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Symplectic Geometry for Optimal Control

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1 INTRODUCTION

One of the main invariants of an extremal in a regular variational problem is its *Morse index*. If the extremal is optimal then its Morse index is zero. In the general case the Morse Index could be interpreted as the minimal number of (independent) additional relations that have to be satisfied by the admissible variations of the given trajectory in order to make it optimal.

It turns out that there an analogue of this index in optimal control. If the set of control parameters is open then the index is easy to define, though much harder to compute. For strongly nondegenerate cases the analogue of the Morse formula was obtained in [8] and [10]. The systematic use of symplectic geometry offers a different way of computing the index, stable under practically any perturbation, thus closing the problem for singular extremals, cf. [4], [5].

In this paper we employ the latter method and compute the index for the problem with constraints on the control parameters. We consider in some detail bang-bang controls and then briefly present a universal formula valid for bang-bang as well as singular parts of the optimal trajectory.

We apply our results to the study of optimal control problems for smooth systems. Necessary conditions for optimality are formulated for controls (including bang-bang controls) that satisfy Pontryagin's Maximum Principle. As an example, we consider the case of a rigid body which is controlled by rotating it with a given velocity around two fixed axes.

2 NONSTATIONARY VECTOR FIELDS AND FLOWS

Let M be a C^∞-manifold, f a complete vector field on M, $e^{tf}, t \in \mathbf{R}$ the corresponding one-parameter group of diffeomorphisms. The values of the field f and of the diffeomorphism e^{tf} at the point $\mu \in M$ will be denoted respectively by $\mu \circ f$ and $\mu \circ e^{tf}$. (Though somewhat unorthodox, these notations will turn out to be very convenient for our purposes, as will be seen later.) The field f acts as a first order differential operator on functions $a \in C^\infty(M)$, and the result of this action is denoted by fa. For a pair of fields f, g we form their commutator (Lie bracket) according to the formula $[f,g]a = (ad_f g)a = f(ga) - g(fa)$. The diffeomorphism e^{tf} also acts on smooth functions: $(e^{tf}a)(\mu) \stackrel{\mathrm{def}}{=} (\mu \circ e^{tf})$. Thus we consider vector fields and diffeomorphisms as $\mathbf{R}$-linear operators of special types on the space $C^\infty(M)$. It is also evident that the composition $g_t = e^{tf} \circ g \circ e^{-tf}$ is a vector field. For an arbitrary $\mu \in M$ we have the equation

$$\tfrac{d}{dt}\mu \circ g_t = \mu \circ ad_f g_t, \qquad g_0 = g,$$

which justifies the notation $e^{tf} \circ g \circ e^{-tf} = e^{t\,\mathrm{ad}_f} g$. By a *nonstationary vector field* on M we mean an arbitrary family $f_t, t \in \mathbf{R}$, of smooth vector fields f_t such that for $\mu \in M$ the composition $\mu \circ f_t$ is locally integrable in t. The differential equation on M defined by a nonstationary field $f_t, t \in \mathbf{R}$ can be written as

$$\frac{d\mu(t)}{dt} = \mu(t) \circ f_t$$

A nonstationary field f_t is called *complete* if there exists a family p_t, $t \in \mathbf{R}$, of diffeomorphisms of M, absolutely continuous in t, such that

$$\forall \mu \in M \qquad \tfrac{d}{dt}\mu \circ p_t = \mu \circ p_t \circ f_t, \qquad \mu \circ p_0 = \mu.$$

The family $p_t, t \in \mathbf{R}$, is unique and is called the *flow* generated by f_t. We denote it by $p_t = \overrightarrow{\exp} \int_0^t f_\tau \, d\tau$. Suppose f_τ is defined for $\tau \in [0,t]$ and is piecewise constant in τ, i.e. $f_\tau = f_i$ for $t_i < \tau \le t_{i+1}$, where $0 = t_0 < t_1 < \cdots < t_l \le t_{l+1} = t$. Then $\overrightarrow{\exp} \int_0^t f_\tau \, d\tau = e^{t_1 f_0} \circ e^{(t_2 - t_1) f_1} \ldots \circ e^{(t - t_l) f_l}$. For a vector field g the family of fields $p_t \circ g \circ p_t^{-1}$ satisfies the equation

$$\frac{d}{dt}(p_t \circ g \circ p_t^{-1}) = p_t \circ ad_{f_t} g \circ p_t^{-1}$$

which justifies the notation $p_t \circ g \circ p_t^{-1} = \overrightarrow{\exp} \int_0^t ad_{f_\tau} \, d\tau g$.

The following result can be considered as a generalization to the nonstationary case of the formula

$$e^{t(f+g)} = \overrightarrow{\exp} \int_0^t e^{\tau \operatorname{ad}_f} g \, d\tau \circ e^{tf}.$$

Proposition 1. Let $f_t, f_t + g_t, t \in \mathbf{R}$, be nonstationary complete vector fields. Then the field $\overrightarrow{\exp} \int_0^t ad_{f_\tau} \, d\tau g_t$, $t \in \mathbf{R}$, is also complete and the following "variational formula" (cf. [2]) holds:

$$\overrightarrow{\exp} \int_0^t (f_\tau + g_\tau) \, d\tau = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau ad_{f_\theta} \, d\theta g_\tau \right) d\tau \circ \overrightarrow{\exp} \int_0^t f_\tau \, d\tau. \quad (1)$$

In the sequel we shall restrict ourselves to flows p_t for nonnegative values of t.

We now define a remarkable action of the group of absolutely continuous substitutions of time which plays a key role in the following. Let $\varphi : [0, +\infty) \to [0, +\infty)$ be an absolutely continuous one-to-one transformation of the time half-axis and $p_t = \overrightarrow{\exp} \int_0^t f_\tau \, d\tau, t \ge 0$. Define $(\varphi * p)_t = \overrightarrow{\exp} \int_0^{\varphi(t)} f_{\varphi^{-1}(\tau)} \, d\tau$, and let $\frac{d\varphi(t)}{dt} = 1 + \alpha(t), t \ge 0$. One can easily see that

$$(\varphi * p) = \overrightarrow{\exp} \int_0^t (1 + \alpha(\tau)) f_\tau \, d\tau. \quad (2)$$

We may start with $\alpha(t)$ rather than with $\varphi(t)$: if $\alpha(t)$ is locally integrable on $[0, +\infty)$, and $\alpha(t) \ge \epsilon - 1 > -1$, then the mapping $t \to t + \int_0^t \alpha(\tau) \, d\tau$ of $[0, +\infty)$ is one-to-one and absolutely continuous. From (2) we obtain that the field $(1 + \alpha(t)) f_t$ is complete if f_t is. For every $t > 0$ let Ω_t be the open set in L_∞ of all functions α such that $\alpha(\tau) \ge \epsilon - 1$ for every $\tau \in [0, t)$ and some $\epsilon > 0$.

Fix a point $\mu_0 \in M$ and define a family of mappings $F_t : \Omega_t \to M$ by

$$F_t : \alpha \longrightarrow \mu_0 \circ \overrightarrow{\exp} \int_0^t (1 + \alpha(\tau)) f_\tau \, d\tau.$$

The mapping F_t is infinitely differentiable from Ω_t to M (cf. [2]), and $F_t(0) = \mu_0 \circ p_t$. In many respects the mapping $G_t : \alpha \to F_t(\alpha) \circ p_t^{-1}$ is more suitable for our purposes than F_t is. Put $h_t = \overrightarrow{\exp} \int_0^\tau ad_{f_\theta}\, d\theta f_\tau$. The variation formula (1) gives

$$G_t(\alpha) = \mu_0 \circ \overrightarrow{\exp} \int_0^t \alpha(\tau) h_\tau \, d\tau, \qquad G_t(0) = \mu_0. \tag{3}$$

We shall need the first and second differentials of G_t at zero. Denote them respectively by

$$G_t' : L_\infty[0,t] \longrightarrow T_{\mu_0}M,$$
$$G_t'' : \ker G_t' \times \ker G_t' \longrightarrow T_{\mu_0}M.$$

Here G_t' is linear, G_t'' is symmetric bilinear and $\ker G_t'$ denotes the kernel of G_t'. We emphasize that the second differential G_t'' is invariantly defined (independently of local coordinates on M) since we consider it only on $\ker G_t'$. For an arbitrary $a \in C^\infty(M)$ we have

$$\begin{aligned}
&\left(\overrightarrow{\exp} \int_0^t \alpha(\tau) h_\tau \, d\tau\right) a \\
&= a + \int_0^t \left(\overrightarrow{\exp} \int_0^\tau \alpha(\theta) h_\theta \, d\theta \circ h_\tau a\right) d\tau \\
&= a + \int_0^t \alpha(\tau) h_\tau a \, d\tau + \int_0^t \left(\int_0^\tau \alpha(\theta) h_\theta \, d\theta\right) \circ \alpha(\tau) h_\tau a \, d\tau \\
&\quad + O\left(\int_0^t |\alpha(\tau)| \, d\tau\right)^3 \\
&= a + \int_0^t \alpha(\tau) h_\tau a \, d\tau + \tfrac{1}{2} \int_0^t \left(\left[\int_0^\tau \alpha(\theta) h_\theta \, d\theta, \alpha(\tau) h_\tau\right] a \, d\tau\right. \\
&\quad + \tfrac{1}{2} \left(\int_0^t \alpha(\tau) h_\tau \, d\tau\right)^2 a + O\left(\int_0^t |\alpha(\tau)| \, d\tau\right)^3.
\end{aligned}$$

Therefore, according to (3)

$$\forall \alpha \in L_\infty[0,t], \qquad G_t'\alpha = \int_0^t \alpha(\tau) \mu_0 \circ h_\tau \, d\tau.$$

$$\forall \alpha_1, \alpha_2 \in \ker G_t', \qquad G_t''(\alpha_1, \alpha_2) = \int_0^t \mu_0 \circ \left[\int_0^\tau \alpha_1(\theta) h_\theta \, d\theta, \alpha_2(\tau) h_\tau\right] d\tau. \tag{4}$$

3 AN OPTIMALITY CONDITION

We now turn to optimal control, and consider the time-optimal problem for the system

$$\dot{\mu} = \mu \circ f(u), \qquad \mu \in M, \qquad \mu(0) = \mu_0, \qquad u \in U \subset \mathbb{R}^r. \tag{5}$$

where the family of smooth vector fields $f(u)$ is continuous in U. As admissible controls we take arbitrary functions belonging to $L^r_\infty[0, +\infty)$ with values in U, and suppose that for every such control $u(.)$ the nonstationary field $f(u(\tau)), \tau \geq 0$, is complete. Since all subsequent considerations are local this assumption does not restrict generality and strongly simplifies notations.

Fix an admissible control $\tilde{u}$ and write $f_\tau = f(\tilde{u}(\tau))$. Let $\tilde{\mu}$ be the trajectory corresponding to $\tilde{u}$: $\frac{d\tilde{\mu}(\tau)}{d\tau} = \tilde{\mu}(\tau) \circ f_\tau$, $\tilde{\mu}(0) = \mu_0$. Suppose that the trajectory $\tilde{\mu}$ satisfies on $[0, t]$ the Maximum Principle, which we formulate in the following form:

There exists a nonzero covector $\psi_0 \in T^*_{\mu_0}M$ such that $\langle \psi_0, \mu_0 \circ f_0 \rangle \leq 0$ and for almost all τ

$$\left\langle \psi_0, \mu_0 \circ \exp \int_0^\tau ad_{f_\theta}\, d\theta (f_\tau - f(u)) \right\rangle \leq 0, \qquad u \in U. \tag{6}$$

The Maximum Principle is the main necessary condition for optimality of a trajectory leading from μ_0 to $\tilde{\mu}(t)$. There are many different ways of obtaining additional optimality conditions. A nontrivial condition could be obtained for arbitrary set $U \subset \mathbb{R}^r$ if one considers the action of the group of time substitutions considered in §2. Indeed, if the trajectory $\tilde{\mu}(\tau), 0 \leq \tau \leq 0$, is optimal for (5) then it is also optimal for the system

$$\dot{\mu} = (1 + \alpha) f_t, \qquad \mu(0) = \mu_0, \qquad \epsilon - 1 \leq \alpha, \qquad \int_0^t \alpha(\tau)\, d\tau \leq 0.$$

Using (3) and the Maximum Principle (6) we obtain

$$\langle \psi_0, \mu_0 \circ h_\tau \rangle = \left\langle \psi_0, \mu_0 \circ \overrightarrow{\exp} \int_0^\tau ad_{f_\theta}\, d\theta f_\tau \right\rangle = \langle \psi_0, \mu_0 \circ f_0 \rangle \leq 0, \tau \in [0, t],$$

and therefore $\langle \psi_0, G'_t \rangle = \int_0^t \langle \psi_0, \mu_0 \circ h_\tau \rangle\, \alpha(\tau)\, d\tau \geq 0$ for every admissible α.

Further optimality conditions could be obtained by considering the second differential G''_t. The nature of these conditions depends heavily on the set of covectors ψ_0 satisfying the Maximum Principle conditions

(6). If ψ_0 is uniquely determined up to a scalar multiple, then an obvious necessary condition for the optimality of $\tilde{\mu}$ is given by the inequality

$$\left\langle \psi_0, G_t''(\alpha,\alpha)\right\rangle \geq 0, \qquad \forall \alpha \in \ker G_t'. \tag{7}$$

which is almost evident. If the linear hull of the covectors ψ_0 satisfying (6) is two dimensional, then the optimality of $\tilde{\mu}$ implies that there exists a nonzero covector for which both the Maximum Principle and inequality (7) hold. This assertion is not so obvious and is a consequence of the fact that the image in $\mathbf{R}^2$ of any quadratic mapping is convex. But if this dimension is more than two then the situation is much more complicated: inequality (7) might be violated for all ψ_0 even for the optimal $\tilde{\mu}$. To obtain appropriate optimality conditions in this case one is led to describing the dependence on ψ_0 of the index of the quadratic form $\left\langle \psi_0, G_t''(\alpha,\alpha)\right\rangle$. We cannot dwell here on the relevant theory, which derives conditions for solving general systems of quadratic equations and inequalities, cf. [5], [6]. For our purposes it is enough to suppose uniqueness of ψ_0 (up to scalar multiples). What has been said can be summarized in the following.

Proposition 2. If the admissible trajectory $\tilde{\mu}(\tau)$, $0 \leq \tau \leq t$, of (5) is time-optimal, then there exists a nonzero covector $\psi_0 \in T^*_{\mu_0}M$ such that for almost all τ

$$\begin{aligned}\langle \psi_0, \mu_0 \circ f_0\rangle \leq 0, \qquad \langle \psi_0, \mu_0 \circ h_\tau\rangle = \\ = \min_{u\in U}\left\langle \psi_0, \mu_0 \circ \overrightarrow{\exp}\int_0^\tau ad_{f_\theta}\, d\theta f(u)\right\rangle\end{aligned} \tag{8}$$

In the case of a unique ψ_0 (determined by (8) up to a scalar multiple) the following additional relation holds.

$$\left\langle \psi_0, G_t''(\alpha,\alpha)\right\rangle = \int_0^t \left\langle \psi_0, \mu_0 \circ \left[\int_0^\tau \alpha(\theta)h_\theta d\theta, \alpha(\tau)h_\tau\right]\right\rangle d\tau \geq 0 \tag{9}$$

$\forall\alpha$ such that $\int_0^t \alpha(\tau)\mu_0 \circ h_\tau d\tau = 0$, where $h_\tau = \overrightarrow{\exp}\int_0^\tau ad_{f_\theta} d\theta f_\tau$.

The given optimality condition is particularly effective for the case of piecewise constant (e.g. bang-bang) controls $\tilde{u}$. In this case the action of the group of time substitutions is reduced to variations of the switching times. Let $\tilde{u}(\tau) = u_i$ for $t_i < \tau \leq t_{i+1}$, where $0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = t$; $f_i = f(u_i)$. Then using the obvious identity $e^{ad_{f_i}} f_i = f_i$, we obtain for

$t_i < \tau \le t_{i+1}$, $i = 0, 1, \ldots, l$,

$$h_\tau = \overrightarrow{\exp} \int_0^{t_i} ad_{f_\theta}\, d\theta f_i = e^{t_1 ad_{f_0}} \circ e^{(t_2-t_1)ad_{f_1}} \circ \cdots e^{(t_i-t_{i-1})ad_{f_{i-1}}} f_i.$$

Thus $h_\tau = h_i$ does not depend on τ on the interval $t_i < \tau \le t_{i+1}$. Let $\alpha_i = \int_{t_i}^{t_{i+1}} \alpha(\tau)d\tau$. Then

$$\left\langle \psi_0, G_t''(\alpha,\alpha) \right\rangle = \sum_{i=0}^{l} \left\langle \psi_0, \mu_0 \circ \left[\sum_{j=0}^{i-1} \alpha_j h_j, \alpha_i h_i \right] \right\rangle, \qquad \sum_{i=0}^{l} \alpha_i \mu_0 \circ h_i = 0. \tag{10}$$

Certainly Proposition 2 has little value if there is no effective procedure for establishing the non-negativity of the quadratic form (9). Actually, we can even solve a more general problem, namely, that of computing the index of this form (i.e. the maximal dimension of a subspace on which the form is negative).

Equality (9) explicitly expresses the form $\left\langle \psi_0, G_t'' \right\rangle$ through the vector fields $h_\tau, 0 \le \tau \le t$. In fact, in this formula only 1-jets of h_τ at μ_0 are used (and even these only partially). Our next goal is the elimination of the unnecessary parameters.

Let Σ denote the quotient space of the space Vect(M) of all smooth vector fields on M, modulo the kernel of the skew-symmetric form

$$(g_1, g_2) \longrightarrow \langle \psi_0, \mu_0 \circ [g_1, g_2] \rangle, \qquad g_1, g_2 \in \mathrm{Vect}(M). \tag{11}$$

Let σ be the nonsingular skew symmetric form on Σ induced by (11). The pair (Σ, σ) constitutes (by definition) a symplectic space. Let $a \in C^\infty(M)$ be such that the differential of a at μ_0 satisfies $d_{\mu_0} a = \psi_0$. It is easy to show that the kernel of the form (11) coincides with the kernel of the linear mapping $g \mapsto (\mu_0 \sqrt{g}, d_{\mu_0}(ga))$ from Vect(M) to $T_{\mu_0}M \oplus T^*_{\mu_0}M$. Furthermore,

$$\langle \psi_0, \mu_0 \circ [g_1, g_2] \rangle = \left\langle d_{\mu_0}(g_2 a), \mu_0 \circ g_1 \right\rangle - \left\langle d_{\mu_0}(g_1 a), \mu_0 \circ g_2 \right\rangle,$$

so that the indicated mapping induces an isomorphism of the symplectic space (Σ, σ) onto $T_{\mu_0}M \oplus T^*_{\mu_0}M$ with the standard skew-scalar product

$$\langle x_1 \oplus \xi_1, x_2 \oplus \xi_2 \rangle \longrightarrow \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle, x_i \in T_{\mu_0}M, \xi_i \in T^*_{\mu_0}M.$$

Let $\dim M = m$, hence $\dim \Sigma = 2m$. The space Σ contains a particular m-dimensional subspace Π_0—the image under the canonical projection $\mathrm{Vect}(M) \to \Sigma$ of all smooth vector fields vanishing at μ_0. The image of the vector field $h_\tau, 0 \leq \tau \leq t$, under this factorization will also be denoted by h_τ. Finally, we introduce one more notation to simplify formulas: we let $Q(\alpha) = \left\langle \psi_0, G_t''(\alpha, \alpha) \right\rangle$, $\alpha \in \ker G_t'$. We have $Q(\alpha) = \int_0^t \sigma(\int_0^\tau h_\theta \alpha(\theta) d\theta, h_\tau \alpha(\tau)) d\tau$ $\int_0^t h_\tau \alpha(\tau) d\tau \in \Pi_0$. For piecewise constant h_τ (cf. (10)), the resulting expression reduces to

$$Q(\alpha) = \sum_{i=1}^{l} \sigma\left(\sum_{j=0}^{i-1} h_j \alpha_j, h_i \alpha_i\right), \qquad \sum_{i=0}^{l} h_i \alpha_i \in \Pi_0. \tag{12}$$

4 REVIEW OF SYMPLECTIC GEOMETRY

The form Q is explicitly expressed through the skew scalar product σ. Therefore the investigation of Q is a problem in symplectic geometry. We shall indicate here some necessary definitions and results from this subject. (A detailed discussion can be found in [9].)

Let S be a subset of Σ. We denote by $S^\perp$ the skew-orthogonal complement to S, $S^\perp = \{x \in \Sigma : \sigma(x, s) = 0 \ \forall s \in S\}$. A subspace $\Gamma \subset \Sigma$ is called *isotropic* if $\Gamma \subset \Gamma^\perp$, and *Lagrangian* if $\Gamma = \Gamma^\perp$. Since Σ is nondegenerate we have $\dim \Gamma^\perp = \operatorname{codim} \Gamma = 2m - \dim \Gamma$. In particular, an arbitrary Lagrangian subspace is of dimension m. Note that the subspace $\Pi_0 \subset \Sigma$ defined in §3 is Lagrangian.

The set $L(\Sigma) = \{\Lambda \subseteq \Sigma; \Lambda^\perp = \Lambda\}$ of all Lagrangian subspaces constitutes a submanifold of the Grassmann manifold of all m-dimensional subspaces in Σ. We will call it the *Lagrange Grassmannian.*

A linear operator $A : \Sigma \to \Sigma$ is called *symplectic* if $\sigma(Ax, Ay) = \sigma(x, y)$ $\forall x, y \in \Sigma$. The set of all symplectic operators constitutes the symplectic group $\mathrm{Sp}(\Sigma)$, which acts transitively on $L(\Sigma)$. Furthermore, the only symplectic invariant of a pair of Lagrangian subspaces Λ_1, Λ_2 is the dimension of their intersection. This means that, if Λ_1', Λ_2' is any other such pair, and $\dim(\Lambda_1 \cap \Lambda_2) = \dim(\Lambda_1' \cap \Lambda_2')$, then there exists $A \in \mathrm{Sp}(\Sigma)$ such that $A\Lambda_1 = \Lambda_1', A\Lambda_2 = \Lambda_2'$. A complete set of invariants of a triple of Lagrangian subspaces $\Lambda_1, \Lambda_2, \Lambda_3$ is given by the dimensions $\dim(\Lambda_1 \cap \Lambda_2 \cap \Lambda_3)$, $\dim(\Lambda_i \cap \Lambda_j)$, $1 \leq i < j \leq 3$, and the *Maslov index* $\mu(\Lambda_1, \Lambda_2, \Lambda_3)$, which can be defined in the following way: $\mu(\Lambda_1, \Lambda_2, \Lambda_3)$ is equal to the signature of the quadratic form $\lambda_1 \oplus \lambda_2 \oplus \lambda_3 \to \sigma(\lambda_1, \lambda_2) + \sigma(\lambda_2, \lambda_3) + \sigma(\lambda_3, \lambda_1)$, defined on the 3-dimensional space $\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$. (We recall that the signature of a quadratic form is defined as the number of positive members minus the number of negative members in the diagonal form.) Evidently, $\mu(\Lambda_1, \Lambda_2, \Lambda_3)$ is skew-symmetric in all three arguments.

To clarify the above relations suppose Λ_0, Λ_1 is a pair of transversal Lagrangian subspaces, hence $\Lambda_0 \cap \Lambda_1 = 0$. We can suppose $\Sigma = \mathbf{R}^m \oplus \mathbf{R}^{m*}, \sigma(q_1 \oplus p_1, q_2 \oplus p_2) = \langle p_2, q_1 \rangle - \langle p_1, q_2 \rangle, \Lambda_0 = \mathbf{R}^m \oplus 0, \Lambda_1 = 0 \oplus \mathbf{R}^{m*}$. Then any m-dimensional subspace $\Delta \subset \Sigma$ transversal to Λ_1 is represented as $\Delta = \{q \oplus S_\Delta\, q : q \in \mathbf{R}^m\}$, where $S_\Delta : \mathbf{R}^m \to \mathbf{R}^{m*}$ is linear. The subspace Δ is Lagrangian iff $S_\Delta = S_\Delta^*$, hence $\dim L(\Sigma) = \frac{m(m+1)}{2}$. Furthermore, any symplectic transformation leaving Λ_0 and Λ_1 unchanged has the form $(q,p) \to (A^{-1}q, A^*p)$, where $A \in GL(\mathbf{R}^m)$. The subspace Δ, characterized by the mapping $S_\Delta : \mathbf{R}^m \to \mathbf{R}^{m*}$, is transformed by this correspondence into the subspace characterized by the mapping $A^* S_\Delta A$. The Maslov index $\mu(\Lambda_0, \Delta, \Lambda_1)$ is equal to the signature of the quadratic form $q \to \langle -S_\Delta q, q \rangle$, $q \in \mathbf{R}^m$. The nontransversal case is also quite simple.

We now return to coordinate-free language. Let $\Sigma \supset \Gamma$ be an isotropic subspace. Denote by $L_\Gamma(\Sigma)$ the set of all Lagrangian subspaces in Σ containing Γ. Passing to the skew-orthogonal complement in the relation $\Gamma \subset \Lambda$ we obtain $\Lambda \subset \Gamma^\perp\ \forall \Lambda \in L_\Gamma(\Sigma)$. It is easy to show that the factorization $\Gamma^\perp \to \Gamma^\perp/\Gamma$ induces a one-to-one mapping of $L_\Gamma(\Sigma)$ on $L(\Gamma^\perp/\Gamma)$. In particular, $L_\Gamma(\Sigma)$ is a closed smooth submanifold in $L(\Sigma)$. We project $L(\Sigma)$ onto the submanifold $L_\Gamma(\Sigma)$, assigning to $\Lambda \in L(\Sigma)$ the Lagrangian subspace $\Lambda^\Gamma = \Lambda \cap \Gamma^\perp + \Gamma$. In general the mapping $\Lambda \to \Lambda^\Gamma$ is discontinuous, but it is smooth on submanifolds $\{\Lambda \in L(\Sigma) : \dim(\Lambda \cap \Gamma) = \text{const}\}$. If $\gamma \in \Sigma$ we shall use the abbreviated notation $\Lambda^\gamma =^{\text{def}} \Lambda^{\mathbf{R}\gamma}$. (It is clear that every one-dimensional subspace is isotropic.) Note that $\dim(\Lambda \cap \Lambda^\gamma) = m-1$ $\forall \gamma \in \Sigma \setminus \Lambda$.

The manifold $L(\Sigma)$ is particularly simple if $\dim \Sigma = 2$. In this case Lagrangian subspaces coincide with arbitrary one-dimensional subspaces and the group $\mathrm{Sp}(\Sigma)$ consists of all linear transformations with unit determinant. Thus $L(\Sigma)$ is diffeomorphic to the circle S^1. The skew scalar product defines a definite orientation of S^1 : for sufficiently close nonzero $\gamma, \gamma' \in \Sigma$ the line $\mathbf{R}\gamma'$ is supposed to be to the positive side of $\mathbf{R}\gamma$ if $\sigma(\gamma', \gamma) > 0$. The Maslov index in this case is an invariant of a triple of points on the oriented circle. Let s_1, s_2, s_3 be three different points on S^1 and suppose that as we move along S^1 in the positive direction from some initial point we meet the selected points in the sequence $s_{i_1}, s_{i_2}, s_{i_3}$. The permutation (i_1, i_2, i_3) depends on the choice of the initial point, but its parity does not. The Maslov index $\mu(s_1, s_2, s_3)$ is equal to 1 or -1 depending on whether the permutation is even or odd.

Finally let us define (for an arbitrary dimension m) a half-integer valued invariant of a triple of Lagrangian subspaces $\Lambda_0, \Lambda_1, \Lambda_2$ by

$$\operatorname{ind}_{\Lambda_0}(\Lambda_1, \Lambda_2) = \tfrac{1}{2}(\mu(\Lambda_0, \Lambda_1, \Lambda_2) - \dim(\Lambda_1 \cap \Lambda_2) + m).$$

One can show that $0 \leq \operatorname{ind}_{\Lambda_0}(\Lambda_1, \Lambda_2) \leq m - \dim(\Lambda_1 \cap \Lambda_2)$. A crucial property of this invariant is the triangle inequality for $\Lambda_i \in L(\Sigma), i = 0, 1, 2, 3$ (cf. [5]):

$$\operatorname{ind}_{\Lambda_0}(\Lambda_1, \Lambda_3) \leq \operatorname{ind}_{\Lambda_0}(\Lambda_1, \Lambda_2) + \operatorname{ind}_{\Lambda_0}(\Lambda_2, \Lambda_3). \tag{13}$$

5 THE INDEX OF THE FORM Q

We now describe an explicit formula for calculating the index $\operatorname{ind} Q$ of the quadratic form $Q(\alpha)$ (which in the finite dimensional case coincides with the number of negative squares in the diagonal form). Proposition 2 proves that the equality $\operatorname{ind} Q = 0$ is necessary for the optimality of $\tilde{u}$. As indicated, this condition is close to be sufficient when $\tilde{u}$ is bang-bang, but in the general situation it evidently needs further strengthening. Therefore in this section we shall consider only the case of a piecewise-constant h_τ, cf. (12). Stronger conditions for optimality of a control that contains both piecewise-constant and singular parts will be considered in §7.

Theorem 1. Let $0 = t < t_1 < \cdots < t_l < t_{l+1} = t$ and $h_\tau = h_i$ on $t_i < \tau \leq t_{i+1}$. Let $\Lambda_{-1} = \Lambda_{l+1} = \Pi_0 \in L(\Sigma), \Lambda_i = \Lambda_{i-1}^{h_i}$, $i = 0, 1, \ldots, l$. Then for the quadratic form $Q(\alpha)$ given by (12) we have

$$\operatorname{ind} Q = \sum_{i=0}^{l+1} \operatorname{ind}_{\Pi_0}(\Lambda_{i-1}, \Lambda_i) + \dim\left(\bigcap_{i=0}^{l+1} \Lambda_i\right) - m. \tag{14}$$

The theorem is proved by induction. During the n-th step we compute the index of $Q(\alpha)$, restricted to the subspace of admissible $\alpha = (\alpha_0, \ldots, \alpha_l)$

such that $\alpha_i = 0$ for $i > n$. The induction step uses the following elementary fact from linear algebra. Let E be a finite dimensional vector space, q a nondegenerate symmetric bilinear form on E. For an arbitrary subspace $V \subset E$ let the index of q, restricted to V, be denoted by $\operatorname{ind}_V q$, and let $V_q^1 = \{e \in E : q(e, V) = 0\}$. Then $\operatorname{ind}_E q = \operatorname{ind}_V q + \operatorname{ind}_{V_q^1} q + \dim(V \cap V_q^1)$.

Note that the quantity l in Theorem 1 denotes the number of switchings of the bang-bang control $\tilde{u}$. From §4 it follows that each of the $l+2$ terms under the summation symbol in (14) is equal to 0, 1/2, or 1. The triangle inequality (13) permits, by reducing the number of summands, to estimate $\operatorname{ind} Q$ from below. For example, $\operatorname{ind}_{\Pi_0}(\Lambda_{i-1}, \Lambda_i) + \operatorname{ind}_{\Pi_0}(\Lambda_i, \Lambda_{i+1}) \geq \operatorname{ind}_{\Pi_0}(\Lambda_{i-1}, \Lambda_{i+1})$. For certain nondegeneracy conditions formula (14) might be even more effective. For example, if $\sigma(h_{i-1}, h_i) \neq 0$, then the symplectic transformation $\lambda \to \lambda + \frac{\sigma(h_i, \lambda)}{\sigma(h_i, h_{i-1})}(h_i - h_{i-1})$, $\lambda \in \Sigma$, maps Λ_{i-1} onto Λ_i. If we choose $\lambda_{i-1} \in \Lambda_{i-1}$ such that $\lambda_{i-1} + h_i \in \Pi_0$ we obtain that $\operatorname{ind}_{\Pi_0}(\Lambda_{i-1}, \Lambda_i)$ takes the values 0, 1/2, 1 when $\sigma(\lambda_{i-1}, h_i)$ is accordingly positive, zero or negative.

More refined results from symplectic geometry enable us to achieve further simplifications and, in particular, to reduce the dimension of Σ. An analysis of formula (14) leads to the following

Corollary. Assume that the conditions of Theorem 1 are valid, and suppose that at least one of the quantities $\operatorname{ind}_{\Pi_0}(\Lambda_{i-1}, \Lambda_i)$, $i = 1, \ldots, l$ is equal to 1. Then $\operatorname{ind} Q > 0$.

Using Theorem 1 and the corollary we can give in many cases upper bounds for the number of switches of bang-bang optimal controls. The situation can be most easily understood when the first switching already violates optimality. Let $\mu_1 = \mu_0 \circ e^{t_1 f_0}$ and $\psi_1 = (e^{-t_1 f_0})^* \psi_0 \in T^*_{\mu_1} M$, so that $\langle \psi_1, \mu_1 \circ g \rangle = \langle \psi_0, \mu_0 \circ e^{t_1 \operatorname{ad} f_0} g \rangle \ \forall g \in T^*_{\mu_1} M$.

Lemma. The equality $\operatorname{ind}_{\Pi_0}(\Lambda_0, \Lambda_1) = 1$ is equivalent to the relations $0 \neq \mu_1 \circ f_1 = \alpha(\mu_1 \circ f_0)$, where $\alpha < 0$ and $\langle \psi_1, \mu_1 \circ [f_1, f_0] \rangle \neq 0$.

As is well known, certain regularity conditions in the classical Calculus of Variations guarantee the absence of optimal trajectories with corners. The previous lemma guarantees that for optimal control, where the trajectories as a rule have corners, there are no trajectories with cusps if certain regularity conditions are satisfied.

6 APPLICATIONS

In this section we give simple applications of the general theory to some special classes of systems. We shall only present the final results and omit intermediary calculations, though the latter are quite instructive. Consider a control system

$$\dot{\mu} = \mu \circ (f + ug), \qquad |u| \leq 1, \qquad f, g \in \mathrm{Vect}(M) \qquad \mu(0) = \mu_0. \quad (15)$$

A number of important results have recently appeared concerning the time-optimal control of such systems in dimensions ≤ 3, cf. [7], [11], [12]. We shall now apply our method to (15).

Suppose $\tilde{u}(\tau)$, $0 \leq \tau \leq t$, is a bang-bang control (hence taking only values ± 1) with l switches, satisfying the Maximum Principle, and write $f_\tau = f + u(\tau)g$. In the above discussion, carried out in a more general setting, the fields h_i, $i = 0, 1, \ldots, l$ and the quadratic form Q were derived from f_τ. A local investigation permits in this situation to replace the fields h_i by the segments of their Taylor series expansions in t, up to the orders which are necessary for computing the index of Q according to (14). Precisely, the following proposition is valid.

Proposition 3. Let $\dim M = 3$ and suppose that the tensor fields $g \wedge [f, g] \wedge [f \pm g, [f, g]]$ do not vanish at $\mu_0 \in M$. Then there exist a neighborhood ϑ_{μ_0} of μ_0 and a time $T > 0$ such that any time-optimal trajectory of (15) corresponding to a bang-bang control, contained in ϑ_{μ_0}, and defined on a time interval of length $\leq T$, has at most two switches.

The above result was obtained in [12] by a completely different method under stronger conditions. We can apply our method even in more degenerate cases and for greater dimensions. To compute the index we only need to solve systems of linear equations, though one should emphasize that the amount of computation sharply increases with the number of switches and the number of terms of the Taylor expansion.

Now we describe one more application which is global in character. Consider the system (15) assuming that $M = SO(3)$—the group of rotations of $\mathbb{R}^3$—and that f, g are left invariant, i.e. $f, g \in so(3)$. Furthermore suppose $\langle f, g \rangle = 0$, where $\langle .,. \rangle$ is an invariant scalar product on $so(3)$. We are still interested in bang-bang time-optimal controls. The problem could be interpreted as follows. A rigid body can be rotated around two fixed axes with equal constant velocities. It is required to bring the body into a given position in minimal time. The only parameter of the problem which could influence the motion is the angle ϕ between the oriented axis, $0 < \phi < \pi$. Without loss of generality we can suppose that $\langle f, f\rangle + \langle g, g\rangle = 1$, and then $\cos\phi = \langle f, f\rangle - \langle g, g\rangle$. The bang-bang trajectories that satisfy the Maximum Principle can have an arbitrary number of switches, and are completely characterized by the condition that the time elapsed between neighboring switches (i.e. the angle of rotation) depends only on the trajectory but not on the number of the switches, and belongs to the interval $[\pi, 2\pi)$. We can deduce from here that for any neighborhood ϑ_{μ_0} and any integer N there exists a bang-bang trajectory satisfying the Maximum Principle for which the first N switching points belong to ϑ_{μ_0}. Such trajectories are not optimal, but they cannot be eliminated by local considerations, since between two neighboring switches they might not stay in ϑ_{μ_0}.

Suppose $\tilde{u}$ is a bang-bang control having $l \geq 2$ switches and satisfying the maximum principle, and suppose further that $\alpha \in [\pi, 2\pi)$ is the time elapsed between two neighboring switches. The composition $e^{\alpha(f-g)} \circ e^{\alpha(f+g)}$ of two consecutive rotations can be represented as a rotation by an angle 2θ around an axis, where $\theta \in (0, \pi)$ is defined by the relation $\tan^2\frac{\theta}{2} = \tan^2\frac{\phi}{2} + (\cot\frac{\alpha}{2}/\cos\frac{\phi}{2})^2$. When α increases from π to 2π the angle θ increases monotonically from ϕ to π. It can be shown that the quadratic form Q in $\tilde{u}$, constructed above satisfies the inequalities $[\frac{l\theta}{\pi}] - 1 \leq \operatorname{ind} Q \leq [\frac{l\theta}{\pi}]$, and consequently *any bang-bang optimal control in our problem has no more than* $[\frac{2\pi}{\phi}]$ *switches.*

For completeness we shall also mention some properties of trajectories with singular arcs (cf. [3]). Suppose the optimal control $u(\tau)$ is singular on the interval $[\tau_0, \tau_1]$ which is maximal with respect to inclusion. Then (a) $u(\tau) \equiv 0$ for $\tau \in (\tau_0, \tau_1]$, (b) $\tau_1 - \tau_0 \leq \pi/\cos\frac{\phi}{2}$, (c) $u(t)$ has no switchings other than τ_o, τ_1.

7 A GENERAL FORMULA FOR THE INDEX

In [4], [5], the index of second variation along a singular arc was computed. In Theorem 1 the same index is computed along a bang-bang arc. In these two cases completely different variations of the initial control are used, though the expressions for the index have similar forms. Here we define a unified "symplectic" expression for the index, which does not need different considerations for bang-bang and singular arcs. However we still restrict the control system under consideration to the form (5) and suppose that $f(u)$ depends on u affinely and $U \subset \mathsf{R}^r$ is a convex polyhedron.

Suppose that the given control $\tilde{u}$ is piecewise continuous and satisfies the Maximum Principle. We shall use the Notations of §3 and suppose that the covector $\psi_0 \in T^*_{\mu_0}M$ is defined by (6) uniquely up to a scalar multiple. For every $\tau \in [0,t]$ denote by $V_\tau \subset U$ the face of minimal dimension of U containing $\tilde{u}(\tau)$; we suppose that V_τ is piecewise constant in τ. Let $\Pi : \mathrm{Vect}(M) \to \Sigma$ be the canonical factorization and put $\Gamma_\tau = span(\Pi\, \overrightarrow{\exp} \int_0^\tau ad_{f_\theta}\, d\theta f_\tau(V_\tau)), \tau \in [0,t]$. As a simple consequence of the generalized Legendre–Clebsch conditions (cf. [1]) we obtain

Proposition 4. If the trajectory $\tilde{\mu}(\tau), 0 \leq \tau \leq t$ is time optimal then $\forall \tau \in [0,t]\ \Gamma_\tau$ is an isotropic subspace of the symplectic space Σ.

Suppose now that Γ_τ, $0 \leq \tau \leq t$, are isotropic and denote by $\mathcal{D}$ the set of all finite subsets of $[0,t]$. Let $D = \{t_1, \dots, t_l\} \in \mathcal{D}, 0 < t_1 < \cdots < t_l$. Define the sequence of Lagrangian subspaces $\Lambda_i(D)$, $i = 0, 1, \dots, l+1$, by $\Lambda_0(D) = \Lambda_{l+1} = \Pi_0, \Lambda_i(D) = \Lambda_{i-1}^{\Gamma_{t_i}}(D)$, $i = 1, \dots, l$.

Theorem 2. For an arbitrary $D \in \mathcal{D}$ put

$$I(D) = \sum_{i=1}^{l+1} \mathrm{ind}_{\Pi_0}(\Lambda_{i-1}(D), \Lambda_i(D)) + \dim\left(\bigcap_{i=0}^{l} \Lambda_i(D)\right) - m \geq 0,$$

where l = number of points in D. Then:

(1) if $D \subset D'$ then $I(D) \leq I(D)'$.

(2) if the trajectory $\tilde{\mu}(\tau), 0 \leq \tau \leq t$, is time optimal, then

$$\forall D \in \mathcal{D}, \qquad I(D) = 0.$$

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