mathrm{Gr}(\mathcal{P})_n\}$ is proved. If a is a real polynomial, we denote by

$$q_a \colon s \longmapsto q\big(a(s)\big)$$

the flow obtained from q by parameter substitution.

LEMMA 4. *Let q be a flow, $\overrightarrow{\ln} q(s) \approx \sum_{n=1}^{\infty} s^{n-1} v_n$. Then, for every integer $n > 0$ and every $w \in \operatorname{span}\{v_1, \ldots, v_{n-1}\}$ there is a polynomial $\alpha(s) = s + \sum_{k=2}^{n} \alpha_k s^k$, such that*

$$\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}} \overrightarrow{\ln} q_\alpha(s) \Big|_{s=0} = v_n + w.$$

Proof. We have

$$\begin{aligned} \vec{q}_\alpha(s) &= \frac{\mathrm{d}}{\mathrm{d}s} \sum_{k=1}^{\infty} \frac{(\alpha(s))^k}{k} v_k \\ &= \sum_{n=1}^{\infty} s^{n-1} \left(v_n + n \sum_{i=1}^{n-1} (r_i^n(\alpha_2, \ldots, \alpha_i) + \alpha_{i+1}) v_{n-i} \right), \end{aligned}$$

where r_i^n is a polynomial of $i-1$ variables. Let $w = \sum_{i=1}^{n-1} \beta_i v_{n-i}$. To prove the lemma, it is sufficient to solve the following 'triangle' system of $n-1$ equations in unknowns $\alpha_2, \ldots, \alpha_n$:

$$r_i^n(\alpha_2, \ldots, \alpha_i) + \alpha_{i+1} = \frac{\beta_i}{n}, \qquad i = 1, \ldots, n-1,$$

which is uniquely solvable.

If we apply Lemma 4 to the flow $p \in \mathcal{P}$, we can conclude that for arbitrary $n > k > 0$ there exists such a polynomial α, that $\omega_p^n + \omega_p^k$ is the coefficient at s^{n-1} in the Taylor series expansion of the vector-function $\overrightarrow{\ln} p_\alpha(s)$. Since $p_\alpha \in \mathrm{Gr}(\mathcal{P})$, there exists such a flow $\widetilde{q} \in \mathrm{Gr}(\mathcal{P})$ that $T_0\widetilde{q} = \omega_p^n + \omega_p^k$, $\mathrm{ord}\,\widetilde{q} = n$. Furthermore, there exists a flow $q \in \mathrm{Gr}(\mathcal{P})$ such that $T_0 q = \omega_p^n$, $\mathrm{ord}\, q = n$. Hence, Lemmas 1–2 imply

$$T_0(\widetilde{q} \circ q^{-1}) = \omega_p^k, \qquad \mathrm{ord}\,(\widetilde{q} \circ q^{-1}) = n.$$

6.

THEOREM 2. *The orbit $\mathcal{O}(\mathcal{P})$ is an immersed analytic submanifold in M, and*

$$T_x\mathcal{O}(\mathcal{P}) = x \circ \Omega \quad \forall x \in \mathcal{O}(\mathcal{P}), \qquad x \circ \Omega \overset{\mathrm{def}}{=} \{x \circ v \mid v \in \Omega\} \subset T_x M.$$

Proof. Since Ω is a subalgebra in $\mathrm{Vect}\, M$, the distribution $x \circ \Omega$, $x \in M$, is generated by analytic vector fields and is involutive. According to Nagano [17], this distribution is completely integrable. Let $N \subset M$ be the maximal integral manifold through x_0 of this distribution. It is sufficient to prove that $\mathcal{O}$ is an open submanifold in N.

First of all, for arbitrary $p \in \mathcal{P}$, $y \in M$, $s \in \mathbb{R}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\, y \circ p(s) = y \circ p(s) \circ \omega_p(s) \subset y \circ p(s) \circ \mathrm{span}\,\{\omega_p^n \mid n \geqslant 1\}.$$

Hence, the curve $x \circ p(s)$, $s \in \mathbb{R}$, is contained in the orbit through y of the distribution $x \circ \Omega$, $x \in M$. From this and from the connectivity of the integral manifolds, we derive the inclusion $\mathcal{O} \subset N$.

Furthermore, let $x \in \mathcal{O}$ and $v_1, \ldots, v_m \in \Omega$ are such that the convex cone spanned by $x \circ V_1, \ldots, x \circ V_m$ coincides with $T_x N$. Let $v_i \in \Omega_{n_i}$, $i = 1, \ldots, m$. According to Theorem 1, there exist $q_i \in \mathrm{Gr}(\mathcal{P})$, $\mathrm{ord}\, q_i = n_i$, such that $T_0 q_i = v_i$. Consider the mapping

$$F\colon\ (s_1, \ldots, s_m) \longmapsto x \circ q_1(s_1^{1/n_1}) \circ \cdots \circ q_m(s^{1/n_m}), \quad s_i \geqslant 0, \quad i = 1, \ldots, m.$$

The image of F is contained in $\mathcal{O}$. On the other hand,

$$F(s_1, \ldots, s_m) = x + \sum_{i=1}^{m} s_i\, x \circ v_i + o\biggl(\sum_{i=1}^{m} s_i\biggr),$$

and we obtain by standard calculations that the image of F contains a neighborhood of x_0 in N.

7. After having discussed the orbits of groups of diffeomorphisms, we now turn to the orbits of groups of flows $\mathrm{Gr}(\mathcal{P})$.

The group operation in $\mathrm{Gr}(\mathcal{P})$ is the pointwise multiplication:

$$(q_1 \circ q_2)(s) \stackrel{\text{def}}{=} q_1(s) \circ q_2(s), \quad s \in \mathbb{R}.$$

Let $s \mapsto \gamma(s) \in M$ be a representative of the germ of an analytic curve on M, $\gamma(0) = x_0$, and s belongs to some neighborhood of zero. Put $(\gamma \circ q)(s) = \gamma(s) \circ q(s)$, $\forall q \in \mathrm{Gr}(\mathcal{P})$. The correspondence

$$q\colon \gamma \longmapsto \gamma \circ q \tag{10}$$

defines an action of the group $\mathrm{Gr}(\mathcal{P})$ on the germs of analytic curves in M at x_0.

We shall study the orbits of the group $\mathrm{Gr}(\mathcal{P})$ not in the infinite-dimensional space of the germs of curves, but in the spaces of n-jets of these curves for $n = 1, 2, \ldots$. We recall that the n-jet of a germ $\gamma(s) \in M$, $\gamma(0)$ is the equivalence class of curves which are tangent to γ for $s = 0$ at least up to the order n. We denote the n-jet of γ by $J^n\gamma$. In local coordinates the germ of a curve in M is identified with the germ of the corresponding vector function of dimension $\dim M$ and the n-jet is identified with the Taylor polynomial of degree n of this function. If we parametrize the jets with the Taylor polynomials, we obtain a natural structure of an $n \dim M$-dimensional manifold on the space of all n-jets of curves at the point x_0. The obtained manifold of the n-jets of curves we denote by $C^n_{x_0}$, and the symbol

$$\mathrm{pr}_n\colon C^n_{x_0} \longrightarrow C^{n-1}_{x_0}$$

will denote the canonical projection of the manifold of the n-jets onto the manifold of $(n-1)$-jets.

The manifold $C^n_{x_0}$ is diffeomorphic to an $n \dim M$-dimensional linear space, but does not have an invariantly defined linear structure (independent of the choice of the local coordinates in M). Nevertheless, some substitute of the linear structure exists and is rather useful.

Let $c \in C^n_{x_0}$, $\xi \in T_{x_0}M$. Furthermore, let the germ of the curve $s \mapsto \gamma(s) \in M$ and the flow $s \mapsto q(s) \in \mathrm{Diff}\, M$ are such that

$$J_n\gamma = c, \qquad \mathrm{ord}\, q = n, \qquad x_0 \circ T_0 q = \xi.$$

Put

$$c + \xi \stackrel{\text{def}}{=} J^n(\gamma \circ q) \in C^n_{x_0}.$$

It is easy to see that the jet $c + \xi$ indeed depends only on c and ξ and does not depend on the choice of the curve γ and the flow q. Furthermore, for all $c \in C^n_{x_0}$ the following relations hold:

(1) $(c + \xi_1) + \xi_2 = c + (\xi_1 + \xi_2)$, $\forall \xi_1, \xi_2 \in T_{x_0}M$;

(2) $\mathrm{pr}_n(c) = \mathrm{pr}_n(c') \iff c' = c + \xi$ for some $\xi \in T_{x_0}M$;

(3) $c + \xi = c \iff \xi = 0$.

The relations (1)–(3), as well as the smooth dependence of $c+\xi$ from (c,ξ), can be proved by writing out these relations in local coordinates. Summing up, we come to the following proposition.

PROPOSITION 1. *The mapping*

$$(c,\xi) \longmapsto c+\xi, \quad c \in C^n_{x_0}, \quad \xi \in T_{x_0}M,$$

defines the structure of an affine bundle on the manifold $C^n_{x_0}$ with the fibre $T_{x_0}M$, the base $C^{n-1}_{x_0}$ and the projection mapping $\mathrm{pr}_n\colon C^n_{x_0} \to C^{n-1}_{x_0}$.

We emphasize that the structure of an affine bundle, and not of a vector bundle, is defined, i.e., the origins in the affine spaces $c+T_{x_0}M$, $c \in C^n_{x_0}$, are not fixed.

8. We consider the action of the group of flows on $C^n_{x_0}$ induced by the action (10) of the group on the space of germs of curves. Let $c \in C^n_{x_0}$, and let $s \mapsto q(s)$ be a flow. Put $c \circ q \overset{\text{def}}{=} J^n(\gamma \circ q)$, where the curve $s \mapsto \gamma(s)$ satisfies $J^n\gamma = c$. The composition $c \circ q$ depends only on c and q, but not on γ.

The following theorem describes the structure of the orbits of the group $\mathrm{Gr}(\mathcal{P})$ in $C^n_{x_0}$, and can be considered as a 'jet precision' of Theorem 2. In the space $C^n_{x_0}$ there is a singled point — the n-jet of the constant curve $\gamma(s) \equiv x_0$. The orbit of this selected jet coincides with the set $\{J^n(x_0 \circ q) \mid q \in \mathrm{Gr}(\mathcal{P})\}$. The orbit of an arbitrary jet $c \in C^n_{x_0}$ is denoted by $c \circ \mathrm{Gr}(\mathcal{P})$. Thus,

$$c \circ \mathrm{Gr}(\mathcal{P}) = \{c \circ q \mid q \in \mathrm{Gr}(\mathcal{P})\}.$$

According to the definition, the mapping $\mathrm{pr}_n\colon C^n_{x_0} \to C^{n-1}_{x_0}$ is equivariant: $\mathrm{pr}_n(c \circ q) = \mathrm{pr}_n(c) \circ q$. Thus,

$$\mathrm{pr}_n\big(c \circ \mathrm{Gr}(\mathcal{P})\big) = \mathrm{pr}_n(c) \circ \mathrm{Gr}(\mathcal{P}).$$

By $\mathcal{C}^n$, we denote the affine bundle $\mathrm{pr}_n\colon C^n_{x_0} \to C^{n-1}_{x_0}$ with the fibre $T_{x_0}M$, cf. Proposition 1.

THEOREM 3. *Let n be a positive integer,*

$$c \in C^n_{x_0}, \qquad \mathcal{O}^n_c = c \circ \mathrm{Gr}(\mathcal{P}) \subset C^n_{x_0}, \qquad \mathcal{O}^{n-1}_c = \mathrm{pr}_n(c) \circ \mathrm{Gr}(\mathcal{P}) \subset C^{n-1}_{x_0};$$

$\mathcal{C}^n \mid \mathcal{O}^{n-1}_c$ is the restriction of the bundle $\mathcal{C}^n$ on the submanifold $\mathcal{O}^{n-1}_c$ of the base. Then, $\mathcal{O}^n_c$ is the total space of the affine subbundle in $\mathcal{C}^n \mid \mathcal{O}^{n-1}_c$ with the fibre $x_0 \circ \Omega_n = \{x_0 \circ v \mid v \in \Omega_n\}$.

Proof. We have to prove that for every $c' \in \mathcal{O}^n_c$ the relation

$$(c'+\xi) \in \mathcal{O}^n_c \iff \xi \in x_0 \circ \Omega_n \tag{11}$$

holds. Let $\xi = x_0 \circ v$ for some $v \in \Omega_n$. According to the Theorem 1, there exists a flow $q \in \mathrm{Gr}(\mathcal{P})$ such that $\operatorname{ord} q = n$ and $T_0 q = v$. From the definitions, we directly obtain $c' \circ q = c' + \xi$. Hence, $(c' + \xi) \in \mathcal{O}_c^n$.

Conversely, suppose $(c' + \xi) \in \mathcal{O}_c^n$ for some $\xi \in T_{x_0}M$. Then $c' + \xi = c' \circ q$ for some $q \in \mathrm{Gr}(\mathcal{P})$. Since $\mathrm{pr}_n(c'+\xi) = \mathrm{pr}_n(c')$, we have $x_0 \circ q(s) = x_0 + s^n \xi + O(s^{n+1})$. Furthermore,

$$x_0 \circ q(s) = x_0 + \int_0^s x_0 \circ q(\tau) \circ \overrightarrow{\ln}\, q(\tau)\, \mathrm{d}\tau.$$

Hence,

$$x_0 \circ \int_0^s \overrightarrow{\ln}\, q(\tau)\, \mathrm{d}\tau = s^n \xi + O(s^{n+1}). \tag{12}$$

LEMMA 5. *For every integer $n > 0$ the following relation holds*

$$\Omega_n = \left\{ \frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}} \overrightarrow{\ln}\, q(s)\Big|_{s=0} \,\Big|\, q \in \mathrm{Gr}(\mathcal{P}) \right\}.$$

Proof. The inclusion of Ω_n into the right-hand set follows from the identity (6), which enables us to compute the $\overrightarrow{\ln}$ of the composition of two flows through the $\overrightarrow{\ln}$ of the factors. The inverse inclusion follows from Theorem 1.

The equality (12) and Lemma 5 imply that $\xi = x_0 \circ v$ for some $v \in \Omega_n$.

COROLLARY. *The conditions of Theorem* 3 *imply*

$$\dim \mathcal{O}_c^n = \sum_{k=1}^{n} \dim (x_0 \circ \Omega_k).$$

The proof by induction over n.

9. The subspaces Ω_n, $n = 1, 2, \ldots$, constitute an increasing filtration of the Lie algebra Ω. Let W_n, $n = 1, 2, \ldots$, be another increasing filtration which is a refinement of Ω. This means that

$$W_n \subset W_{n+1}, \qquad W_n \subset V_n, \qquad [W_i, W_j] \subset W_{i+j} \quad \forall n, i, j.$$

We correspond to the filtration W and to any positive integer m, the subgroup

$$\mathrm{Gr}_W^m(\mathcal{P}) \stackrel{\mathrm{def}}{=} \left\{ q \in \mathrm{Gr}(\mathcal{P}) \,\Big|\, \frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}} \overrightarrow{\ln}\, q(s)\Big|_{s=0} \in W_k,\ k = 1, \ldots, m \right\} \tag{13}$$

of the group $\mathrm{Gr}(\mathcal{P})$. Formula (6) implies that $\mathrm{Gr}_W^m(\mathcal{P})$ is indeed a subgroup in $\mathrm{Gr}(\mathcal{P})$. It turns out that for every subgroup of the type (13) a theorem, similar to the Theorem 3, is valid.

PROPOSITION 2. *Let $W_1 \subset W_2 \ldots$, be an increasing filtration of the Lie algebra V, which is a refinement of the filtration V_i, $i = 1, 2, \ldots$. Furthermore, let*

$$c \in C^n_{x_0}, \qquad \mathcal{O}^{n,m}_c(W) = c \circ \mathrm{Gr}^m_W(\mathcal{P}) \subset C^n_{x_0},$$

$$\mathcal{O}^{n-1,m}_c(W) = \mathrm{pr}_n(c) \circ \mathrm{Gr}^m_W(\mathcal{P}) \subset C^{n-1}_{x_0}, \quad n \leqslant m.$$

Then the orbit $\mathcal{O}^{n,m}_c(W)$ is the total space of the affine subbundle in $C^n \mid \mathcal{O}^{n-1,m}_c(W)$ with the fibre $x_0 \circ W_n$.

Proof. Suppose $c' \in \mathcal{O}^{n,m}_c(W)$. We have to prove that

$$(c' + \xi) \in \mathcal{O}^{n,m}_c(W) \iff \xi \in x_0 \circ W_n.$$

That the left inclusion implies the right one is proved similarly to the proof of the corresponding assertion in Theorem 3, cf. the proof of the formula (12).

Conversely, let $\xi = x_0 \circ w$, $w \in W_n$. According to Theorem 1, there exists $q \in \mathrm{Gr}(\mathcal{P})$ such that $\mathrm{ord}\, q = n$, $T_0 q = w$. Let $k(q)$ be the least among the numbers i for which

$$\frac{\mathrm{d}^{i-1}}{\mathrm{d}s^{i-1}} \vec{\ln}\, q(s)\Big|_{s=0} \notin W_i.$$

Evidently, $k(q) > n$. If $k(q) > m$ then $q \in \mathrm{Gr}^m_W(\mathcal{P})$, otherwise we can find a $q' \in \mathrm{Gr}(\mathcal{P})$ for which

$$\mathrm{ord}\, q' = k(q), \qquad T_0 q' = -\frac{1}{k(q)!} \frac{\mathrm{d}^{k(q)-1}}{\mathrm{d}s^{k(q)-1}} \vec{\ln}\, q(s)\Big|_{s=0}.$$

Then,

$$\mathrm{ord}\,(q \circ q') = n, \qquad T_0(q \circ q') = w \quad \text{and} \quad k(q \circ q') > k(q).$$

If $k(q \circ q')$ still does not exceed m, we choose a corresponding q'', etc. In any case, by a finite number of steps, we come to a flow in $\mathrm{Gr}^m_W(\mathcal{P})$ of order n with the tangent field w, and this completes the proof.

3. Orbits of Semigroups

1. According to Theorem 2, the orbit $\mathcal{O}(\mathcal{P})$ is a submanifold in M. Clearly, the attainable sets $\mathcal{A}_t(\mathcal{P})$ are contained in $\mathcal{O}(\mathcal{P})$ for every $t > 0$. An evident necessary condition for local controllability is the equality of the dimensions of $\mathcal{O}(\mathcal{P})$ and M. In this case, $\mathcal{O}(\mathcal{P})$ is an open set in M. In the future, all constructions are carried out in $\mathcal{O}(\mathcal{P})$, therefore we shall assume, to simplify the notations, that $\mathcal{O}(\mathcal{P}) = M$. This convention does not restrict the generality and will be fulfilled in the sequel without further mention.

THEOREM 1. *The attainable set $\mathcal{A}_t(\mathcal{P})$ has interior points for every $t > 0$, and is contained in the closure of its interior $\mathcal{A}_t(\mathcal{P}) \subset \overline{\operatorname{int} \mathcal{A}_t(\mathcal{P})}$.*

The theorem almost immediately follows from the following lemma.

LEMMA 1. *For every $x \in M$ there exist $p_1, \ldots, p_k \in \mathcal{P}$ such that the mapping*

$$(s_1, \ldots, s_k) \longmapsto x \circ p_1(s_1) \circ \cdots \circ p_k(s_k) \tag{1}$$

of $\mathbb{R}^k$ into M has regular points, i.e., points, where the rank of the differential is equal to $\dim M$ *(the number k depends, in general, on x).*

Suppose that Lemma 1 is proved, $x \in \mathcal{A}_t(\mathcal{P})$, and take the positive ortant in $\mathbb{R}^k$,

$$\mathbb{R}^k_+ = \{(s_1, \ldots, s_k) \mid s_i > 0,\, i = 1, \ldots, k\}.$$

Then, we can find a neighborhood of the origin in $\mathbb{R}^k$, O_0, such that the image of $O_0 \cap \mathbb{R}^k_+$ under the mapping (1) is contained in $\mathcal{A}_t(\mathcal{P})$. The analyticity of the mapping and the existence of the regular points imply, that the regular points constitute an open everywhere dense subset in $\mathbb{R}^k$. Hence, such points exist in $O_0 \cap \mathbb{R}^k_+$. At the same time, the image of a small neighborhood of a regular point is open.

Proof of Lemma 1. Suppose the contrary, that for a point $x \in M$ every mapping of the form (1) does not possess regular points. Let the maximal rank of the differentials of these mappings be equal to $n < \dim M$, and is attained in some point $(\bar{s}_1, \ldots, \bar{s}_k)$. Since the rank is lower-semicontinuous, this rank is identically equal to n in some neighborhood O of $(\bar{s}_1, \ldots, \bar{s}_k)$. According to the rank theorem, the image of O under the mapping (1) is an n-dimensional submanifold $N \subset M$.

Let $y \in N$, $p \in \mathcal{P}$, then $y \circ p(s) \in N$ for all s sufficiently close to zero. Indeed, the differential of the mapping

$$(s_1, \ldots, s_k, s) \longmapsto x \circ p_1(s_1) \circ \cdots \circ p_k(s_k) \circ p(s), \quad (s_1, \ldots, s_k) \in O,$$

is of rank n for sufficiently small s, hence, its image is an n-dimensional manifold which contains N. Therefore, N is an open set in this manifold.

We shall now show that the fields ω^i_p, $i = 1, 2, \ldots$, are tangent to N: $y \circ \omega^i_p \in T_y N$ for all $y \in N$. Indeed, if we suppose the contrary, then we can find a minimal i such that the relation $y \circ \omega^i_p \in T_y N$ is not valid for all $y \in N$. Hence, this relation is not valid on some open subset of N, and therefore in every point of this open set, we have $y \circ \omega_p(s) \notin T_y N$ for every sufficiently small $s \neq 0$. At the same time, differentiating the relation $y \circ p(s) \in N$ with respect to s, we obtain $y \circ p(s) \circ \omega_p(s) \in T_{y \circ p(s)} N$. Thus, ω^i_p is tangent to the manifold N, $i = 1, 2, \ldots$, $p \in \mathcal{P}$. But in this case, all the fields from $\operatorname{Lie}\{\omega^i_p \mid i > 0,\ p \in \mathcal{P}\} = \Omega$ are tangent to the manifold N, which is contradictory to the Theorem 2.2.

2.

DEFINITION. We say that the point $x \in \mathcal{A}_t(\mathcal{P})$ is *normally attainable* if there exist such points

$$p_i \in \mathcal{P}, \qquad \overline{s}_i > 0, \quad i = 1, \ldots, k, \quad \sum_{i=1}^{k} \overline{s}_i < t,$$

so that

$$x = x_0 \circ p_1(\overline{s}_1) \circ \cdots \circ p_k(\overline{s}_k)$$

and $(\overline{s}_1, \ldots, \overline{s}_k)$ is a regular point of the mapping

$$(s_1, \ldots, s_k) \longmapsto x_0 \circ p_1(s_1) \circ \cdots \circ p_k(s_k).$$

It is easily seen that every normally attainable set is an interior point of $\mathcal{A}_t(\mathcal{P})$. The opposite statement is not so evident, although it is true.

PROPOSITION 1. *Let* $x \in \operatorname{int}(\mathcal{A}_\tau(\mathcal{P}))$ *for some* $\tau < t$. *Then* x *is a normally attainable point of* $\mathcal{A}_t(\mathcal{P})$.

Proof. Put $\varepsilon = t - \tau$. Let O_x be a neighborhood of $x \in M$ which is contained in $\operatorname{int}(\mathcal{A}_{t-\varepsilon}(\mathcal{P}))$. Lemma 1 and the following lemma considerations applied to the family of flows $\mathcal{P}^{-1} = \{s \mapsto p(s)^{-1} \mid p \in \mathcal{P}\}$, imply the existence of

$$y \in O_x, \quad p_i \in \mathcal{P}, \quad \overline{\tau}_i > 0, \quad i = 1, \ldots, l, \qquad \sum_{i=1}^{l} \overline{\tau}_i < \varepsilon,$$

such that $y = x \circ p_l(\overline{\tau}_l) \circ \cdots \circ p_1(\overline{\tau}_1)$ and $(\overline{\tau}_1, \ldots, \overline{\tau}_l)$ is a regular point of the mapping

$$(\tau_1, \ldots, \tau_l) \longmapsto x \circ p_l(\tau_l)^{-1} \circ \cdots \circ p_1(\tau_1)^{-1}. \tag{2}$$

Put

$$\tau = (\tau_1, \ldots, \tau_l), \qquad P(\tau) = p_1(\tau_1) \circ \cdots \circ p_l(\tau_l).$$

We have $x = y \circ P(\overline{\tau})$. Differentiating the identity $P(\tau) \circ P(\tau)^{-1} = \mathrm{id}$ gives

$$\frac{\partial}{\partial \tau_i}\left(P(\tau)^{-1}\right) = -P(\tau) \circ \frac{\partial}{\partial \tau_i}\left(P(\tau)^{-1}\right) \circ P(\tau).$$

From here we deduce that $\overline{\tau}$ is a regular point of the mapping $\tau \mapsto y \circ P(\tau)$. Indeed, we have

$$\begin{aligned}\frac{\partial}{\partial \tau_i} y \circ P(\tau)\Big|_{\tau=\overline{\tau}} &= -y \circ P(\overline{\tau}) \circ \frac{\partial}{\partial \tau_i} P(\tau)^{-1}\Big|_{\tau=\overline{\tau}} \circ P(\overline{\tau}) \\ &= -x \circ \frac{\partial}{\partial \tau_i} P(\tau)^{-1}\Big|_{\tau=\overline{\tau}} \circ P(\overline{\tau}),\end{aligned}$$

and the regularity of the mapping (2) is equivalent to the fact that among the vectors

$$x \circ \frac{\partial}{\partial \tau_i} P(\tau)^{-1}\Big|_{\tau=\overline{\tau}}$$

there are $\dim M$ linearly independent vectors. Since the operator $P(\overline{\tau})$ is invertible, the same number of linearly independent vectors are contained among the vectors

$$\frac{\partial}{\partial \tau_i} y \circ P(\tau)\Big|_{\tau=\overline{\tau}}.$$

Furthermore, since $y \in \mathcal{A}_{t-\varepsilon}(\mathcal{P})$, there exist

$$q_i \in \mathcal{P}, \qquad \overline{s}_i > 0, \quad i = 1, \dots, k, \quad \sum_{i=1}^{k} s_i < t - \varepsilon,$$

such that $y = x_0 \circ q_1(\overline{s}_1) \circ \cdots \circ q_k(\overline{s}_k)$. Put

$$s = (s_1, \dots, s_k), \qquad Q(s) = q_1(s_1) \circ \cdots \circ q_k(s_k),$$

then $x = x_0 \circ Q(\overline{s}) \circ P(\overline{\tau})$, where $(\overline{s}, \overline{\tau})$ is a regular point of the mapping $(s, \tau) \mapsto x_0 \circ Q(s) \circ P(\tau)$.

PROPOSITION 2. *The family of flows $\mathcal{P}$ is locally controllable iff the family $\mathcal{P}^{-1}$ is locally controllable.*

Proof. Suppose $\mathcal{P}^{-1}$ is locally controllable. Then for every $t > 0$, the set $\mathcal{A}_{t/2}(\mathcal{P}^{-1})$ contains a neighborhood of x_0, in which a normally attainable point $x \in \mathcal{A}_{t/2}(\mathcal{P})$ is contained. We have

$$x_0 \circ p_l(\overline{\tau}_l)^{-1} \circ \cdots \circ p_1(\overline{\tau}_1)^{-1} = x = x_0 \circ q_1(\overline{s}_1) \circ \cdots \circ q_k(\overline{s}_k),$$

where

$$p_i, q_j \in \mathcal{P}, \qquad \overline{\tau}_i, \overline{s}_j > 0, \qquad \sum_{i=1}^{l} \tau_i < \frac{t}{2}, \qquad \sum_{j=1}^{k} s_k < \frac{t}{2}.$$

Furthermore, $(\overline{s}_1, \dots, \overline{s}_k)$ is a regular point of the mapping

$$(s_1, \dots, s_k) \longmapsto x_0 \circ q_1(s_1) \circ \cdots \circ q_k(s_k).$$

Hence,

$$x_0 = x_0 \circ q_1(\overline{s}_1) \circ \cdots \circ q_k(\overline{s}_k) \circ p_1(\overline{\tau}_1) \circ \cdots \circ p_l(\overline{\tau}_l),$$

where $(\overline{s}_1, \dots, \overline{s}_k,\ \overline{\tau}_1, \dots, \overline{\tau}_l)$ is a regular point of the mapping

$$(s_1, \dots, s_k, \tau_1, \dots, \tau_l) \longmapsto x_0 \circ q_1(s_1) \circ \cdots \circ q_k(s_k) \circ p_1(\tau_1) \circ \cdots \circ p_l(\tau_l).$$

3. We recall that A denotes the space of all real polynomials without free coefficients. Let

$$A_+ = \left\{ a \in A \,\middle|\, a = \sum_{i=k}^{l} \alpha_i s^i,\ \alpha_k > 0;\ 0 < k \leqslant l \right\}$$

be a subset in A consisting of polynomials with positive first nonzero coefficient. Put

$$\begin{aligned} \mathrm{Sgr}(\mathcal{P}) = \{ s \mapsto p_1(a_1(s)) \circ \cdots \circ p_k(a_k(s)) \mid p_i \in \mathcal{P}, \\ a_i \in A_+,\ i = 1, \ldots, k;\ k > 0 \}, \end{aligned} \tag{3}$$

which is a subsemigroup of the group of flows $\mathrm{Gr}(\mathcal{P})$, cf. (2.1). Evidently, for all $q \in \mathrm{Sgr}(\mathcal{P})$, $t > 0$ there exists $s > 0$ such that $x_0 \circ q(\tau) \in \mathcal{A}_t(\mathcal{P})$ for $0 \leqslant \tau \leqslant s$. It should be clear that the orbits of the semigroup $\mathrm{Sgr}(\mathcal{P})$ permit us to estimate the lower bounds for the attainable sets. For an arbitrary integer $m > 0$, we put

$$\mathrm{Gr}^m_<(\mathcal{P}) = \left\{ q \in \mathrm{Gr}(\mathcal{P}) \,\middle|\, \frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}} \overrightarrow{\ln} q(s) \Big|_{s=0} \in \Omega_{k-1},\ k = 1, \ldots, m \right\}, \tag{4}$$

where, by definition, $\Omega_0 = 0$. The group (4) is a special case of the group Gr^m_W, cf. (2.13), for the filtration $W_k = \Omega_{k-1}$.

THEOREM 2. *For a given positive integer n consider*

$$c_0 \in C^n_{x_0}, \qquad \mathcal{O}^n = c_0 \circ \mathrm{Gr}(\mathcal{P}), \qquad \mathcal{O}^n_+ = c_0 \circ \mathrm{Sgr}(\mathcal{P}).$$

Denote by $\mathrm{ri}\,\mathcal{O}^n_+$ *the set of points in* $\mathcal{O}^n_+$ *which are interior relative to the manifold* $\mathcal{O}^n$. *Then,*

(1) $\mathcal{O}^n_+ \subset \overline{\mathrm{ri}\,\mathcal{O}^n_+}$;

(2) $\forall c \in \mathrm{ri}\,\mathcal{O}^n_+$ *the following inclusion is valid*: $c \circ \mathrm{Gr}^n_<(\mathcal{P}) \subset \mathrm{ri}\,\mathcal{O}^n_+$.

Proof. Assertion (1) is a special case of Theorem 1. Indeed, every flow $q(\cdot)$ can be considered as a diffeomorphism of the space $C^n_{x_0}$, acting according to the rule $c \mapsto c \circ q(\cdot)$. Hence, $\mathcal{O}^n_+$ is the attainable set (for arbitrary time) of the family of flows on $C^n_{x_0}$,

$$\tau \longmapsto p(\tau\, a(\cdot)), \quad p \in \mathcal{P}, \quad a \in A_+, \tag{5}$$

and the manifold $\mathcal{O}^n$ is an orbit of the same family of flows.

(2) Let $c \in \mathrm{ri}\,\mathcal{O}^n_+$, $c' \in c \circ \mathrm{Gr}^n_<(\mathcal{P})$. We have to prove that $c' \in \mathrm{ri}\,\mathcal{O}^n_+$. By induction over n, the problem is reduced to the case $\mathrm{pr}_n c = \mathrm{pr}_n c'$. Hence, according to Proposition 2.2, $c' = c + \xi$ for some $\xi \in x_0 \circ \Omega_{n-1}$. Furthermore, the mapping

$\widehat{c} \mapsto \widehat{c} + \xi$, $\widehat{c} \in \mathcal{O}^n$, is a diffeomorphism of the manifold $\mathcal{O}^n$. Therefore, $\operatorname{ri}\mathcal{O}^n_+ + \xi$ is an open subset in $\mathcal{O}^n$. Thus, it remains to prove that $c + \xi \in \mathcal{O}^n_+$, $\forall c \in \operatorname{ri}\mathcal{O}^n_+$. Proposition 1 (cf. also (5)) implies the existence of $p_1, \dots, p_k \in \mathcal{P}$, $a_1, \dots, a_k \in A_+$, such that $c = c_0 \circ p_1(a_1(\cdot)) \circ \cdots \circ p_k(a_k(\cdot))$, and the vector $(1, \dots, 1) \in \mathbb{R}^k$ is a regular point of the mapping

$$(\tau_1, \dots, \tau_k) \longmapsto c_0 \circ p_1\big(\tau_1 a_1(\cdot)\big) \circ \cdots \circ p_k\big(\tau_k a_k(\cdot)\big)$$

of $\mathbb{R}^k$ into $\mathcal{O}^n$. According to Theorem 2.3, $\operatorname{pr}_n c + x_0 \circ \Omega_{n-1} \subset \operatorname{pr}_n \mathcal{O}^n$. The implicit function theorem implies the existence of smooth functions $\varepsilon \mapsto \tau_i(\varepsilon)$, $i = 1, \dots, k$, defined for sufficiently small ε and satisfying the conditions

$$\tau_i(0) = 1, \quad i = 1, \dots, k;$$

$$\operatorname{pr}_n\big(c_0 \circ p_1\big(\tau_1(\varepsilon) a_1(\cdot)\big) \circ \cdots \circ p_k(\tau_k(\varepsilon) a_k(\cdot))\big) = \operatorname{pr}_n c + \varepsilon\xi.$$

Let $s \mapsto \gamma_0(s)$ be a curve in M such that $J^n\gamma_0(\cdot) = c$. Put

$$\gamma(s) = \gamma_0(s) \circ p_1\big(\tau_1(s) a_1(s)\big) \circ \cdots \circ p_k\big(\tau_k(s) a_k(s)\big).$$

Then $J^n\gamma(\cdot) = c + \xi$. The jet $J^n\gamma(\cdot)$ is unchanged if we substitute in the definition of the curve $\gamma(\cdot)$ the smooth functions $s \mapsto \tau_i(s)$ by their Taylor polynomials of order n. Hence, $J^n\gamma(\cdot) \in \mathcal{O}^n_+$.

4. We recall that in the space of jets $C^n_{x_0}$ we have the singled point — the jet of the constant curve $\gamma(s) \equiv x_0$. The n-jet of the constant curve is denoted by $J^n x_0$, and for the corresponding orbits of the group and the semigroup, in the sequel we shall use the notations

$$J^n x_0 \circ \operatorname{Gr}(\mathcal{P}) = \mathcal{O}^n_0, \qquad J^n x_0 \circ \operatorname{Sgr}(\mathcal{P}) = \mathcal{O}^n_{0+}.$$

Respectively, $\operatorname{ri}\mathcal{O}^n_{0+}$ is the subset in $\mathcal{O}^n_{0+}$, consisting of interior points relative to $\mathcal{O}^n_0$.

PROPOSITION 3. *Let for a given $n > 0$, $x_0 \circ \Omega_n = T_{x_0}M$. Then, for every $t > 0$ and every curve γ in M, which satisfies the condition $J^n\gamma \in \operatorname{ri}\mathcal{O}^n_{0+}$, there exists a $\tau > 0$, such that $\gamma(s) \in \operatorname{int} \mathcal{A}_t(\mathcal{P})$ for all $s \in (0, \tau)$.*

Proof. Proposition 1 implies the existence of $p_1, \dots, p_k \in \mathcal{P}$, and $a_1, \dots, a_k \in A_+$, such that $J^n\gamma = J^n x_0 \circ p_1(a_1(\cdot)) \circ \cdots \circ p_k(a_k(\cdot))$, and the vector $(1, \dots, 1) \in \mathbb{R}^k$ is a regular point of the mapping

$$(\tau_1, \dots, \tau_k) \longmapsto J^n x_0 \circ p_1\big(\tau_1 a_1(\cdot)\big) \circ \cdots \circ p_k\big(\tau_k a_k(\cdot)\big)$$

of $\mathbb{R}^k$ into $\mathcal{O}^n_0$. According to Theorem 2.3, $J^n\gamma + x_0 \circ \Omega_n \subset \mathcal{O}^n_0$. At the same time, $x_0 \circ \Omega_n = T_{x_0}M$. The implicit function theorem implies the existence of smooth functions $\xi \mapsto \tau_i(\xi)$, $i = 1, \ldots, k$, defined for all $\xi \in T_{x_0}M$, sufficiently close to zero, which satisfy the following conditions

$$\tau_i(0) = 1, \quad i = 1, \ldots, k;$$
$$J^n x_0 \circ p_1\big(\tau_1(\xi)a_1(\cdot)\big) \circ \cdots \circ p_k\big(\tau_k(\xi)a_k(\cdot)\big) = J^n\gamma + \xi. \tag{6}$$

Put

$$F(\xi, s) = x_0 \circ p_1\big(\tau_1(\xi)a_1(s)\big) \circ \cdots \circ p_k\big(\tau_k(\xi)a_k(s)\big),$$

where (ξ, s) belongs to a neighborhood of zero in $T_{x_0}M \times \mathbb{R}$. The equality (6) implies that in arbitrary local coordinates in M, the following relation holds:

$$F(\xi, s) - \gamma(s) = s^n\xi + s^{n+1}r(\xi, s),$$

where $r(\cdot, \cdot)$ is a smooth mapping. Hence, for every sufficiently small s, the image of the mapping $\xi \mapsto F(\xi, s)$ contains a neighborhood of the point $\gamma(s)$.

5. For every integer $n > 0$ put

$$E_n(\mathcal{P}) = \big\{\xi \in T_{x_0}M \mid \exists q \in \operatorname{Sgr}(\mathcal{P}),\ x_0 \circ q(s) = x_0 + s^n\xi + o(s^n)\big\}.$$

An equivalent definition:

$$E_n(\mathcal{P}) = \big\{\xi \in T_{x_0}M \mid J^n x_0 + \xi \in \mathcal{O}^n_{0+}\big\}.$$

PROPOSITION 4. *Suppose, $E_n(\mathcal{P}) = T_{x_0}M$ for some $n > 0$. Then $\mathcal{P}$ is locally controllable at x_0.*

Proof. Let $\xi_1, \ldots, \xi_k$ be nonzero vectors in $T_{x_0}M$, such that

$$\sum_{i=1}^{k} \xi_i = 0, \qquad \operatorname{span}\{\xi_1, \ldots, \xi_k\} = T_{x_0}M.$$

There exist $q_1, \ldots, q_k \in \operatorname{Sgr}(\mathcal{P})$, such that

$$J^n(x_0 \circ q_i) = J^n x_0 + \xi_i, \quad i = 1, \ldots, k.$$

For arbitrary $s \in \mathbb{R}$, $\tau_i > 0$, $i = 1, \ldots, k$, we put

$$F(\tau_1, \ldots, \tau_k; s) = x_0 \circ q_1(\tau_1^{1/n}s) \circ \cdots \circ q_k(\tau_k^{1/n}s).$$

Then,

$$F(\tau_1, \ldots, \tau_k; s) = x_0 + s^n \sum_{i=1}^{k} \tau_i\xi_i + s^{n+1}r(\tau_1, \ldots, \tau_k; s),$$

where r is a smooth mapping. Hence, there exists a neighborhood of the point $(1, \ldots, 1)$ in $\mathbb{R}^k$ such that its image under the mapping

$$(\tau_1, \ldots, \tau_k) \longmapsto F(\tau_1, \ldots, \tau_k; s)$$

contains a neighborhood of x_0 for all s, sufficiently close to zero.

PROPOSITION 5. *For every integer $n > 0$, the set $E_n(\mathcal{P})$ is a convex co* $T_{x_0}M$. *If $E_k(\mathcal{P}) = -E_k(\mathcal{P})$, $\forall k < n$, then*

$$\overline{E_n(\mathcal{P})} \supset x_0 \circ \Omega_{n-1}, \qquad \operatorname{span} E_n(\mathcal{P}) = x_0 \circ \Omega_n.$$

Proof. If

$$x_0 \circ q(s) = x_0 + s^n \xi + o(s^n),$$

then

$$x_0 \circ q(\alpha s) = x_0 + s^n \alpha^n \xi + o(s^n) \quad \forall \alpha \geqslant 0;$$

if

$$x_0 \circ q_i(s) = x_0 + s^n \xi_i + o(s^n), \quad i = 1, 2,$$

then

$$x_0 \circ q_1(s) \circ q_2(s) = x_0 + s^n(\xi_1 + \xi_2) + o(s^n).$$

Hence, $E_n(\mathcal{P})$ is a convex cone in $T_{x_0}M$.

Let $E_k(\mathcal{P}) = -E_k(\mathcal{P})$ for $k < n$. We shall prove the existence of a curve γ_n i such that

$$J^n \gamma_n \in \operatorname{ri} \mathcal{O}^n_{0+}, \qquad J^{n-1}\gamma_n = J^{n-1}x_0.$$

If such a curve exists, then the inclusion $\overline{E_n(\mathcal{P})} \supset x_0 \circ \Omega_{n-1}$ follows fror Theorem 2 and Proposition 2.2, if applied to the filtration $W_k = \Omega_{k-1}$, $k = 1,$ and the equality $\operatorname{span} E_n(\mathcal{P}) = x_0 \circ \Omega_n$ — from the Theorem 2.3.

Theorem 2 guarantees that $\operatorname{ri} \mathcal{O}^n_{0+}$ is not empty. Let the curve γ_1 satisf relations

$$J^n \gamma_1 \in \operatorname{ri} \mathcal{O}^n_{0+}, \qquad J^1\gamma_1 = J^1 x_0 + \xi_1.$$

Then $\xi_1 \in E_1(\mathcal{P})$. Since $E_1(\mathcal{P}) = -E_1(\mathcal{P})$, we can find $q_1 \in \operatorname{Sgr}(\mathcal{P})$, suc $J^1(x_0 \circ q_1) = J^1 x_0 - \xi_1$. Put $\gamma_2 = \gamma_1 \circ q_1$. Then

$$J^n \gamma_2 \in \operatorname{ri} \mathcal{O}^n_{0+}, \qquad J^1\gamma_2 = J^1_{x_0}, \qquad J^2\gamma_0 = J^2 x_0 + \xi_2.$$

Hence, $\xi_2 \in E_2(\mathcal{P})$. If $E_2(\mathcal{P}) = -E_2(\mathcal{P})$, then we can find $q_2 \in \operatorname{Sgr}(\mathcal{P})$, suc $J^2(x_0 \circ q_2) = J^1_{x_0} - \xi_2$. Put $\gamma_3 = \gamma_2 \circ q_2$, etc.

COROLLARY. *Let for a given $n > 0$ $x_0 \circ \Omega_n = T_{x_0}M$. If $E_k(\mathcal{P}) = -E_k($ $k = 1, \dots, n$, then $x_0 \in \operatorname{int} \mathcal{A}_t(\mathcal{P})$ $\forall t > 0$, hence, $\mathcal{P}$ is locally controllable at x_0*

Proof. Proposition 5 implies that $E_n(\mathcal{P}) = x_0 \circ \Omega_n = T_{x_0}M$. Hence, Proposit implies the local controllability of $\mathcal{P}$.

6. For every integer $m > 0$ denote by $\mathcal{V}^m$ the Lie algebra consisting of polynomials in $s \in \mathbb{R}$ of the form

$$v(s) = \sum_{k=1}^{m} s^k v_k, \quad v_k \in \Omega_k, \quad k = 1, \dots, m,$$

with the Lie multiplication

$$[v, w](s) = \sum_{k=1}^{m} s^k \sum_{i=1}^{k-1} [v_i, w_{k-i}] \quad \forall v, w \in \mathcal{V}^m.$$

The following proposition is easily obtained from Theorem 1.1 and formula (2.6).

PROPOSITION 6. *The space $\mathcal{V}^m$ coincides with the set of Taylor polynomials of degree m of vector functions of the form*

$$s \longmapsto \int_0^s \overrightarrow{\ln}\, q(\tau)\, \mathrm{d}\tau, \quad q \in \mathrm{Gr}(\mathcal{P}).$$

For every analytic curve $s \mapsto v(s) \in \operatorname{Vect} M$, $v(0) = 0$, denote by $\overrightarrow{\exp}_m(v)$ the Taylor polynomial of degree m of the mapping

$$s \longmapsto \overrightarrow{\exp} \int_0^s \dot{v}(\tau)\, \mathrm{d}\tau \approx \mathrm{id} + \sum_{k=1}^{m} \int \dots \int_{\Delta_s^k} \dot{v}(\tau_k) \circ \dots \circ \dot{v}(\tau_1)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_k,$$

where $\dot{v} = \mathrm{d}v/\mathrm{d}s$, cf. (2.3). Thus, $\overrightarrow{\exp}_m(v)$ is a polynomial of degree m in s, which has differential operators on M as coefficients. Let $v, v' \in \mathcal{V}^m$ and

$$\overrightarrow{\exp}_m(v) = \mathrm{id} + \sum_{k=1}^{m} s^k D_k, \qquad \overrightarrow{\exp}_m(v') = \mathrm{id} + \sum_{k=1}^{m} s^k D'_k.$$

It is easily seen that the multiplication

$$\left(\overrightarrow{\exp}_m(v), \overrightarrow{\exp}_m(v')\right) \longmapsto \mathrm{id} + \sum_{k=1}^{m} s^k \sum_{i=1}^{k-1} D_i \circ D'_{k-i}$$

defines on the set $\overrightarrow{\exp}_m(\mathcal{V})$ the structure of a connected Lie group, with the Lie algebra $\mathcal{V}^m$. It is easy to observe that the restriction $\overrightarrow{\exp}_m \mid \mathcal{V}^m$ is a diffeomorphism of $\mathcal{V}^m$ on the Lie group $\overrightarrow{\exp}_m(\mathcal{V}^m)$.

Finally, denote by $\vec{\mathcal{V}}_+^m$ the set of Taylor polynomials of degree m of vector functions of the form

$$s \longmapsto \int_0^s \overrightarrow{\ln}\, q(\tau)\, \mathrm{d}\tau, \quad q \in \mathrm{Sgr}(\mathcal{P}).$$

Thus, $\vec{\mathcal{V}}_+^m \subset \mathcal{V}^m$, and $\overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m)$ is a subsemigroup of the Lie group $\overrightarrow{\exp}_m(\mathcal{V}^m)$, which generates this group.

THEOREM 3. *For every $m > 0$ the subset $\vec{\mathcal{V}}_+^m$ has interior points in the space $\mathcal{V}^m$ and, even more, is contained in the closure of its interior:* $\vec{\mathcal{V}}_+^m \subset \overline{\operatorname{int} \vec{\mathcal{V}}_+^m}$.

Proof. The Lie group $\overrightarrow{\exp}_m(\mathcal{V}^m)$ acts transitively on itself by left translations. This standard action turns the semigroup $\overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m)$ into the semigroup of diffeomorphisms of the manifold $\overrightarrow{\exp}_m(\mathcal{V}^m)$, and the orbit of this semigroup of diffeomorphisms through the point id coincides with the initial semigroup. Every 'diffeomorphism' $\overrightarrow{\exp}_m(v)$, $v \in \vec{\mathcal{V}}_+^m$, is imbedded into the 'flow' $\tau \mapsto \overrightarrow{\exp}_m(v_\tau)$, where $v_\tau(s) = v(\tau s)$. Now we can apply to this family of 'flows' Theorem 1. All 'attainable sets' from id coincide with $\overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m)$. Hence

$$\overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m) \subset \overline{\operatorname{int}(\overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m))}.$$

But in this case we have $\vec{\mathcal{V}}_+^m \subset \overline{\operatorname{int} \vec{\mathcal{V}}_+^m}$.

Suppose for $n > 0$ we have $E_k(\mathcal{P}) = -E_k(\mathcal{P})$ for $k = 1, \ldots, n$. For every integer $m > 0$ put

$$\mathcal{V}_0^m = \left\{ \left(\sum_{k=1}^m s^k v_k \right) \in \mathcal{V}^m \mid x_0 \circ v_i = 0 \quad \text{for } 1 \leqslant i \leqslant \min(m, n) \right\},$$

which is a linear subspace in $\mathcal{V}^m$.

PROPOSITION 7. *For every $m > 0$ the relation*

$$\mathcal{V}_0^m \cap \operatorname{int} \vec{\mathcal{V}}_+^m \neq \varnothing$$

holds.

Proof. Let $q \in \operatorname{Gr}(\mathcal{P})$, and suppose

$$T^m q = \mathrm{id} + \sum_{i=1}^m \frac{s^i}{i!} \frac{\mathrm{d}^i}{\mathrm{d}s^i} q \Big|_{s=0} \tag{7}$$

is the Taylor polynomial of degree m. It is easily seen that the mapping $q \mapsto T^m q$, $q \in \operatorname{Gr}(\mathcal{P})$, is a homomorphism of $\operatorname{Gr}(\mathcal{P})$ onto $\overrightarrow{\exp}_m(\mathcal{V}^m)$, where $\operatorname{Sgr}(\mathcal{P})$ is mapped onto $\overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m)$.

Let $q_0 \in \operatorname{Sgr}(\mathcal{P})$ be such that $T^m q_1 \in \operatorname{int} \overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m)$. Suppose that

$$J^1(x_0 \circ q_1) = J^1 x_0 + \xi_1, \quad \xi_1 \in T_{x_0} M.$$

Then $\xi_1 \in E_1$. Since $E_1 = -E_1$, we can find $p_1 \in \operatorname{Sgr}(\mathcal{P})$ such that

$$J^1(x_0 \circ p_1) = J^1 x_0 - \xi_1.$$

Put $q_2 = q_1 \circ p_1$, then $J^1(x_0 \circ q_2) = J^1 x_0$. Furthermore,

$$T^m q_2 = (T^m q_1) \circ (T^m p_1) \in \operatorname{int} \overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m).$$

Suppose

$$J^2(x_0 \circ q_2) = J^2 x_0 + \xi_2, \qquad \xi_2 \in T_{x_0} M.$$

Then $\xi_2 \in E_2$. If $2 \leqslant n$, then we can find $p_2 \in \mathrm{Sgr}(\mathcal{P})$ such that

$$J^2(x_0 \circ p_2) = J^2 x_0 - \xi_2.$$

Put $q_3 = q_2 \circ p_2$, then $J^2(x_0 \circ q_3) = J^2 x_0$. Furthermore,

$$T^m q_3 = (T^m q_2) \circ (T^m p_2) \in \operatorname{int} \overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m).$$

Continuing this process, we can find $q_n \in \mathrm{Sgr}(\mathcal{P})$ such that

$$J^n(x_0 \circ q_n) = J^n x_0, \qquad T^m q_n \in \operatorname{int} \overrightarrow{\exp}_m(\vec{\mathcal{V}}_+^m).$$

Since $\overrightarrow{\exp}_m \mid \mathcal{V}^m$ is a diffeomorphism, there exists a unique

$$v \in \operatorname{int} \vec{\mathcal{V}}_+^m, \qquad v = \sum_{i=1}^m s^i v_i,$$

which satisfies the condition $\overrightarrow{\exp}_m(v) = T^m q_n$. The jet $J^m(x_0 \circ q_n)$ depends only on $T^m q_n$. Since $\mathrm{ord}\,(x_0 \circ q_n) > n$, we obtain $x_0 \circ v_i = 0$ for $1 \leqslant i \leqslant \min(m, n)$.

4. Obstructions to Local Controllability

1. Let $p \in \mathcal{P}$ and let $p(s) \approx \mathrm{id} + \sum_{n=1}^\infty s^n D_p^n$ be the Taylor series expansions of the mapping $s \mapsto p(s)$. Then, for every $n > 0$, D_p^n is a differential operator on M of degree $\leqslant n$, and $D_p^n - (1/n)\omega_p^n$ is a polynomial in $\omega_p^1, \ldots, \omega_p^{n-1}$ in the associative algebra of differential operators on M, cf. (2.3). We introduce the formal power series

$$\Lambda_p(s) = \sum_{n=1}^\infty s^n \Lambda_p^n = \ln\left(\mathrm{id} + \sum_{n=1}^\infty s^n D_p^n\right) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\left(\sum_{i=1}^\infty s^i D_p^i\right)^n.$$

PROPOSITION 1. *For arbitrary $p \in \mathcal{P}$, $n > 0$, the relation*

$$\Lambda_p^n = \frac{1}{n}\omega_p^n + \sum_{k=1}^{n-1} \sum_{|i|=n} \alpha_{k,i}^n [\omega_p^{i_k}, [\ldots, [\omega_p^{i_1}, \omega_p^{i_0}] \ldots]$$

holds, where $i = (i_0, \ldots, i_k)$ is a multiindex, $|i| = i_1 + \ldots + i_k$, $\alpha_{k,i}^n$ are rational numbers (constants not depending on p).

Proof. We have, cf. (2.3),

$$\Lambda_p(s) = \ln\left(\mathrm{id} + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_s^n} \omega_p(\tau_n) \circ \cdots \circ \omega_p(\tau_1)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n\right)$$

$$= \int_0^s \omega_p(\tau)\, \mathrm{d}\tau + \sum_{n=2}^{\infty} \int \cdots \int_{\Delta_s^n} \pi_n\big(\omega_p(\tau_n), \ldots, \omega_p(\tau_1)\big) \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n,$$

$$\pi_n(z_n, \ldots, z_1) = \sum_{\rho} \frac{(-1)^{d(\rho)}}{n^2 \binom{n-1}{d(\rho)}} [z_{\rho(n)}, [\ldots, [z_{\rho(2)}, z_{\rho(1)}] \ldots],$$

where the summation is over all permutations

$$\rho\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}, \qquad d(\rho) = \#\big\{1 \leqslant i < n \mid \rho(i) > \rho(i+1)\big\}.$$

The proof that the π_n are commutator polynomials, with explicit expressions different from those given here, are found in [1]. The above formula is proved in [5].

For every flow $q(s) \approx \mathrm{id} + \sum_{i=1}^{\infty} s^i D_i$ and every integer $m > 0$ we put

$$\ln q = \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\sum_{i=1}^{\infty} s^i D_i\right)^n = T^m \ln q + O(s^{m+1}).$$

Thus, $T^m \ln q$ is a segment of the power series for $\ln q$ of the length m.

COROLLARY (to Propositions 1 and 3.6).

(1) *For every $m > 0$ the relation*

$$\{T^m \ln q \mid q \in \mathrm{Gr}\,(\mathcal{P})\} = \left\{\sum_{i=1}^{m} s^k v_k \mid v_k \in \Omega_k\right\} = \mathcal{V}_m$$

holds.

(2) *For every flow q the first nonzero terms of the series in the powers of s for the expressions* $\ln q(s)$ *and* $\int_0^s \overrightarrow{\ln} q(\tau)\, \mathrm{d}\tau$ *have the same order and are equal.*

2. Denote by $\mathcal{L}$ the free Lie algebra over $\mathbb{R}$ with the set of free generators λ_p^i, $p \in \mathcal{P}$, $i = 1, 2, \ldots$. We emphasize that λ_p^i are not vector fields but indexed free generators, hence

$$\mathcal{L} = \mathrm{Lie}\,\big\{\lambda_p^i \mid p \in \mathcal{P},\ i = 1, 2, \ldots\big\}.$$

For arbitrary integers $n \geqslant m > 0$, we put

$$\mathcal{L}(n, m) = \mathrm{span}\left\{[\lambda_{p_m}^{i_m}, [\ldots, [\lambda_{p_2}^{i_2}, \lambda_{p_1}^{i_1}] \ldots] \,\middle|\, p_j \in \mathcal{P},\ \sum_{j=1}^{m} i_j = n\right\}.$$

Then $\mathcal{L} = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \mathcal{L}(n,m)$, and the subspaces $\mathcal{L}(n,m)$ define a bigrading of the Lie algebra $\mathcal{L}$: if $\vartheta_i \in \mathcal{L}(n_i, m_i)$, $i = 1,2$, then $[\vartheta_1, \vartheta_2] \in \mathcal{L}(n_1+n_2, m_1+m_2)$.

Let

$$\mathfrak{A} = \mathrm{Ass}\{\lambda_p^i \mid p \in \mathcal{P},\ i = 1,2\dots\}$$

be a free associative algebra with the set of free generators λ_p^i, $p \in \mathcal{P}$, $i = 1,2,\dots$. In the sequel, without additionally mentioning it we shall suppose that $\mathcal{L}$ is realized in the standard way, as a subspace in $\mathfrak{A}$ consisting of all commutator polynomials in λ_p^i with the commutator product as the Lie product: $[\vartheta_1, \vartheta_2] = \vartheta_1\vartheta_2 - \vartheta_2\vartheta_1$.

Denote by $\mathrm{Aut}(\mathcal{P})$ the group of all bijections of the set $\mathcal{P}$. We have a natural left action of the group $\mathrm{Aut}(\mathcal{P})$ on the set $\mathfrak{A}$. This action identifies every $\sigma \in \mathrm{Aut}(\mathcal{P})$ with the automorphism $\vartheta \mapsto \sigma\vartheta$ of $\mathfrak{A}$ where

$$\sigma(\lambda_{p_1}^{i_1} \dots \lambda_{p_k}^{i_k}) = \lambda_{\sigma(p_1)}^{i_1} \dots \lambda_{\sigma(p_k)}^{i_k}$$

for every monomial $\lambda_{p_1}^{i_1} \dots \lambda_{p_k}^{i_k} \in \mathfrak{A}$. It is easily seen that the action of $\mathrm{Aut}(\mathcal{P})$ preserves the spaces $\mathcal{L}(n,m)$, in other words,

$$\sigma\vartheta \in \mathcal{L}(n,m) \quad \forall \sigma \in \mathrm{Aut}(\mathcal{P}), \quad \vartheta \in \mathcal{L}(n,m), \quad n \geqslant m > 0.$$

Finally, let $J\colon \mathfrak{A} \longrightarrow \mathfrak{A}$ be an antiautomorphism of the algebra $\mathfrak{A}$, acting according to the rule $J(\lambda_{p_1}^{i_1} \dots \lambda_{p_k}^{i_k}) = \lambda_{p_k}^{i_k} \dots \lambda_{p_1}^{i_1}$. It is easily seen that $J\sigma = \sigma J$, $\forall \sigma \in \mathrm{Aut}(\mathcal{P})$.

PROPOSITION 2. *For arbitrary $n \geqslant m > 0$, $\vartheta \in \mathcal{L}(n,m)$ the relation*

$$J\vartheta = (-1)^{m-1}\vartheta \tag{1}$$

holds.

The relation (1) seems to be well known. It is an easy consequence of general symmetry relations for Lie polynomials, which are given in [3, 5]. But the proof of this special case is much easier, and we give here a sketch of the proof.

Proof. Consider the group generated by formal series

$$e^{\alpha\lambda_p^i} = 1 + \sum_{n=1}^{\infty} \frac{(\alpha\lambda_p^i)^k}{n!}, \quad \alpha \in \mathbb{R}, \quad p \in \mathcal{P}, \quad i = 1,2,\dots,$$

with the usual multiplication.

Let $\vartheta \in \mathcal{L}(n,m)$, then there exist $\alpha_j \in \mathbb{R}$, $\lambda_{p_j}^{i_j}$, $j = 1,\dots,k$, such that

$$e^{\tau\alpha_1\lambda_{p_1}^{i_1}} \dots e^{\tau\alpha_k\lambda_{p_k}^{i_k}} = 1 + \tau^m\vartheta + O(\tau^{m+1}). \tag{2}$$

The proof proceeds in the standard way used in Lie theory, which permits us to realize the commutators in the Lie algebra as the leading terms of commutators in the group. Applying the involution J to the relation (2), we obtain

$$e^{\tau\alpha_k\lambda_{p_k}^{i_k}}\dots e^{\tau\alpha_1\lambda_{p_1}^{i_1}} = 1 + \tau^m J\vartheta + O(\tau^{m+1}),$$

$$\left(e^{-\tau\alpha_k\lambda_{p_k}^{i_k}}\dots e^{-\tau\alpha_1\lambda_{p_1}^{i_1}}\right)\left(e^{\tau\alpha_1\lambda_{p_1}^{i_1}}\dots e^{\tau\alpha_k\lambda_{p_k}^{i_k}}\right) = 1.$$

Hence $\tau^m\vartheta + (-\tau)^m J\vartheta = 0$.

3. Let $\Lambda\colon \mathcal{L} \to \Omega$ be a Lie algebra homomorphism, defined by the relations $\Lambda(\lambda_p^i) = \Lambda_p^i$, $p \in \mathcal{P}$, $i = 1, 2, \dots$. Proposition 1 implies:

$$\Lambda\left(\sum_{i=1}^{n}\sum_{j=1}^{i}\mathcal{L}(i,j)\right) = \Omega_n, \quad \forall n > 0. \tag{3}$$

Let $\vartheta \in \mathcal{L}$, the value of the field $\Lambda(\vartheta)$ at x_0 will be denoted by ϑ_{x_0}; thus, $\vartheta_{x_0} = x_0 \circ \Lambda(\vartheta)$.

It turns out that the mapping $x_0 \circ \Lambda\colon \vartheta \mapsto \vartheta_{x_0}$ defines the family $\mathcal{P}$ up to the change of variables in M. Let $\mathcal{P}'$ be a family of germs of analytic flows at $x_0' \in M$, $\ln p' = \sum_{n=1}^{\infty} s^n \Lambda_{p'}^n$, $\forall p' \in \mathcal{P}'$;

$$\Lambda'\colon \mathcal{L}' \longrightarrow \operatorname{Lie}\left\{\Lambda_{p'}^n \mid p \in \mathcal{P}',\, n = 1, 2, \dots\right\}$$

be a homomorphism of the free Lie algebra

$$\mathcal{L}' = \operatorname{Lie}\left\{\lambda_{p'}^n \mid p' \in \mathcal{P}',\ n = 1, 2, \dots\right\},$$

defined by the relations $\Lambda'(\lambda_{p'}^n) = \Lambda_{p'}^n$.

PROPOSITION 3. *Let $\psi\colon \mathcal{P} \leftrightarrow \mathcal{P}'$ be a bijection, $\psi_*\colon \mathcal{L} \to \mathcal{L}'$ be a monomorphism of Lie algebras defined by the relations $\psi_*(\lambda_p^n) = \lambda_{\psi(p)}^n$, $p \in \mathcal{P}$, $n = 1, 2, \dots$. The following two assertions are equivalent.*

(1) *There exists a diffeomorphism $\Psi\colon O_{x_0} \to O'_{x_0'}$, defined on a neighborhood O_{x_0} of x_0, such that $\Psi(x_0) = x_0'$ and $\Psi \circ p \circ \Psi^{-1} = \psi(p)$, $\forall p \in \mathcal{P}$.*

(2) $\ker(x_0 \circ \Lambda) = \ker(x_0' \circ \Lambda' \circ \psi_*)$.

Proof. The implication (1) $\Longrightarrow$ (2) easily follows from the definitions. To prove the opposite implication, consider the family of germs of flows $(p, \psi(p))$, $p \in \mathcal{P}$, of the manifold $M \times M$ at the point (x_0, x_0'). Theorem 2.2 and Proposition 1 imply that the orbit $\underline{\mathcal{O}}$ of this family through (x_0, x_0') is a submanifold in $M \times M$ and

$$T_{(x_0,x_0')}\underline{\mathcal{O}} = \{(x_0 \circ \Lambda(\vartheta),\ x_0' \circ \Lambda' \circ \psi_*(\vartheta) \mid \vartheta \in \mathcal{L}\} \subset T_{x_0}M \times T_{x_0}M.$$

The assertion (2) implies that the space $T_{(x_0,x_0')}\underline{\mathcal{O}}$ is $\dim M$-dimensional and is projected without singularities on both factors of $M \times M$. Hence some coordinate neighborhood of (x_0, x_0') in $\underline{\mathcal{O}}$ is a graph of the diffeomorphism $\Psi\colon \mathcal{O}_{x_0} \longrightarrow \mathcal{O}'_{x_0'}$. It is easy to show that for the diffeomorphism Ψ, the assertion (2) is valid.

We have proved Proposition 3 which generalizes Proposition 1.1 of the Introduction.

THEOREM 1. *Let Σ be a finite subgroup in* $\mathrm{Aut}(\mathcal{P})$ *and $n > 0$. Suppose that for every k, l subject to the relation $2l+1 \leqslant k \leqslant n$, and every $\vartheta \in \mathcal{L}(k, 2l+1)$, for which $\sigma\vartheta = \vartheta$, $\forall \sigma \in \Sigma$, the inclusion $\vartheta_{x_0} \in x_0 \circ \Omega_{k-1}$ holds. Then $x_0 \circ \Omega_n = E_n(\mathcal{P})$.*

The proof is based on two propositions which are also of independent interest.

PROPOSITION 4. *Suppose there exist $q \in \mathrm{Sgr}(\mathcal{P})$, $T^n \ln q = \sum_{k=1}^n s^k v_k$, such that*

$$J^n(x_0 \circ q) \in \mathrm{ri}\, \mathcal{O}^n_{0+}, \qquad x_0 \circ v_k \in x_0 \circ \Omega_{k-1}, \quad k = 1, \dots, n.^\star$$

Then $J^n x_0 \in \mathrm{ri}\, \mathcal{O}^n_{0+}$ and $E_n(\mathcal{P}) = x_0 \circ \Omega_n$.

Proof. We have $x_0 \circ v_1 = 0$, hence

$$J^2(x_0 \circ q) = J^2 x_0 + x_0 \circ v_2.$$

Since $x_0 \circ v_2 \in x_0 \circ \Omega_1$, there exists $p_2 \in \mathrm{Gr}^n_{<}(\mathcal{P})^{\star\star}$ such that $\mathrm{ord}\, p_2 = 2$ and $x_0 \circ T_0 p_2 = -x_0 \circ v_2$.

Put $q_2 = q \circ p_2$. According to the Theorem 3.2,

$$J^n(x_0 \circ q_2) \in \mathrm{ri}\, \mathcal{O}^n_{0+} \quad \text{and} \quad J^2(x_0 \circ q_2) = J^2 x_0.$$

Let $T^n \ln q_2 = \sum_{k=1}^n s^k v_k^2$. We have $\ln q_2 = \ln(e^{\ln q} \circ e^{\ln p_2})$. Proposition 1 and the Campbell–Hausdorff formula imply

$$(v_k^2 - v_k) \in \Omega_{k-1}, \quad k = 1, \dots, n.$$

Thus

$$x_0 \circ v_k^2 \in x_0 \circ \Omega_{k-1}, \quad \forall k \leqslant n, \quad J^3(x_0 \circ q_2) = J^3 x_0 + x_0 \circ v_3^2.$$

There exists $p_3 \in \mathrm{Gr}^n_{<}(\mathcal{P})$ such that $\mathrm{ord}\, p_3 = 3$ and $x_0 \circ T_0 p_3 = -x_0 \circ v_3^2$. Put

$$q_3 = q_2 \circ p_3, \qquad T^n \ln q_3 = \sum_{k=1}^n s^k v_k^3.$$

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* The set $\mathrm{ri}\, \mathcal{O}^n_{0+}$ is defined in No. 4, §3.

** The definition of the group $\mathrm{Gr}^n_{<}(\mathcal{P})$ is given by the formula (3.4)

Then $J^4(x_0 \circ q_3) = J^4 x_0 + x_0 \circ v_4^3$. Proceeding in the described way, we finally get to a flow q_n such that

$$J^n(x_0 \circ q_n) \in \operatorname{ri} O_{0+}^n, \qquad J^n(x_0 \circ q_n) = J^n x_0.$$

Hence

$$E_n(\mathcal{P}) = \left\{\xi \in T_{x_0}M \mid J^n x_0 + \xi \in \mathcal{O}_0^n\right\} = x_0 \circ \Omega_n.$$

PROPOSITION 5 (cf. [23]). *Let L be a Lie algebra, G be a finite group of linear transformations of L. Let $\#G = m+1$ and $l = (m+1)^n - 1$, $n > 0$. Then there exists a sequence $g_1, \dots, g_l \in G$ such that $\forall z \in L$ the polynomial $T^n \ln\left(e^{sz} e^{sg_1(z)} \dots e^{sg_l(z)}\right)$ is invariant under the group G. In this expression, the exponent*

$$e^{sz} = 1 + \sum_{n=1}^{\infty} \frac{s^n}{n!} z^n$$

is a power series with coefficients in universal enveloping algebra UL, and $T^n \ln(\cdot)$ denotes the segment of the length n of the power series in $\ln(\cdot)$.

Proof. Let $g_1, \dots, g_m$ be all nonunit elements of G, g_0 be the unit. Then

$$\ln\left(e^{sz} e^{sg_1(z)} \dots e^{sg_m(z)}\right) = s \sum_{i=0}^{m} g_i(z) + O(s^2).$$

Evidently, the element $\sum_{i=0}^{m} g_i(z)$ is invariant under G. Further, we use induction over n.

Suppose that

$$\ln\left(e^{sz} e^{sg_1(z)} \dots e^{sg_l(z)}\right) = \sum_{i=1}^{n+1} s^i z_i + O(s^{n+2}) = Z,$$

where the elements $z_1, \dots, z_n$ are invariant under G. Then

$$\ln\left(e^{Z} e^{g_1(Z)} \dots e^{g_m(Z)}\right) = (m+1) \sum_{i=1}^{n} s^i z_i + s^n \sum_{j=0}^{m} g_j(z_{n+1}) + O(s^{n+2}),$$

where r_i are commutator polynomials in $z_1, \dots, z_{i-1}$. Since such polynomials of invariant elements are themselves invariant, to make the induction step it is sufficient to extend the sequence $g_1, \dots, g_l$ by the sequence

$$g_1 g_1, \dots, g_1 g_l, g_2 g_1, \dots, g_2 g_l, \dots, g_m g_1, \dots, g_m g_l.$$

Now we go to the proof of Theorem 1. Let $q \in \operatorname{Sgr}(\mathcal{P})$ satisfy the the condition $J^n(x_0 \circ q) \in \operatorname{ri} \mathcal{O}_{0+}^n$. The flow q, as an element of the semigroup $\operatorname{Sgr}(\mathcal{P})$, is represented as, cf. (3.3),

$$q(s) = p_1(a_1(s)) \circ \dots \circ p_k(a_k(s)), \quad p_i \in \mathcal{P}, \quad a_i \in A_+, \quad i = 1, \dots, k.$$

For every $\sigma \in \Sigma \subset \mathrm{Aut}(\mathcal{P})$ put

$$q_\sigma(s) = \sigma(p_1)\big(a_1(s)\big) \circ \ldots \circ \sigma(p_k)\big(a_k(s)\big).$$

Let

$$\zeta(s) = \ln\left(e^{\lambda_{p_1}(a_1(s))} \ldots e^{\lambda_{p_k}(a_k(s))}\right),$$

where

$$\lambda_p(\tau) = \sum_{i=1}^{\infty} \tau^i \lambda_p^i, \quad p \in \mathcal{P}.$$

Proposition 5 implies the existence of a sequence $\sigma_1, \ldots, \sigma_l \in \Sigma$ such that the polynomial

$$T^n \ln\left(e^{\zeta(s)} e^{\sigma_1(\zeta(s))} \ldots e^{\sigma_l(\zeta(s))}\right) = \sum_{i=1}^{n} s^i \vartheta_i, \quad \vartheta_i \in \sum_{m=1}^{i} \mathcal{L}(i,m),$$

is invariant under the group Σ. Put $q^a = q \circ q_{\sigma_1} \circ \cdots \circ q_{\sigma_l}$, then

$$J^n(x_0 \circ q^a) \in \mathrm{ri}\, O^n_{0+} \quad \text{and} \quad \sum_{i=1}^{n} s^i \Lambda(\vartheta_i) = T^n \ln q^a.$$

If $\vartheta_i \in \sum_j \mathcal{L}(i, 2j+1)$, $i = 1, \ldots, n$, then, under the hypothesis of the theorem, we have

$$x_0 \circ \Lambda(\vartheta_i) = (\vartheta_i)_{x_0} \in x_0 \circ \Omega_{i-1}, \quad i = 1, \ldots, n,$$

and the assertion of the theorem follows from Proposition 4. Otherwise, we need some additional work. We shall use the inclusion, cf. Proposition 2, $(\vartheta_i + J\vartheta_i) \in \sum_j \mathcal{L}(i, 2j+1)$ for all i.

LEMMA 1. *Let*

$$\Omega[s] = \left\{ \sum_{i=1}^{k} s^i v_i \mid v_i \in \Omega_i,\ i = 1, \ldots, k;\ k > 0 \right\}$$

be a Lie algebra over the ring of polynomials in s, *and let*

$$\Omega_<[s] = \left\{ \sum_{i=1}^{k} s^i v_i \mid v_i \in \Omega_i,\ x_0 \circ v_i \in x_0 \circ \Omega_{i-1},\ i = 1, \ldots, k;\ k > 0 \right\}.$$

Then the subspace $\Omega_<[s]$ *is a Lie subalgebra in* $\Omega[s]$.

Proof. Let

$$v \in \Omega_i, \quad x_0 \circ v \in x_0 \circ \Omega_{i-1}, \qquad w \in \Omega_j, \quad x_0 \circ w \in x_0 \circ \Omega_{j-1}.$$

Then

$$v = v^0 + v^-, \qquad w = w^0 + w^-,$$

where

$$x_0 \circ v^0 = x_0 \circ w^0 = 0, \quad v^- \in \Omega_{i-1}, \quad w^- \in \Omega_{j-1}.$$

We have

$$x_0 \circ [v, w] = x_0 \circ [v^0, w^-] + x_0 \circ [v^-, w].$$

At the same time $[v^0, w^-], [v^-, w] \in \Omega_{i+j-1}$, since $0 \subset \Omega_1 \subset \Omega_2 \subset \dots$ is an increasing filtration of the Lie algebra Ω.

For every $\sigma \in \Sigma$ put

$$\overline{q}_\sigma(s) = \sigma(p_k)(a_k(s)) \circ \cdots \circ \sigma(p_1)(a_1(s)),$$

and let $\overline{q}^a = \overline{q}_{\sigma_l} \circ \cdots \circ \overline{q}_{\sigma_1}$. We recall that

$$\Lambda\left(\sum_{i=1}^{n} s^i \vartheta_i\right) = T^n \ln q^a.$$

It is easily seen that

$$\Lambda\left(\sum_{i=1}^{n} s^i J\vartheta_i\right) = T^n \ln \overline{q}^a.$$

Hence

$$\Lambda\left(T^n \ln\left(e^{\Sigma s^i \vartheta_i} e^{\Sigma s^i J\vartheta_i}\right)\right) = T^n \ln(q^a \circ \overline{q}^a).$$

Let

$$T^n \ln\left(e^{\Sigma s^i \vartheta_i} e^{\Sigma s^i J\vartheta_i}\right) = \sum_{i=1}^{n} s^i \eta_i.$$

Suppose that $(\vartheta_i)_{x_0} \in x_0 \circ \Omega_{i-1}$ for $i \leqslant i_0$. The Campbell–Hausdorff formula, Proposition 2, and Lemma 1 imply $(\eta_i)_{x_0} \in x_0 \circ \Omega_{i-1}$ for $i \leqslant i_0 + 1$. Furthermore, η_i are invariant under the group Σ for $i = 1, \dots, n$, and $J^n(x_0 \circ q^a \circ \overline{q}^a) \in \operatorname{ri} \mathcal{O}^n_{0+}$.

The transition from the flow q^a to the flow $q^a \circ \overline{q}^a$ can be considered as the induction step. Repetition of this construction leads to a flow which satisfies the conditions of Proposition 4.

Remark. Theorem 1 and Proposition 3.4 imply Theorem 1.3 formulated in the Introduction. Indeed, it is enough to formulate this theorem for rational numbers $\alpha = k/m > 0$. In this case, the hypothesis of Theorem 1.3 is identical to the hypothesis of Theorem 1 when applied to the family $\mathcal{P} = \{e^{s^k f + s^m g}, e^{s^k f - s^m g}\}$, consisting of two flows.

4. Theorem 1 formulates sufficient conditions for the cone $E_n(\mathcal{P})$ to be a-priori maximal. The cones $E_n(\mathcal{P})$, $n = 1, 2 \ldots$, consist of vectors tangent to the curves

$$s \longmapsto x_0 \circ q(s), \quad q \in \mathrm{Sgr}(\mathcal{P}). \tag{4}$$

But the vectors, which are tangent in a reasonable sense to the attainable sets $\mathcal{A}_t$, $t > 0$, are not necessarily tangent to the curves (4). To obtain a deeper insight into the local structure of the attainable sets, we have to go beyond the cones $E_n(\mathcal{P})$.

DEFINITION. A subspace $R \subset T_{x_0}M$ is called *reachable for* $\mathcal{P}$ if there exists a sequence of mappings

$$\varphi_n \colon V \longrightarrow \left\{ p_1(s_1) \circ \cdots \circ p_l(s_l) \mid p_i \in \mathcal{P},\, s_i \geqslant 0,\, \sum_{i=1}^{l} s_i < t_n,\, l > 0 \right\},$$

V is a neighborhood of zero in R, $t_n \to 0$, such that

(1) the mapping $(x, \xi) \mapsto x \circ \varphi_n(\xi)$ is continuously differentiable near $(x_0, 0)$ and tends to the constant mapping $(x, \xi) \mapsto x_0$ for $n \to \infty$ in the C^1-topology;

(2) $\alpha_n(x_0 \circ \varphi_n(\xi) - x_0) \to \xi$ uniformly in ξ for some $\alpha_n \to \infty$.

PROPOSITION 6. *If R_1, R_2 are reachable subspaces for $\mathcal{P}$, then the subspace $R_1 + R_2$ is also reachable; if $T_{x_0}M$ is a reachable space, then $\mathcal{P}$ is locally controllable at x_0.*

Proof. We can suppose without a loss of generality that $R_1 \cap R_2 = 0$. If φ_n^1, φ_n^2 are mappings which ensure the reachability of the subspaces R_1 and R_2, then the mapping $(\xi_1 + \xi_2) \mapsto \varphi_n^1(\xi_1) \circ \varphi_n^2(\xi_2)$ ensures the reachability of the subspace $R_1 + R_2$. If the space $T_{x_0}M$ is reachable, then the local controllability follows from the implicit function theorem.

The proof of the following assertion is identical to the proof of Proposition 3.4.

PROPOSITION 7. *For every $n > 0$ the subspace $E_n(\mathcal{P}) \cap (-E_n(\mathcal{P}))$ is reachable for $\mathcal{P}$.*

Our next goal is to describe a purely algebraic method for constructing reachable subspaces. To do this, we have to fix some terminology.

Up to now, we meant under a grading of a Lie algebra gradings by nonnegative integers. From now on we shall have to consider algebras which are graded by positive real numbers. We shall say that a Lie algebra L is graded if it is represented as a direct sum $L = \bigoplus_{r\geqslant 0} L_r$, where the set $\{r \geqslant 0 \mid L_r \neq 0\}$ has no limit points and $[L_{r_1}, L_{r_2}] \subset L_{r_1+r_2}$. Furthermore, put $L_{<r} = \sum_{t<r} L_t$.

Let $L = \mathrm{Lie}(\mathcal{S})$ be a free Lie algebra with the set of free generators $\mathcal{S}$. Put $L^1 = \operatorname{span}\mathcal{S} \subset L$. For every integer $n > 0$ define a space L^n by the relation $L^n = [L, L^{n-1}]$. Then

$$L = \bigoplus_{n=1}^{\infty} L^n, \qquad [L^i, L^j] \subset L^{i+j} \quad \forall i, j.$$

The grading thus obtained will be called a *canonical grading* of the free Lie algebra L.

Suppose now that L is not simply a free Lie algebra, but a free (not necessarily canonically) graded Lie algebra, $L = \bigoplus_{r\geqslant 0} L_r$. We call L a *free graded Lie algebra* if $L^1 = \sum_{r\geqslant 0} L^1 \cap L_r$. Then

$$L = \bigoplus_{n=1}^{\infty} \bigoplus_{r\geqslant 0} (L^n \cap L_r),$$

and the subspaces $L^n \cap L_r$ define a bigrading of L.

By $\mathfrak{G}(L)$ we define the group of all automorphisms and antiautomorphisms of the Lie algebra L which preserve the bigrading. Thus

$$\begin{aligned}\mathfrak{G}(L) = \{g\colon L \longrightarrow L \mid g[\vartheta_1, \vartheta_2] = \pm[g(\vartheta_1), g(\vartheta_2)],\\ g(L^n \cap L_r) = L^n \cap L_r,\ \forall \vartheta_1, \vartheta_2 \in L,\ r, n > 0\}.\end{aligned} \tag{5}$$

As above, we denote by $J\colon L \to L$ an automorphism defined by the relation $J\vartheta = (-1)^{n-1}\vartheta$, $\forall \vartheta \in L^n$. It is easily seen that $Jg = gJ$, $\forall g \in \mathfrak{G}$. From this and from the fact that L is a free Lie algebra, we obtain the following proposition.

PROPOSITION 8. *Let $\mathfrak{G}_0(L)$ be a subgroup in $\mathfrak{G}(L)$ consisting of all automorphisms of the Lie algebra L contained in $\mathfrak{G}(L)$, Then*

(1) $\mathfrak{G}(L) = \mathfrak{G}_0(L) \cup J\mathfrak{G}_0(L)$,

(2) *the mapping $g \mapsto g \mid L^1$, $g \in \mathfrak{G}_0(L)$ defines the isomorphism $\mathfrak{G}_0(L)$ on $\times_{r\geqslant 0} \mathrm{GL}(L_r \cap L^1)$. In particular,*

$$\mathfrak{G} \approx \Big(\mathop{\times}_{r\geqslant 0} \mathrm{GL}(L_r \cap L^1)\Big) \times \mathbb{Z}_2.$$

DEFINITION. By a *system of vector fields on M*, we call a triple $\underline{L} = (L, h, G)$, where L is a free graded Lie algebra, $h\colon L \to \operatorname{Vect} M$ is a homomorphism of Lie algebras, G is a finite subgroup in $\mathfrak{G}(L)$ which contains J; in particular, $G = G_0 \cup JG_0$, where $G_0 = G \cap \mathfrak{G}_0(L)$.

Put

$$\mathrm{Inv}(\underline{L}) = \{\vartheta \in L \mid g(\vartheta) = \vartheta \ \forall g \in G\}.$$

Let $r(\underline{L})$ be a supremum of such $r > 0$ that for every positive $r' < r$ and every $\vartheta \in L_{r'} \cap \mathrm{Inv}(\underline{L})$ the following inclusion holds

$$x_0 \circ h(\vartheta) \in x_0 \circ h(L_{<r'}) \subset T_{x_0}M.$$

Put $R(\underline{L}) = x_0 \circ h(L_{<r(\underline{L})})$. Important examples of systems of vector fields represent the systems

$$\underline{\mathcal{L}_\Sigma} = \{\mathcal{L}, \Lambda, \Sigma \cup J\Sigma\}, \quad \Sigma \subset \mathrm{Aut}(\mathcal{P}). \tag{6}$$

Using the introduced notations we can reformulate Theorem 1 in the following way: *If for a finite subgroup* $\Sigma \subset \mathrm{Aut}(\mathcal{P})$ *we have* $n < r(\underline{\mathcal{L}_\Sigma})$, *then* $E_n(\mathcal{P}) = x_0 \circ \Omega_n$.

COROLLARY. *The spaces* $R(\underline{\mathcal{L}_\Sigma})$, $\Sigma \subset \mathrm{Aut}(\mathcal{P})$ *are reachable for* $\mathcal{P}$.

It turns out that other systems of vector fields $\underline{L}$ exist such that $R(\underline{L})$ are normally accessible spaces for $\mathcal{P}$. These systems could be constructed from $\underline{\mathcal{L}_\Sigma}$ with the help of special procedures which we are now going to describe.

DEFINITION. Let $\underline{L} = (L, h, G)$, $\underline{\widehat{L}} = (\widehat{L}, \widehat{h}, \widehat{G})$ be systems of vector fields. We shall say that the system $\underline{L}$ *is induced by the system* $\underline{\widehat{L}}$ if there exist a linear mapping $\Phi\colon L^1 \hookrightarrow \widehat{L}$ and a homomorphism $\varphi\colon G_0 \hookrightarrow \widehat{G}$, such that $\Phi \circ g = \varphi(g) \circ \Phi$, $\forall g \in G_0$, $h|_{L^1} = \widehat{h} \circ \varphi$, and

$$\mathrm{Inv}(\underline{\widehat{L}}) \cap \widehat{L}_{<r} \subset \Phi(L^1 \cap L_{<r}) \subset \widehat{L}_{<r}, \quad \forall r > 0. \tag{7}$$

Furthermore, for a free graded Lie algebra L and $\nu \geqslant 0$, such that the set $\{t - \nu n \mid L^n \cap L_t \neq 0\}$ is contained in $\mathbb{R}_+$ and has no limit points, we put

$$L_r(\nu) = \sum_{t - \nu n = r} (L^n \cap L_t), \quad \forall r \geqslant 0. \tag{8}$$

We have $L = \bigoplus_{r \geqslant 0} L_r(\nu)$. The grading $L_r(\nu)$, $r \geqslant 0$, will be called the ν *modification of the grading* L_r, $r \geqslant 0$, or simply a ν-grading, if the initial grading (corresponding to $\nu = 0$) is given. The Lie algebra L, equipped with the ν-grading, is denoted by $L(\nu)$, for $\nu = 0$ the argument will be omitted.

A system of vector fields $\underline{L}(\nu) = (L(\nu), h, G)$ is called the ν-*modification of the system* $\underline{L}(\nu) = (L, h, G)$.

THEOREM 2. *Let* $n \geqslant 1$, $\nu_1, \ldots, \nu_n$ *be a sequence of nonnegative numbers*, $\nu_1 + \ldots + \nu_n \leqslant 1$, *and* $\underline{L_1}, \ldots, \underline{L_n}$ *be a sequence of systems of vector fields, where* $\underline{L_1} = \underline{\mathcal{L}_\Sigma}$ *for some finite subgroup* $\Sigma \subset \mathrm{Aut}(\mathcal{P})$. *Suppose that the system* $\underline{L_{i+1}}$ *is induced by the system* $\underline{L_i}(\nu_i)$ *for* $i = 1, \ldots, n$. *Then* $R(\underline{L_n}(\nu_n))$ *is a reachable space for* $\mathcal{P}$.

5. The proof of Theorem 2 will be given in the next section. Now we shall give some implications of this general theorem. First of all, we obtain, for $n = 1$, the following essential generalization of the corollary to Theorem 1 formulated above.

COROLLARY. *The subspace $R(\mathcal{L}_\Sigma(\nu))$ is reachable for $\mathcal{P}$, $\forall \nu \in [0,1]$ and for every finite subgroup $\Sigma \subset \operatorname{Aut}(\mathcal{P})$.*

DEFINITION. Let $\underline{L} = (L, h, G)$, $\underline{\widehat{L}} = (\widehat{L}, \widehat{h}, \widehat{G})$ be two systems of vector fields. We shall say that $\underline{L} \subset \underline{\widehat{L}}$, or that $\underline{L}$ is a *subsystem of the system* $\underline{\widehat{L}}$, if

$$L \subset \widehat{L}, \qquad L^1 \cap \widehat{L}_r = L^1 \cap L_r, \quad \forall r \geqslant 0, \quad G_0 \subset \widehat{G}, \quad h = \widehat{h}\big|_L.$$

THEOREM 3. *Let $\underline{L} = (L, h, G) \subset \underline{\mathcal{L}_\Sigma}$ for some finite subgroup $\Sigma \subset \operatorname{Aut}(\mathcal{P})$, and $\operatorname{Inv}(\underline{\mathcal{L}_\Sigma}) \subset L$. Then $R(\underline{L}(\nu))$ is a reachable space for $\mathcal{P}$ for all $\nu \in [0,1]$.*

Proof. We can assume that $\nu \in (0,1)$ since $R(\underline{L}(\nu))$ is semicontinuous from below in ν. Let $\widetilde{L}$ be a free Lie algebra such that $\widetilde{L}^1 \cap \widetilde{L}_r = L_r$; $\widetilde{G_0}$ be a subgroup in $\mathfrak{G}_0$, consisting of automorphisms which have restrictions on L coinciding with the elements of G_0; let $\Phi\colon \widetilde{L} \to \mathcal{L}$ be a homomorphism induced by the imbedding $L \hookrightarrow \mathcal{L}$; $\widetilde{h} = h \circ \Phi$.

Put $\underline{\widetilde{L}} = (\widetilde{L}, \widetilde{h}, \widetilde{G}_0 \cup J\widetilde{G}_0)$. It is easily seen that the system of vector fields $\underline{\widetilde{L}}$ is induced by the system $\underline{\mathcal{L}_\Sigma}$. Therefore, Theorem 3 follows from Theorem 2 and from the following assertion.

LEMMA 2. *For all $\nu \in (0,1)$ the relation $(\underline{L}(\nu)) = R(\underline{\widetilde{L}}(\nu))$ holds.*

Proof. It is easy to show that

$$\Big(L_r \cap \sum_{i \geqslant k} L^i\Big) \supset \Phi(\widetilde{L}_r \cap \widetilde{L}^k) \supset L_r \cap L^k, \qquad L^k \cap \Phi(\widetilde{L}^k \cap \operatorname{Inv}(\underline{\widetilde{L}})) = L^k \cap \operatorname{Inv}(L).$$

Hence

$$\Phi(\widetilde{L}_{<r}(\nu)) = L_{<r}(\nu), \qquad L_r(\nu) \cap \Phi(\widetilde{L}_r(\nu) \cap \operatorname{Inv}(\underline{\widetilde{L}})) = L_r(\nu) \cap \operatorname{Inv}(\underline{L}).$$

Since $\widetilde{h} = h \circ \Phi$ we obtain $r(\underline{\widetilde{L}}(\nu)) = r(\underline{L}(\nu))$ and $R(\underline{\widetilde{L}}(\nu)) = R(\underline{L}(\nu))$.

We apply now Theorem 3 to the case when $\mathcal{P}$ consists of two one-parameter subgroups in $\operatorname{Diff} M$,

$$\mathcal{P} = \{p_+, p_-\}, \quad p_\pm(s) = e^{s(f \pm g)}, \quad f, g \in \operatorname{Vect} M, \tag{9}$$

and $\Sigma = \operatorname{Aut}(\mathcal{P})$ is a two-element group of permutations of a two-element set $\mathcal{P}$. We have

$$\Lambda^1_{p\pm} = \omega^1_{p\pm} = f \pm g, \qquad \Lambda^i_{p\pm} = \omega^i_{p\pm} = 0 \quad \text{for} \quad i > 1.$$

Furthermore,

$$\mathcal{L} = \operatorname{Lie}\{\lambda^i_{p\pm} \mid i = 1, 2 \ldots\}.$$

Since $\Lambda(\lambda^i_{p\pm}) = \Lambda^i_{p\pm} = 0$ for $i > 1$, there is no need to consider the whole Lie algebra $\mathcal{L}$, but it is sufficient to restrict ourselves to its Lie subalgebra, generated by elements $\lambda^1_{p\pm}$. Put

$$y = \frac{1}{2}(\lambda^1_{p+} - \lambda^1_{p-}), \qquad z = \frac{1}{2}(\lambda^1_{p+} + \lambda^1_{p-}).$$

Then

$$\Lambda(y) = g, \qquad \Lambda(z) = f, \qquad \mathrm{Lie}\,\{\lambda^1_{p\pm}\} = \mathrm{Lie}\,\{y, z\}.$$

The only nonunit element $\sigma \in \mathrm{Aut}\,(\mathcal{P})$ acts according to the formulas $\sigma y = -y$, $\sigma z = z$. Hence, $\mathrm{Inv}\,(\underline{\mathcal{L}_\Sigma}) \cap \mathrm{Lie}\,\{y, z\}$ is a linear hull of all commutator monomials in y, z, of even degree in y and odd degree in z. Furthermore,

$$\mathcal{L}_k \cap \mathrm{Lie}\,\{y, z\} = \mathcal{L}^k \cap \mathrm{Lie}\,\{y, z\} \quad \forall k > 0.$$

As a very special case of Theorem 3 (not to mention Theorem 2), we obtain the following proposition.

PROPOSITION 9. *Let* $\nu \in [0, 1]$, $r \geqslant 0$, *and let* Π^1 *be a set of bihomogeneous commutator polynomials in two variables* y, z, *such that:*

(a) *The elements of the set* Π^1 *are free generators of the Lie subalgebra* $\mathrm{Lie}\,\Pi^1 \subset \mathrm{Lie}\,\{y, z\}$ *generated by these elements;**

(b) $\mathrm{Lie}\,\Pi^1$ *contains all commutator monomials which are of even degree in the first variable and of odd degree in the second.*

Put

$$\Pi^{k+1} = [\Pi, \Pi^k], \qquad d_\nu(\pi) = \deg \pi - \nu k, \quad k = 1, 2, \ldots,$$

$$\Pi = \bigcup_{k=1}^{\infty} \Pi^k.$$

Suppose that for all $l \geqslant 0$ *and every* $\pi \in \Pi^{2l+1}$, *which is of even degree in the first variable and of odd degree in the second, and satisfies the condition* $d_\nu(\pi) \leqslant r$, *the inclusion*

$$x_0 \circ \pi(g, f) \in \mathrm{span}\,\{x_0 \circ \pi'(g, f) \mid \pi \in \Pi,\ d_\nu(\pi') < d_\nu(\pi)\}$$

holds. Then the space

$$\mathrm{span}\,\{x_0 \circ \pi(g, f) \mid \pi \in \Pi,\ d_\nu(\pi) \leqslant r\}$$

is reachable for the family of flows (9).

* Notice that every Lie subalgebra of a free Lie algebra is free.

5. Fast-Switching Variations

1. Let $s \mapsto p(s)$ be an analytic curve in $\operatorname{Diff} M$ (a flow on M). From a given $\mathcal{P}$, we constructed other curves, using polynomial substitutions of the parameter and pointwise multiplication of curves. In this section, we consider one additional construction — we correspond a sequence of curves of the type

$$s \longmapsto \underbrace{p(s_n s) \circ \cdots \circ p(s_n s)}_{n \text{ times}}, \quad s_n \longrightarrow 0 \quad (n \to \infty)$$

to a given curve $p(\cdot)$.

With these sequences, we can work in the same way as with analytic curves in $\operatorname{Diff} M$, and we start by introducing the relevant system of notions.

DEFINITION. Let $Q = \{q_n(\cdot)\}_{n=1}^{\infty}$ be a sequence of flows on M, $q_n(s) \in \operatorname{Diff} M$, $s \in \mathbb{R}$. We shall call Q a *tame sequence* if $\forall m > 0$ there exists an expansion

$$q_n(s) = \mathrm{id} + \sum_{i=1}^{m} n^{-i} D_Q^i(s) + o(sn^{-m}), \quad \frac{s}{n^m} \to 0, \tag{1}$$

where $D_Q^i(s)$ are differential operators on M polynomially depending on s, $D_Q^i(0) = 0$.

Let $Q = \{q_n\}_{n=1}^{\infty}$, $Q' = \{q'_n\}_{n=1}^{\infty}$ be two tame sequences of flows. Put

$$Q^{-1} = \{q_n^{-1}\}_{n=1}^{\infty}, \qquad Q \circ Q' = \{q_n \circ q'_n\}_{n=1}^{\infty}.$$

It is easily seen that the sequences Q^{-1} and $Q \circ Q'$ are also tame. Hence, the termwise multiplication and taking the inverse, define the structure of a group on the set of all tame sequences.

Define $\operatorname{ord} Q = \min\{i \mid D_Q^i(\cdot) \neq 0\}$ and call this number the *order of the sequence* Q. Let $\operatorname{ord} Q = k$ and put $T_0 Q(s) = D_Q^k(s)$. It is easily seen that $T_0 Q(s) \in \operatorname{Vect} M$, $s \in \mathbb{R}$. Finally, suppose that

$$Q_\tau(s) = \mathrm{id} + \sum_{i=1}^{\infty} \tau^i D_Q^i(s)$$

is a formal power series in τ, and $T^m Q_\tau(s) = \mathrm{id} + \sum_{i=1}^{m} \tau^i D_Q^i(s)$ is a segment of the length m of this series.

Consider the formal series

$$\overrightarrow{\ln} Q_\tau(s) = Q_\tau(s)^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}\tau} Q_\tau(s) = \sum_{i=1}^{\infty} \tau^{i-1} \omega_Q^i(s), \qquad \ln Q_\tau(s) = \sum_{i=1}^{\infty} \tau^i \Lambda_Q^i(s).$$

It is easily seen that the differential operators ω^i_Q are in fact vector fields on M. Proposition 4.1 implies that Λ^j_Q are commutator polynomials in ω^i_Q, in particular, $\Lambda^j_Q \in \operatorname{Vect} M$. Furthermore, the first nonzero terms of the series $\ln Q_\tau(s)$ and

$$\int_0^T \vec{\ln}\, Q_\tau(s)\,\mathrm{d}\tau = \sum_{i=1}^{\infty} \frac{\tau^i}{i}\,\omega^i_Q(s)$$

coincide and are equal to $\tau^{\operatorname{ord} Q} T_0 Q(s)$.

2. Denote A^2 the space of all real polynomials in two variables without free terms: $a(0,0) = 0$, $\forall a \in A^2$. If $a \in A^2$ and $s \mapsto q(s)$ is a flow, then $s \mapsto q(a(\tau, s)) \circ q(a(\tau, 0))^{-1}$ is again a flow, $\forall \tau \in \mathbb{R}$. If $\{q_n(\cdot)\}_{n=1}^{\infty}$ is a tame sequence of flows then $\{q_n(a(n^{-1}, \cdot)) \circ q_n(a(n^{-1}, 0))^{-1}\}_{n=1}^{\infty}$ is again a tame sequence of flows.

Let $\mathfrak{Q}$ be a set of tame sequences of flows. We shall denote by $\operatorname{Gr}(\mathfrak{Q})$ the group generated by all tame sequences of flows of the form

$$\left\{q_n\left(a(n^{-1}, \cdot)\right) \circ q_n\left(a(n^{-1}, 0)\right)^{-1}\right\}_{n=1}^{\infty},$$

where

$$\left\{q_n(\cdot)\right\}_{n=1}^{\infty} \in \mathfrak{Q}, \quad a \in A^2. \tag{2}$$

For every integer $m \geqslant 0$ we put

$$\Omega_m(\mathfrak{Q}) = \operatorname{span}\left\{ [\omega^{i_k}_{Q_k}(s_k), [\ldots, [\omega^{i_1}_{Q_1}(s_1), \omega^{i_0}_{Q_0}(s_0)] \ldots] \mid \sum_{j=0}^{k} i_j \leqslant m,\ Q_j \in \mathfrak{Q},\ s_j \in \mathbb{R};\ k \geqslant 0 \right\},$$

$$\Omega(\mathfrak{Q}) = \bigcup_{m=1}^{\infty} \Omega_m(\mathfrak{Q}) = \operatorname{Lie}\left\{\omega^i_Q(s) \mid Q \in \mathfrak{Q},\ s \in \mathbb{R},\ i = 1, 2, \ldots\right\}.$$

Proposition 4.1 implies that

$$\Omega_m(\mathfrak{Q}) = \operatorname{span}\left\{ [\Lambda^{i_k}_{Q_k}(s_k), [\ldots, [\Lambda^{i_1}_{Q_1}(s_1), \Lambda^{i_0}_{Q_0}(s_0)] \ldots] \mid \sum_{j=0}^{k} i_j \leqslant m,\ Q_j \in \mathfrak{Q},\ s_j \in \mathbb{R};\ k \geqslant 0. \right\}.$$

THEOREM 1. *For every set $\mathfrak{Q}$ of tame sequences and every integer $m > 0$, the following relation holds*

$$\Omega_m(\mathfrak{Q}) = \left\{T_0 P(s) \mid P \in \operatorname{Gr}(\mathfrak{Q}),\ \operatorname{ord} P = m,\ s \in \mathbb{R}\right\}.$$

Proof. It proceeds analogous to the proof of Theorem 2.1. Let

$$\mathcal{T}_m = \{T_0 P(s) \mid P \in \mathrm{Gr}(\Omega),\ \operatorname{ord} P = m,\ s \in \mathbb{R}\}.$$

Formula (2.6) implies that $\Omega_m(\Omega) \supset \mathcal{T}_m$. To prove the opposite inclusion, it is sufficient to show that

$$\omega_Q^k(s) \in \mathcal{T}_m, \quad k = 1, \ldots, m, \quad s \in \mathbb{R}, \quad Q \in \Omega,$$

and use Lemmas 2.1–2.3.* The inclusion $\omega_Q^m(s) \in \mathcal{T}_m$, $s \in \mathbb{R}, Q \in \Omega$ is proved exactly as the inclusions $\omega_P^m \in \{T_0 q \mid q \in \mathrm{Gr}(\mathcal{P})_m\}$ from Theorem 2.1. But to deduce from here the inclusions $\omega_Q^k(s) \in \mathcal{T}_m$ for $k < m$, we have to use a different reasoning. First of all, substituting, if necessary, the sequence $Q = \{q_n(\cdot)\}_{n=1}^{\infty}$ by the sequence $\{s \mapsto q_n(s^{m+1})\}_{n=1}^{\infty}$, we can suppose without a loss of generality that $\omega_Q^k(s) = o(s^m)$, $s \to 0$, $\forall k \leqslant m$. Since $\mathcal{T}_m$ is a linear space the relation $\mathrm{d}^i/\mathrm{d}s^i \omega_Q^k(s) \in \mathcal{T}_m$, $\forall s \in \mathbb{R}$, is equivalent for all $i \leqslant m$ to the relation $\omega_Q^k(s) \in \mathcal{T}_m$ $\forall s \in \mathbb{R}$.

Now we apply induction on k. Suppose that the assertion is proved for $k' < k$. Let

$$\widehat{Q} = \left\{s \mapsto q_n(s + n^{-(m-k)}) \circ q_n(n^{-(m-k)})^{-1}\right\}_{n=1}^{\infty} \in \mathrm{Gr}(\Omega).$$

The coefficient at τ^m of the series $\overrightarrow{\ln}\, \widehat{Q}_\tau$ coincides with the coefficient at τ^{m-1} of the series $\overrightarrow{\ln}\, Q_\tau(s + \tau^{m-k})$ and is equal to

$$\omega_Q^m(s) + \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}s}\omega_Q^k(s) + \sum_{1<i<\frac{m}{m-k}} \frac{1}{i!(m-(m-k)i)}\frac{\mathrm{d}^i}{\mathrm{d}s^i}\omega_Q^{m-(m-k)i}(s). \tag{3}$$

Considering the sequence $\widehat{Q}$ instead of Q, we conclude that the field (3) belongs to $\mathcal{T}_m$. Since $\omega_Q^m(s) \in \mathcal{T}_m$ and since, according to the induction hypothesis $\mathrm{d}^i/\mathrm{d}s^i\omega_Q^{k'}(s) \in \mathcal{T}_m$ for $k' < k$, we obtain the relation $\mathrm{d}/\mathrm{d}s\,\omega_Q^k(s) \in \mathcal{T}_m$, $\forall s \in \mathbb{R}$, which implies the inclusion $\omega_Q^k(s) \in \mathcal{T}_m$.

3. Let $Q = \{q_n(\cdot)\}_{n=1}^{\infty}$ be a tame sequence of flows. For every s, put $Q(s) = \{q_n(s)\}_{n=1}^{\infty}$. The sequence of diffeomorphisms $Q(s)$ is also called tame.

Let $\tau \mapsto \gamma(\tau)$ be a smooth curve in M, $\gamma(0) = x_0$. There exists a smooth curve $\widehat{\gamma}$ in M such that $\widehat{\gamma}(n^{-1}) = \gamma(n^{-1}) \circ Q(s)$, $n = 1, 2, \ldots$. The curve $\widehat{\gamma}$ is not uniquely defined, nevertheless, it is easily seen that the jet $J^m\widehat{\gamma}$ depends only on $J^m\gamma$ and on the polynomial $T^m Q_\tau(s)$, $\forall m > 0$. Put $(J^m\gamma) \circ Q(s) = J^m\widehat{\gamma}$. The mapping

$$c \longmapsto c \circ Q(s), \quad c \in C_{x_0}^m, \tag{4}$$

* It is worth mentioning that the relation $T_0 q(a(\cdot)) = \alpha_1^{\operatorname{ord} q} T_0 q$ appearing in Lemma 2.1 is useless in the considered situation. But Lemmas 2.1–2.2 imply that $\mathcal{T}_m$ is an additive group. Since $\mathcal{T}_m$ is at the same time arcwise connected, we deduce that it is a linear space.

is a diffeomorphism of the manifold of jets $C^m_{x_0}$. Additionally, we have

$$\left(c \circ Q'(s')\right) \circ Q(s)^{-1} = c \circ \left(Q'(s') \circ Q(s)^{-1}\right)$$

for all tame sequences of diffeomorphisms $Q(s)$ and $Q'(s')$. In other words, mappings of the form (4) define an action of the group of tame sequences of diffeomorphisms on the manifold $C^m_{x_0}$.

Let $\mathfrak{Q}$ be a set of tame sequences of flows, $c \in C^m_{x_0}$. We denote

$$c\mathcal{O}^m(\mathfrak{Q}) = \left\{c \circ P(s) \mid P \in \mathrm{Gr}\,(\mathfrak{Q}),\ s \in \mathbb{R}\right\}, \qquad c\mathcal{O}^{m-1}(\mathfrak{Q}) = \mathrm{pr}_m\left(c\mathcal{O}^m(\mathfrak{Q})\right).$$

Theorem 2.2 implies that $c\mathcal{O}^m(\mathfrak{Q})$ is an immersed analytic submanifold in $C^m_{x_0}$, and $c\mathcal{O}^{m-1}(\mathfrak{Q})$ in $C^{m-1}_{x_0}$. We also recall that we denoted by $\mathcal{C}^m$ the affine bundle $\mathrm{pr}_m \colon C^m_{x_0} \to C^{m-1}_{x_0}$, cf. Section 2, No. 8.

PROPOSITION 1. *Let $c \in C^m_{x_0}$ and let $\mathcal{C}^m \mid c\mathcal{O}^{m-1}(\mathfrak{Q})$ be a restriction of the bundle $\mathcal{C}^m$ on the submanifold $c\mathcal{O}^{m-1}(\mathfrak{Q})$ in the base. Then $c\mathcal{O}^m(\mathfrak{Q})$ is the total space of an affine subbundle in $\mathcal{C}^m \mid c\mathcal{O}^{m-1}(\mathfrak{Q})$ with the fibre $x_0 \circ \Omega_m(\mathfrak{Q})$.*

The proposition is obtained from Theorem 1 exactly as Theorem 2.3 is obtained from Theorem 2.1.

By analogy with the group of flows (3.4), we introduce the group of tamed sequences of flows

$$\mathrm{Gr}^m_{<}(\mathfrak{Q}) = \left\{P \in \mathrm{Gr}\,(\mathfrak{Q}) \,\middle|\, \frac{\mathrm{d}^{k-1}}{\mathrm{d}\tau^{k-1}} \overrightarrow{\ln} P_\tau(s)\bigg|_{\tau=0} \in \Omega_{k-1}(\mathfrak{Q}),\ s \in \mathbb{R};\ k \leqslant m\right\}.$$

Let $c \in C^m_{x_0}$, and denote

$$c\mathcal{O}^m_{<}(\mathfrak{Q}) = \left\{c \circ P(s) \mid P \in \mathrm{Gr}^m_{<}(\mathfrak{Q}),\ s \in \mathbb{R}\right\}, \qquad c\mathcal{O}^{m-1}_{<}(\mathfrak{Q}) = \mathrm{pr}_m\left(c\mathcal{O}^m_{<}(\mathfrak{Q})\right).$$

The following proposition is easily obtained from Theorem 1, cf. Proposition 2.2.

PROPOSITION 2. *For every $c \in C^m_{x_0}$ the manifold $c\mathcal{O}^m_{<}(\mathfrak{Q})$ is the total space of an affine subbundle in $\mathcal{C}^m \mid c\mathcal{O}^{m-1}_{<}(\mathfrak{Q})$ with the fibre $x_0 \circ \Omega_{m-1}(\mathfrak{Q})$.*

4. Let $Q = \{q_n(\cdot)\}_{n=1}^\infty$ be a tame sequence of flows, $a \in A^2$, $s \in \mathbb{R}$. Put

$$Q_a(s) = \left\{q_n\left(a(n,s)\right)\right\}_{n=1}^\infty, \tag{5}$$

which is a tame sequence of diffeomorphisms. Finally, for every $c \in C^n_{x_0}$ and every set of tame sequences of flows $\mathfrak{Q}$, we put

$$c\mathcal{O}^m_{+}(\mathfrak{Q}) = \left\{c \circ Q^1_{a_1}(s_1) \circ \cdots \circ Q^k_{a_k}(s_k) \mid Q^i \in \mathfrak{Q},\ a_i \in A^2_{+},\ s_i \geqslant 0;\ k > 0\right\},$$

where

$$A^2_{+} = \left\{a \in A^2 \mid a(\tau, s) \geqslant 0 \text{ for } \tau \geqslant 0,\ s \geqslant 0\right\}.$$

It is easily seen that $c\mathcal{O}^m_{+}(\mathfrak{Q}) \subset c\mathcal{O}^m(\mathfrak{Q})$. For $c = J^m x_0$, i.e., for the m-jet of a constant curve, we shall omit the symbol c in the expressions $c\mathcal{O}^m(\mathfrak{Q})$, $c\mathcal{O}^m_{<}(\mathfrak{Q})$, $c\mathcal{O}^m_{+}(\mathfrak{Q})$.

PROPOSITION 3. *Let* $c \in C^m_{x_0}$, *and denote* $\mathrm{ri}\,(c\mathcal{O}^m_+(\Omega))$ *the set of points in* $c\mathcal{O}^m_+(\Omega)$ *which are interior relative to the manifold* $c\mathcal{O}^m(\Omega)$. *Then*

(1) $c\mathcal{O}^m_+(\Omega) \subset \overline{\mathrm{ri}\,(c\mathcal{O}^m_+(\Omega))}$;

(2) $\forall c' \in \mathrm{ri}\,(c\mathcal{O}^m_+(\Omega))$ *the inclusion* $c'\mathcal{O}^m_<(\Omega) \subset \mathrm{ri}\,(c\mathcal{O}^m_+(\Omega))$ *holds.*

Proof of the Proposition 3 is similar to the proof of Theorem 3.2. The only difference consists in considering not the family of flows (3.5), but the family of flows $s \mapsto Q_a(s)$, $Q \in \Omega$, $a \in A^2_+$ on $C^m_{x_0}$, (the tame sequence $Q_a(s)$ is considered here as a diffeomorphism $c \mapsto c \circ Q_a(s)$ of the manifold $C^m_{x_0}$).

The role of the set $O^m_+(\Omega)$ is clarified by the following

PROPOSITION 4. *Let* $J^m x_0 \in \mathrm{ri}\,(\mathcal{O}^m_+(\Omega))$. *Then* $x_0 \circ \Omega_m(\Omega)$ *is a reachable space for the family of flows*

$$\{s \longmapsto q_k(\alpha_k s) \mid k > N,\ \{q_n(\cdot)\}_{n=1}^{\infty} \in \Omega\} \tag{6}$$

for every sequence of real numbers $\alpha_k \to \infty\ (k \to \infty)$ *and every* $N > 0$.

Proof. Proposition 3.1 implies the existence of

$$Q^i \in \Omega, \qquad a_i \in A^2_+, \qquad \overline{s}_i > 0, \quad i = 1, \dots, k,$$

such that

$$J^n_{x_0} = J^n x_0 \circ Q^1_{a_1}(\overline{s}_1) \circ \cdots \circ Q^k_{a_k}(\overline{s}_k)$$

and the vector $(\overline{s}_1, \dots, \overline{s}_k) \in \mathbb{R}^k$ is a regular point of the mapping

$$(s_1, \dots, s_k) \longmapsto J^n x_0 \circ Q^1_{a_1}(s_1) \circ \cdots \circ Q^k_{a_k}(s_k)$$

from $\mathbb{R}^k$ into $\mathcal{O}^m(\Omega)$. Proposition 1 implies that

$$J^m x_0 + x_0 \circ \Omega_m(\Omega) \subset \mathcal{O}^m(\Omega).$$

According to the implicit function theorem, there exists a neighborhood of zero V in $x_0 \circ \Omega_n$ and smooth functions $\xi \mapsto s_i(\xi)$, $\xi \in V$, $i = 1, \dots, k$, such that $s_i(0) = \overline{s}_i$ and

$$J^m x_0 \circ Q^1_{a_1}(s_1(\xi)) \circ \cdots \circ Q^k_{a_k}(s_k(\xi)) = J^m x_0 + \xi. \tag{7}$$

Let $Q^i = \{q^i_n(\cdot)\}_{n=1}^{\infty}$ and put

$$\varphi_n(\xi) = q^1_n\big(a_1(n, s_1(\xi))\big) \circ \cdots \circ q^k_n\big(a_k(n, s_k(\xi))\big), \quad \xi \in V, \quad n = 1, 2, \dots.$$

The relation (7) together with (5) imply

$$n^m\big(x_0 \circ \varphi_n(\xi) - x_0\big) = \xi + n^{-1}\rho_n(\xi),$$

where $\rho_n(\xi)$ is uniformly bounded with respect to n and ξ. In this case the sequence of mappings φ_n guarantees that the space $x_0 \circ \Omega_m(\Omega)$ is reachable for the family of flows (6).

5. In accordance with the notation of the Lie algebra $\mathcal{V}^m$ (cf. Section 3, No. 6), we denote $\mathcal{V}^m(\Omega)$ the Lie algebra consisting of all polynomials in τ of the form $v(\tau) = \sum_{k=1}^m \tau^k v_k$, $v_k \in \Omega_k(\Omega)$, $k = 1, \ldots, m$, with the Lie multiplication

$$[v, w](\tau) = \sum_{k=1}^{m} \tau^k \sum_{i=1}^{k-1} [v_i, w_{k-1}], \quad \forall v, w \in \mathcal{V}^m(\Omega).$$

PROPOSITION 5. *The following relations hold*

$$\mathcal{V}^m(\Omega) = \left\{ T^m \int_0^\tau \vec{\ln} P_{\tau'}(s) \, d\tau' \mid P \in \mathrm{Gr}(\Omega), \ s \in \mathbb{R} \right\}$$
$$= \left\{ T^m \ln P_\tau(s) \mid P \in \mathrm{Gr}(\Omega), \ s \in \mathbb{R} \right\},$$

where, as usually, the symbol T^m applied to a series in powers of τ denotes a segment of this series of the length m.

Proof. The first equality is easily obtained from Theorem 1, and the second one from Proposition 4.1.

Consider the semigroup of tame sequences of diffeomorphisms (not flows!),

$$\mathrm{Sg}(\Omega) = \left\{ Q^1_{a_1}(s_1) \circ \cdots \circ Q^k_{a_k}(s_k) \mid Q^i \in \Omega, \ a_i \in A^2_+, \ s_i \geqslant 0, \ k > 0 \right\}.$$

Put

$$\mathcal{V}^m_+(\Omega) = \left\{ T^m \ln P_\tau \mid P \in \mathrm{Sg}(\Omega) \right\} \subset \mathcal{V}^m(\Omega),$$

$$\vec{\mathcal{V}}^m_+(\Omega) = \left\{ T^m \int_0^\tau \vec{\ln} P_{\tau'} \, d\tau' \mid P \in \mathrm{Sg}(\Omega) \right\} \subset \mathcal{V}^m(\Omega).$$

PROPOSITION 6. *The subsets $\mathcal{V}^m_+(\Omega)$, $\vec{\mathcal{V}}^m_+(\Omega)$ of the space $\mathcal{V}^m(\Omega)$ have interior points, and even more, they are contained in the closure of their interior.*

The assertion about the set $\vec{\mathcal{V}}^m_+$ is proved as in Theorem 3.3, and the assertion about the set $\mathcal{V}^m_+$ can be deduced from the assertion about $\vec{\mathcal{V}}^m_+$ with the aid of Proposition 4.1, or it can be proved directly, if we substitute in all our reasonings the chronological exponent by the usual exponent and use the Campbell–Hausdorff formula instead of (2.6).

6. Let

$$\mathcal{L}(\Omega) = \mathrm{Lie} \left\{ \lambda^{ij}_Q \mid Q \in \Omega, \ i, j = 1, 2, \ldots \right\},$$

$$\mathfrak{A}(\Omega) = \mathrm{Ass} \left\{ \lambda^{ij}_Q \mid Q \in \Omega, \ i, j = 1, 2, \ldots \right\}$$

be a free Lie algebra and a free associative algebra with the set of free generators λ_Q^{ij}. As usual, we identify $\mathcal{L}(\Omega)$ with a subspace in Ass(Ω), consisting of all commutator polynomials in λ_Q^{ij}, and we put

$$[\vartheta_1, \vartheta_2] = \vartheta_1\vartheta_2 - \vartheta_2\vartheta_1 \quad \forall \vartheta_1, \vartheta_2 \in \mathcal{L}(\Omega).$$

As above, we denote by $\mathrm{Aut}(\Omega)$ the group of bijections of the set Ω and identify $\sigma \in \mathrm{Aut}(\Omega)$ with the automorphism $\vartheta \mapsto \sigma\vartheta$ of the algebra $\mathfrak{A}(\Omega)$, which acts according to the rule: $\sigma\lambda_Q^i = \lambda_{\sigma(Q)}^i$, $\forall Q \in \Omega$, $i = 1, 2, \ldots$.

For every integer $k > 0$ we put

$$\mathcal{L}_k(\Omega) = \mathrm{span}\left\{ [\lambda_{Q_l}^{i_lj_l}, [\ldots, [\lambda_{Q_2}^{i_2j_2}, \lambda_{Q_1}^{i_1j_1}]\ldots] \,\middle|\, Q_\iota \in \Omega, \sum_{\iota=1}^{l} i_\iota = k,\, l > 0 \right\},$$

$$\mathcal{L}^k(\Omega) = \left[\mathcal{L}^1(\Omega), \mathcal{L}^{k-1}(\Omega)\right], \quad \text{where} \quad \mathcal{L}^1(\Omega) = \mathrm{span}\left\{\lambda_Q^{ij} \mid Q \in \Omega,\, i, j > 0\right\}.$$

Denote $\Lambda\colon \mathcal{L}(\Omega) \to \mathrm{Vect}\, M$ a homomorphism of Lie algebras defined by the relation

$$\sum_j s^j \Lambda(\lambda_Q^{ij}) = \Lambda_Q^i(s) \quad \forall Q \in \Omega, \quad i = 1, 2, \ldots, \quad s \in \mathbb{R}.$$

THEOREM 2. *Let Σ be a finite subgroup in* $\mathrm{Aut}(\Omega)$ *and $m > 0$. Suppose that for every $\vartheta \in \mathcal{L}_k(\Omega) \cap \mathcal{L}^{2l+1}(\Omega)$, where $2l + 1 \leqslant k \leqslant m$, which satisfies the relation $\sigma\vartheta = \vartheta$, $\forall\sigma \in$* $\mathrm{Aut}(\Omega)$, *the relation $x_0 \circ \Lambda(\vartheta) \in x_0 \circ \Omega_{k-1}(\Omega)$ holds.*

Then $x_0 \circ \Omega_m(\Omega)$ is a reachable space for the family of flows

$$\left\{s \mapsto q_k(\alpha_k s) \mid k > N,\ \{q_n(\cdot)\}_{n=1}^{\infty} \in \Omega\right\}$$

for every sequence of reals $\alpha_k \to \infty$ $(k \to \infty)$ $\forall N > 0$.

Proof. According to Proposition 4, it is sufficient to prove that $J^m x_0 \in \mathrm{ri}\,(\mathcal{O}_+^m(\Omega))$. The following assertion is obtained from Propositions 2 and 3 in the same way as the analogous Proposition 4.4 — from Proposition 2.2 and Theorem 3.2.

PROPOSITION 7. *Suppose there exists*

$$P \in \mathrm{Sg}(\Omega), \qquad T^m \ln P_\tau = \sum_{k=1}^{m} \tau^k v_k,$$

such that

$$J^m(x_0 \circ P) \in \mathrm{ri}\left(\mathcal{O}_+^m(\Omega)\right), \qquad x_0 \circ v_k \in x_0 \circ \Omega_{k-1}(\Omega), \quad k = 1, \ldots, m.$$

Then $J^m x_0 \in \mathrm{ri}\,(\mathcal{O}_+^m(\Omega))$.

To complete the proof of Theorem 2, we have to establish the existence of a polynomial

$$\vartheta(\tau) = \sum_{k=1}^{m} \tau^k \vartheta_k, \quad \vartheta_k \in \mathcal{L}_k(\Omega) \cap \sum_i \mathcal{L}^{2i+1}(\Omega),$$

such that

(1) $\sigma\vartheta(\tau) = \vartheta(\tau)$, $\forall \sigma \in \Sigma$,

(2) $\Lambda(\vartheta(\tau)) = T^m \ln P_\tau$,

where $P \in \mathrm{Sg}(\Omega)$, $J^m(x_0 \circ P) \in \mathrm{ri}(\mathcal{O}_+^m(\Omega))$.

The next proposition states even more, and the additional information will be used in the sequel.

PROPOSITION 8. *There exists a family of elements $\vartheta(\tau, s) \in \mathcal{L}(\Omega)$, polynomially depending on τ, s, such that*

$$\sigma\vartheta(\tau, s) = \vartheta(\tau, s), \quad \vartheta(\tau, s) \in \sum_i \mathcal{L}^{2i+1}(\Omega), \quad \forall \sigma \in \Sigma, \quad \tau, s \in \mathbb{R},$$

and for every $s > 0$ the relations

(1) $\Lambda(\vartheta(\cdot, s)) \in \mathrm{int}\, \mathcal{V}_+^m$,

(2) $\Lambda(\vartheta(\tau, s)) = T^m \ln P_\tau(s)$,

hold, where $P(s) \in \mathrm{Sg}(\Omega)$, $J^m(x_0 \circ P(s)) \in \mathrm{ri}(\mathcal{O}_+^m(\Omega))$.

Proof. First of all, Proposition 3.6 implies the existence of a $\widehat{P} \in \mathrm{Sg}(\Omega)$ such that

$$J^m(x_0 \circ \widehat{P}) \in \mathrm{ri}(\mathcal{O}_+^m(\Omega)), \qquad T^m \ln \widehat{P} \in \mathrm{int}\, \mathcal{V}_+^m. \tag{8}$$

Let

$$\widehat{P} = Q_{a_1}^1(s_1) \circ \cdots \circ Q_{a_l}^l(s_l), \quad a_i \in A_+^2, \quad s_i > 0,$$

then for $\forall s > 0$ the tame sequence of diffeomorphisms $\widehat{P}(s) = Q_{a_1}^1(ss_1) \circ \cdots \circ Q_{a_l}^l(ss_l)$ also satisfies the inclusions (8). Put

$$\widehat{\vartheta}(\tau, s) = T^m \ln \left(e^{\vartheta_1(\tau,s)} \ldots e^{\vartheta_l(\tau,s)}\right),$$

where

$$\Lambda(\vartheta_i(\tau, s)) = \ln Q_{a_i\tau}^i(ss_i), \quad i = 1, \ldots, l.$$

As above, let $J\colon \mathfrak{A}(\Omega) \longrightarrow \mathfrak{A}(\Omega)$ be an antiautomorphism, defined by the relation

$$J(\lambda_{Q_1}^{i_1 j_1} \ldots \lambda_{Q_k}^{i_k j_k}) = \lambda_{Q_k}^{i_k j_k} \ldots \lambda_{Q_1}^{i_1 j_1}.$$

We recall, cf. Proposition 4.2, that

$$J\vartheta = (-1)^{k-1}\vartheta, \quad \forall \vartheta \in \mathcal{L}^k(\Omega).$$

We have

$$J(e^{\vartheta}) = e^{J\vartheta}, \qquad J(e^{\vartheta_1}\dots e^{\vartheta_l}) = e^{J\vartheta_l}\dots e^{J\vartheta_1},$$

$$\Lambda\big(J\widehat{\vartheta}(\tau, s)\big) = T^m \ln\big(Q^l_{a_1\tau}(ss_l)\circ\cdots\circ Q^1_{a_1\tau}(ss_1)\big). \tag{9}$$

Furthermore, let $\sigma \in \operatorname{Aut}(\mathcal{P})$, then $\sigma(e^{\vartheta}) = e^{\sigma\vartheta}$ and

$$\Lambda\big(\sigma\widehat{\vartheta}(\tau,\tau)\big) = T^m \ln\big(\sigma(Q^1)_{a_1\tau}(ss_1)\circ\cdots\circ\sigma(Q^l)_{a_1\tau}(ss_l)\big). \tag{10}$$

Consider the group $G = \Sigma \cup J\Sigma$. Proposition 4.5 guarantees the existence of a sequence of elements of this finite group $g_1, \dots, g_n$ such that for

$$\vartheta(\tau, s) = T^m \ln\left(e^{\widehat{\vartheta}(\tau,s)}e^{g_1\widehat{\vartheta}(\tau,s)}\dots e^{g_n\widehat{\vartheta}(\tau,s)}\right)$$

the relations

$$g\vartheta(\tau, s) = \vartheta(\tau, s), \quad \forall g \in G,$$

hold. In particular, $\vartheta(\tau, s) \in \sum_i \mathcal{L}^{2i+1}(\Omega)$ since $J \in G$. Relations (9), (10) imply the relation $\Lambda(\vartheta(\tau, s)) = T^m \ln P_\tau(s)$ for some $P(s) \in \operatorname{Sg}(\Omega)$, which, together with $\widehat{P}$, satisfy the inclusions (8).

7. *Proof of Theorem* 4.2. Without loss of generality, we assume that ν_i are rationals, $0 < \nu_i < 1$. To begin with, we prove the theorem for a sequence consisting of a single system of vector fields $\underline{\mathcal{L}_\Sigma}$. Let

$$\theta = \frac{\nu}{1-\nu} = \frac{k}{l},$$

where k, l are positive integers. For all $p \in \mathcal{P}$, $n > 0$, $s \in \mathbb{R}$, we put

$$p^\nu_n(s) = \underbrace{p(n^{-k-l}s)\circ\cdots\circ p(n^{-k-l}s)}_{n^k \text{ times}}.$$

Then

$$\ln p^\nu_n(s) \approx n^{l\theta} \ln p(n^{-l-l\theta}s) = \sum_{i=1}^{\infty} n^{-\frac{l}{1-\nu}(i-\nu)} s^i \Lambda^i_p.$$

Consider the family of tame sequences of flows

$$\{p^\nu_n(\cdot)\}^\infty_{n=1}, \quad p \in \mathcal{P}, \tag{11}$$

which satisfies the conditions of Theorem 2 for all

$$m < \frac{l}{1-\nu}\, r\big(\underline{\mathcal{L}}_\Sigma(\nu)\big).$$

Hence, $R(\underline{\mathcal{L}}_\Sigma(\nu))$ is a reachable space for $\mathcal{P}$.

Let Ω be a family of tame sequences of flows. We shall say that Ω is a *$\mathcal{P}$-admissible family*, if for arbitrary $\{q_n\}_{n=1}^\infty \in \Omega$, $s > 0$, we have a representation

$$q_n(s) = p_1(s_1) \circ \cdots \circ p_{k_n}(s_{k_n}), \quad p_i \in \mathcal{P}, \qquad \sum_{i=1}^{k_n} s_i \longrightarrow 0 \quad (n \to \infty).$$

A typical example of a $\mathcal{P}$-admissible family is represented by the family (11).

We shall say that the system of vector fields $\underline{L} = (L, h, G)$ is *$\mathcal{P}$-admissible* if for all $N > 0$ there exist a $\mathcal{P}$-admissible family Ω of tame sequences of flows, a homomorphism Γ: $G_0 \to \operatorname{Aut}(\Omega)$, and $\alpha > 0$ such that

(a) $h(L^1 \cap L_{<r}) = \Lambda(\mathcal{L}^1(\Omega) \cap \mathcal{L}_{<\alpha r}(\Omega))$, $0 < r \leqslant N$,

(b) for arbitrary $\eta \in L$, $\vartheta \in \mathcal{L}(\Omega)$, which satisfy the relations $h(\eta) = \Lambda(\vartheta)$, $\forall g \in G_0$ the equality $h(g\eta) = \Lambda(\Gamma(g)\vartheta)$ holds.

A simple example of a $\mathcal{P}$-admissible system of fields is given by $\underline{\mathcal{L}}_\Sigma(\nu)$, where the family (11) is the corresponding $\mathcal{P}$-admissible family of tame sequences.

Let $\underline{L}$ be a $\mathcal{P}$-admissible system of fields, and Ω, Γ, α satisfy conditions (a), (b). It is easy to show that for all $k \leqslant \alpha N$, $l \geqslant 0$, and for an arbitrary $\Gamma(G_0)$-invariant element $\vartheta \in \mathcal{L}_k(\Omega) \cap \mathcal{L}^{2l+1}(\Omega)$ there exists a G-invariant element $\eta \in (L_{k/\alpha} + L_{<k/\alpha})$ such that $h(\eta) = \Lambda(\vartheta)$. Hence, Ω satisfies the conditions of Theorem 2 for $\Sigma = \Gamma(G_0)$ and $\forall m < r(\underline{L})$. Therefore, $R(\underline{L})$ is a reachable space for $\mathcal{P}$.

To complete the proof of Theorem 4.2, it is sufficient to show that every system of vector fields induced by a $\mathcal{P}$-admissible system is itself $\mathcal{P}$-admissible and that the $\mathcal{P}$-admissibility of the system $\underline{L}$ implies the $\mathcal{P}$-admissibility of the system $\underline{L}(\mu)$ for every positive rational

$$\mu < \inf\left\{\frac{t}{n} \mid L^n \cap L_t \neq 0\right\}.$$

Let

$$\theta = \frac{\mu}{1-\mu} = \frac{k}{l}.$$

Suppose that a $\mathcal{P}$-admissible family of tame sequences of flows Ω satisfies conditions (a), (b) for some N, α, Γ. For all $Q = \{q_n\}_{n=1}^\infty \in \Omega$ we put

$$Q_\mu = \Big\{\underbrace{q_{n^{k+l}} \circ \cdots \circ q_{n^{k+l}}}_{n^k \text{ times}}\Big\}.$$

It is easy to show that the $\mathcal{P}$-admissible family of tame sequences $\Omega_\mu = \{Q_\mu \mid Q \in \Omega\}$ satisfies conditions (a), (b), if we substitute N by $(1-\mu)N$, $\underline{L}$ by $\underline{L}(\mu)$, α by $l\alpha/(1-\mu)$, and if we consider instead of Γ the homomorphism $\Gamma_\mu(g)\colon G_0 \to \operatorname{Aut}(\Omega_\mu)$, where

$$\Gamma_\mu(g)\colon Q_\mu \longmapsto (\Gamma(g)Q)_\mu, \quad \forall g \in G_0, \quad Q \in \Omega.$$

Finally, let the system of vector fields $\underline{\widetilde{L}} = (\widetilde{L}, \widetilde{h}, \widetilde{G})$ be induced by $\underline{L}$, i.e., there exists a linear mapping $\Phi\colon \widetilde{L}^1 \hookrightarrow L$ and a homomorphism $\varphi\colon \widetilde{G}_0 \hookrightarrow G$, such that

$$\Phi \circ \widetilde{g} = \varphi(\widetilde{g}) \circ \Phi, \quad \forall \widetilde{g} \in \widetilde{G}_0, \quad \widetilde{h}\big|_{\widetilde{L}^1} = h \circ \Phi.$$

and

$$\operatorname{Inv}(\underline{L}) \cap L_{<r} \subset \Phi\big(\widetilde{L}^1 \cap \widetilde{L}_{<r}\big) \subset L_{<r}, \quad \forall r > 0. \tag{12}$$

As above, we suppose that Ω is a $\mathcal{P}$-admissible family of tame sequences of flows satisfying conditions (a), (b). Let $\widetilde{\Omega}$ be the set of all tame sequences of flows $\widetilde{P} = \{\widetilde{p}_n(\cdot)\}_{n=1}^\infty$ such that

$$\widetilde{P}(s) = \{\widetilde{p}_n(s)\}_{n=1}^\infty \in \operatorname{Sg}(\Omega)$$

and

$$T^m \ln \widetilde{P}_\tau(s) \in \widetilde{h}(\widetilde{L}^1), \quad \forall s > 0, \quad \tau \in \mathbb{R}, \quad m \leqslant \alpha N.$$

Consider the space of vector polynomials

$$\mathcal{W} = \left\{ \sum_{i=1}^{m} \tau^i \widetilde{h}(\vartheta_i) \,\middle|\, \vartheta_i \in \widetilde{L}^1 \cap \widetilde{L}_{<r} \text{ for } i < \alpha r \right\}.$$

Proposition 8, inclusions (12), and the equality $\widetilde{h}|_{\widetilde{L}^1} = h \circ \Phi$, imply that the set

$$\{T^m \ln \widetilde{P}_\tau(s) \mid \widetilde{P} \in \widetilde{\Omega}, \ s > 0\} \subset \mathcal{W}$$

has interior points in $\mathcal{W}$. Hence,

$$\widetilde{h}\big(\widetilde{L}^1 \cap \widetilde{L}_{<r}\big) = \Lambda\big(\mathcal{L}^1(\widetilde{\Omega}) \cap \mathcal{L}_{<\alpha r}(\widetilde{\Omega})\big), \quad 0 < r \leqslant N.$$

Put

$$\mathfrak{I} = \big\{(Q^1, \dots, Q^k; a_1, \dots, a_k) \mid Q_i \in \Omega, \ a_i \in A_+^2, \ k > 0; \ Q^1_{a_1} \circ \dots \circ Q^k_{a_k} \in \widetilde{\Omega}\big\}$$

and denote

$$P_I = Q^1_{a_1} \circ \dots \circ Q^k_{a_k} \quad \text{for} \quad I = (Q^1, \dots, Q^k; a_1, \dots, a_k) \in \mathfrak{I}.$$

Then $\widetilde{\Omega} = \{P_I \mid I \in \mathfrak{I}\}$. In other words, the family $\widetilde{\Omega}$ is indexed by the set $\mathfrak{I}$. At this place, we have to distinguish between the set of tame of sequences and the family of tame sequences. The difference arises when the elements corresponding to different values of the index coincide.

Let

$$\sigma \in \Gamma \circ \varphi(\widetilde{G}_0), \qquad I = (Q^1, \dots, Q^k; a_1, \dots, a_k) \in \mathfrak{I},$$

then $\sigma I = (\sigma(Q^1), \ldots, \sigma(Q^k); a_1, \ldots, a_k)$ also belongs to I. This follows from the fact that the group $\varphi(\widetilde{G}_0)$ preserves $\Phi(\widetilde{L}^1)$. We obtain a homomorphism $\Gamma \circ \varphi\colon \widetilde{G_0} \to \mathrm{Aut}(\widetilde{\Omega})$, $\Gamma \circ \varphi(g)\colon P_I \mapsto P_{\Gamma \circ \varphi(g) I}$ which satisfies condition (b).

Thus, $\widetilde{L}$ is a $\mathcal{P}$-admissible system of fields, where $\widetilde{\Omega}$ is the corresponding $\mathcal{P}$-admissible family of tame sequences of flows.

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