Lie-algebraic stability criteria for switched systems

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Abstract

It was recently shown that a family of exponentially stable linear systems whose matrices generate a solvable Lie algebra possesses a quadratic common Lyapunov function, which implies that the corresponding switched linear system is exponentially stable for arbitrary switching. In this paper we prove that the same properties hold under the weaker condition that the Lie algebra generated by given matrices can be decomposed into a sum of a solvable ideal and a subalgebra with a compact Lie group. The corresponding local stability result for nonlinear switched systems is also established. Moreover, we demonstrate that if a Lie algebra fails to satisfy the above condition, then it can be generated by a family of stable matrices such that the corresponding switched linear system is not stable. Relevant facts from the theory of Lie algebras are collected at the end of the paper for easy reference.

1 Introduction

A switched system can be described by a family of continuous-time subsystems and a rule that orchestrates the switching between them. Such systems arise, for example, when different controllers are being placed in the feedback loop with a given process, or when a given process exhibits a switching behavior caused by abrupt changes of the environment. For a discussion of various issues related to switched systems, see the recent survey article [13].

To define more precisely what we mean by a switched system, consider a family $\{f_p : p \in \mathcal{P}\}$ of sufficiently regular functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, parameterized by some index set $\mathcal{P}$. Let $\sigma : [0,\infty) \to \mathcal{P}$ be a piecewise constant function of time, called a *switching signal.* A *switched system* is then given by the following system of differential equations in $\mathbb{R}^n$:

$$\dot{x} = f_\sigma(x). \tag{1}$$

We assume that the state of (1) does not jump at the switching instants, i.e., the solution $x(\cdot)$ is everywhere continuous. Note that infinitely fast switching (chattering), which calls for a concept of generalized solution, is not considered in this paper. In the particular case when all the individual subsystems are linear (i.e., $f_p(x) = A_p x$ where $A_p \in \mathbb{R}^{n\times n}$ for each $p \in \mathcal{P}$), we obtain a *switched linear system*

$$\dot{x} = A_\sigma x. \tag{2}$$

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This paper is concerned with the following problem: find conditions on the individual subsystems which guarantee that the switched system is asymptotically stable for an arbitrary switching signal σ. In fact, a somewhat stronger property is desirable, namely, asymptotic or even exponential stability that is uniform over the set of all switching signals. Clearly, all the individual subsystems must be asymptotically stable, and we will assume this to be the case throughout the paper. Note that it is not hard to construct examples where instability can be achieved by switching between asymptotically stable systems (Section 4 contains one such example), so one needs to determine what additional requirements must be imposed. This question has recently generated considerable interest, as can be seen from the work reported in [9, 12, 16, 17, 18, 19, 21, 22].

Commutation relations among the individual subsystems play an important role in the context of the problem posed above. This can be illustrated with the help of the following example. Consider the switched linear system (2), take $\mathcal{P}$ to be a finite set, and suppose that the matrices A_p commute pairwise: $A_pA_q = A_qA_p$ for all $p, q \in \mathcal{P}$. Then it is easy to show directly that the switched linear system is exponentially stable, uniformly over all switching signals. Alternatively, one can construct a quadratic common Lyapunov function for the family of linear systems

$$\dot{x} = A_p x, \qquad p \in \mathcal{P} \tag{3}$$

as shown in [18], which is well known to lead to the same conclusion.

In this paper we undertake a systematic study of the connection between the behavior of the switched system and the commutation relations among the individual subsystems. In the case of the switched linear system (2), a useful object that reveals the nature of these commutation relations is the Lie algebra $\mathfrak{g} := \{A_p : p \in \mathcal{P}\}_{LA}$ generated by the matrices A_p, $p \in \mathcal{P}$ (with respect to the standard Lie bracket $[A_p, A_q] := A_pA_q - A_qA_p$). The observation that the structure of this Lie algebra is relevant to stability of (2) goes back to the paper by Gurvits [9]. That paper studied the discrete-time counterpart of (2) taking the form

$$x(k+1) = A_{\sigma(k)}x(k) \tag{4}$$

where σ is a function from nonnegative integers to a finite index set $\mathcal{P}$ and $A_p = e^{L_p}$, $p \in \mathcal{P}$ for some matrices L_p. Gurvits conjectured that if the Lie algebra $\{L_p : p \in \mathcal{P}\}_{LA}$ is nilpotent (which means that Lie brackets of sufficiently high order equal zero), then the system (4) is asymptotically stable for any switching signal σ. He was able to prove this conjecture for the particular case when $\mathcal{P} = \{1, 2\}$ and the third-order Lie brackets vanish: $[L_1, [L_1, L_2]] = [L_2, [L_1, L_2]] = 0$.

It was recently shown in [12] that the switched linear system (2) is exponentially stable for arbitrary switching if the Lie algebra $\mathfrak{g}$ is solvable (see Section A.3 for the definition). The proof relied on the facts that matrices in a solvable Lie algebra can be simultaneously put in the upper-triangular form (Lie's Theorem) and that a family of linear systems with stable upper-triangular matrices has a quadratic common Lyapunov function. For the result to hold, the index set $\mathcal{P}$ does not need to be finite (although a suitable compactness assumption is required). One can derive the corresponding result for discrete-time systems in similar fashion, thereby confirming and directly generalizing the statement conjectured by Gurvits (because every nilpotent Lie algebra is solvable).

In the present paper we continue the line of work initiated in the above references. Our main theorem is a direct extension of the one proved in [12]. The new result states that one still has exponential stability for arbitrary switching if the Lie algebra $\mathfrak{g}$ is a semidirect sum of a solvable ideal and a subalgebra with a compact Lie group (which amounts to saying that all the matrices in this second subalgebra have purely imaginary eigenvalues). The corresponding local stability result for the nonlinear switched system (1) is also established. Being formulated in terms of the original data, such Lie-algebraic stability criteria have an important advantage over results that depend on a particular choice of coordinates, such as the one reported in [16]. Moreover, we demonstrate that the above condition is in some sense the strongest one that can be given on the Lie algebra level. Loosely speaking, we show that if a Lie algebra does not satisfy this condition, then it could be generated by a switched linear system that is not stable.

More precisely, the main contributions of the paper can be summarized as follows (see the appendix for an overview of relevant definitions and facts from the theory of Lie algebras). Given a matrix Lie algebra $\hat{\mathfrak{g}}$ which contains the identity matrix, we are interested in the following question: Is it true that any set of stable generators for $\hat{\mathfrak{g}}$ gives rise to a switched system that is exponentially stable, uniformly over all switching signals? We discover that this property depends only on the structure of $\hat{\mathfrak{g}}$ as a Lie algebra, and not on the choice of a particular matrix representation of $\hat{\mathfrak{g}}$. The following equivalent characterizations of the above property can be given:

1. The factor algebra $\hat{\mathfrak{g}}$ mod $\mathfrak{r}$, where $\mathfrak{r}$ denotes the radical, is a compact Lie algebra.
2. The Killing form is negative semidefinite on $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$.
3. The Lie algebra $\hat{\mathfrak{g}}$ does not contain any subalgebras isomorphic to $sl(2,\mathbb{R})$.

We will also show how the investigation of stability (in the above sense) of a switched linear system in $\mathbb{R}^n$, $n > 2$, whose associated Lie algebra is low-dimensional, can be reduced to the investigation of stability of a switched linear system in $\mathbb{R}^2$. For example, take $\mathcal{P} = \{1, 2\}$, and define $\tilde{A}_i := A_i - \frac{1}{n}\text{trace}(A_i)I$, $i = 1, 2$. Assume that all iterated Lie brackets of the matrices $\tilde{A}_1$ and $\tilde{A}_2$ are linear combinations of $\tilde{A}_1$, $\tilde{A}_2$, and $[\tilde{A}_1, \tilde{A}_2]$. This means that if we consider the Lie algebra $\mathfrak{g} = \{A_1, A_2\}_{LA}$ and add to it the identity matrix (if it is not already there), the resulting Lie algebra $\hat{\mathfrak{g}}$ has dimension at most 4. In this case, the following algorithm can be used to verify that the switched linear system generated by A_1 and A_2 is uniformly exponentially stable or, if this is not possible, to construct a second-order switched linear system whose uniform exponential stability is equivalent to that of the original one.

Step 1. If $[\tilde{A}_1, \tilde{A}_2]$ is a linear combination of $\tilde{A}_1$ and $\tilde{A}_2$, stop: the system is stable. Otherwise, write down the matrix of the Killing form for the Lie algebra $\tilde{\mathfrak{g}} := \{\tilde{A}_1, \tilde{A}_2\}_{LA}$ relative to the basis given by $\tilde{A}_1$, $\tilde{A}_2$, and $[\tilde{A}_1, \tilde{A}_2]$. (This is a symmetric 3×3 matrix; see Section A.4 for the definition of the Killing form.)

Step 2. If this matrix is degenerate or negative definite, stop: the system is stable. Otherwise, continue.

Step 3. Find three matrices h, e, and f in $\tilde{\mathfrak{g}}$ with commutation relations $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$ (this is always possible in the present case). We can then write $\tilde{A}_i = \beta_i e + \gamma_i f + \delta_i h$, where $\alpha_i, \beta_i, \gamma_i$ are constants, $i = 1, 2$.

Step 4. Compute the largest eigenvalue of h. It will be an integer; denote it by k. Then the given system is stable if and only if so is the switched linear system generated by the 2×2 matrices

$$\hat{A}_1 := \begin{pmatrix} \frac{\text{trace}(A_1)}{nk} - \delta_1 & -\beta_1 \\ -\gamma_1 & \frac{\text{trace}(A_1)}{nk} + \delta_1 \end{pmatrix}, \quad \hat{A}_2 := \begin{pmatrix} \frac{\text{trace}(A_2)}{nk} - \delta_2 & -\beta_2 \\ -\gamma_2 & \frac{\text{trace}(A_2)}{nk} + \delta_2 \end{pmatrix}.$$

All the steps in the above reduction procedure involve only elementary matrix operations (addition, multiplication, and computation of eigenvalues and eigenvectors). Details and justification are given in Section 4.

Before closing the Introduction, we make one more remark to further motivate the work reported here and point out its relationship to a more classical branch of control theory. Assume that $\mathcal{P}$ is a finite set, say, $\mathcal{P} = \{1, \ldots, m\}$. The switched system (1) can then be recast as

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i \tag{5}$$

where the admissible controls are of the form $u_k = 1$, $u_i = 0$ $\forall i \neq k$ (this corresponds to $\sigma = k$). In particular, the switched linear system (2) gives rise to the bilinear system

$$\dot{x} = \sum_{i=1}^{m} A_i x u_i.$$

It is intuitively clear that asymptotic stability of (1) for arbitrary switching corresponds to lack of controllability for (5). Indeed, it means that for any admissible control function the resulting solution trajectory must approach the origin. Lie-algebraic techniques have received a lot of attention in the context of the controllability problem for systems of the form (5). As for the literature on stability analysis of switched systems, despite the fact that it is vast and growing, Lie-algebraic methods do not yet seem to have penetrated it. The present work can be considered as a step towards filling this gap.

The rest of the paper is organized as follows. In Section 2 we establish a sufficient condition for stability (Theorem 2) and discuss its various implications. In Section 3 we prove a converse result (Theorem 4). Section 4 contains a detailed analysis of switched systems whose associated Lie algebras are isomorphic to the Lie algebra $gl(2, \mathbb{R})$ of real 2×2 matrices. This leads, among other things, to the reduction algorithm sketched above and to a different (and arguably more illuminating) proof of Theorem 4. To make the paper self-contained, in the appendix we provide an overview of relevant facts from the theory of Lie algebras.

2 Sufficient conditions for stability

The switched system (1) is called (locally) *uniformly exponentially stable* if there exist positive constants M, c and μ such that for any switching signal σ the solution of (1) with $\|x(0)\| \leq M$ satisfies

$$\|x(t)\| \leq ce^{-\mu t}\|x(0)\| \quad \forall t \geq 0. \tag{6}$$

The term "uniform" is used here to describe uniformity with respect to switching signals. If there exist positive constants c and μ such that the estimate (6) holds for any switching signal σ and any initial condition $x(0)$, then the switched system is called *globally uniformly exponentially stable*. Similarly, one can also define the property of uniform *asymptotic* stability, local or global. For switched linear systems all the above concepts are equivalent (see [15]). In fact, as shown in [1], in the linear case global uniform exponential stability is equivalent to the seemingly weaker property of asymptotic stability for any switching signal.

In the context of the switched linear system (2), we will always assume that $\{A_p : p \in \mathcal{P}\}$ is a *compact* (with respect to the usual topology in $\mathbb{R}^{n\times n}$) set of real $n \times n$ matrices with eigenvalues in the open left half-plane. Let $\mathfrak{g}$ be the Lie algebra defined by $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ as before. The following stability criterion was established in [12]. It will be crucial in proving Theorem 2 below.

Theorem 1 [12] *If $\mathfrak{g}$ is a solvable Lie algebra, then the switched linear system (2) is globally uniformly exponentially stable.*

Remark 1. The proof of this result given in [12] relies on a construction of a quadratic common Lyapunov function for the family of linear systems (3). The existence of such a function actually implies global uniform exponential stability of the time-varying system $\dot{x} = A_\sigma x$ with σ not necessarily piecewise constant. This observation will be used in the proof of Theorem 2.

The above condition can always be checked directly in a finite number of steps if $\mathcal{P}$ is a finite set. Alternatively, one can use the standard criterion for solvability in terms of the Killing form. Similar criteria exist for checking the other conditions to be presented in this paper—see Sections A.3 and A.4 for details.

We now consider a Levi decomposition of $\mathfrak{g}$, i.e., we write $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{r}$ is the radical and $\mathfrak{s}$ is a semisimple subalgebra (see Section A.4). Our first result is the following generalization of Theorem 1.

Theorem 2 *If $\mathfrak{s}$ is a compact Lie algebra, then the switched linear system (2) is globally uniformly exponentially stable.*

PROOF. For an arbitrary $p \in \mathcal{P}$, write $A_p = r_p + s_p$ with $r_p \in \mathfrak{r}$ and $s_p \in \mathfrak{s}$. Let us show that r_p is a stable matrix. Writing

$$e^{(r_p+s_p)t} = e^{s_p t} B_p(t) \tag{7}$$

we have the following equation for $B_p(t)$:

$$\dot{B}_p(t) = e^{-s_p t} r_p e^{s_p t} B_p(t), \qquad B_p(0) = I. \tag{8}$$

To verify (8), differentiate the equality (7) with respect to t, which gives

$$(r_p + s_p)e^{(r_p+s_p)t} = s_p e^{s_p t} B_p + e^{s_p t} \dot{B}_p.$$

Using (7) again, we have

$$r_p e^{s_p t} B_p + s_p e^{s_p t} B_p = s_p e^{s_p t} B_p + e^{s_p t} \dot{B}_p$$

hence (8) holds. Define $c_p(t) := e^{-s_p t} r_p e^{s_p t}$. Clearly, $\mathrm{spec}(c_p(t)) = \mathrm{spec}(r_p)$ for all t. It is well known that for any two matrices A and B one has

$$e^{-A} B e^{A} = e^{\mathrm{ad} A}(B) = B + [A, B] + \frac{1}{2}[A, [A, B]] + \ldots \tag{9}$$

hence we obtain the expansion

$$c_p(t) = r_p + [s_p t, r_p] + \frac{1}{2}[s_p t, [s_p t, r_p]] + \ldots$$

Since $[\mathfrak{s}, \mathfrak{r}] \subseteq \mathfrak{r}$, we see that $c_p(t) \in \mathfrak{r}$. According to Lie's Theorem, there exists a basis in which all matrices from $\mathfrak{r}$ are upper-triangular. Combining the above facts, it is not hard to check that $\mathrm{spec}(B_p(t)) = e^{t\mathrm{spec}(r_p)}$. Now it follows from (8) that $\mathrm{spec}(r_p)$ lies in the open left half of the complex plane. Indeed, as $t \to \infty$ we have $e^{(r_p+s_p)t} \to 0$ because the matrix A_p is stable. Since $\mathfrak{s}$ is compact, there exists a constant $C > 0$ such that we have $|e^s x| \geq C|x|$ for all $s \in \mathfrak{s}$ and $x \in \mathbb{R}^n$, thus we cannot have $e^{s_p t} x \to 0$ for $x \neq 0$. Therefore, $B_p(t) \to 0$, and so r_p is stable.

Since $p \in \mathcal{P}$ was arbitrary, we see that all the matrices r_p, $p \in \mathcal{P}$ are stable. Theorem 1 implies that the switched linear system generated by these matrices is globally uniformly exponentially stable. Moreover, the same property holds for matrices in the extended set $\bar{\mathfrak{r}} := \{\bar{A} : \exists\, p \in \mathcal{P} \text{ and } s \in \mathfrak{s} \text{ such that } \bar{A} = e^{-s} r_p e^{s}\}$. This is true because the matrices in this set are stable and because they belong to $\mathfrak{r}$ (the last statement follows from the expansion (9) again since $[\mathfrak{s}, \mathfrak{r}] \subseteq \mathfrak{r}$). Now, the transition matrix of the original switched linear system (2) at time t takes the form

$$\Phi(t, 0) = e^{(r_{p_k}+s_{p_k})t_k} \cdots e^{(r_{p_1}+s_{p_1})t_1} = e^{s_{p_k} t_k} B_{p_k}(t_k) \cdots e^{s_{p_1} t_1} B_{p_1}(t_1)$$

where t_1, $t_1 + t_2$, $\ldots$, $t_1 + t_2 + \cdots + t_{k-1} < t$ are switching instants, $t_1 + \cdots + t_k = t$, and as before $\dot{B}_{p_i}(t) = e^{-s_{p_i} t} r_{p_i} e^{s_{p_i} t} B_{p_i}(t)$, $i = 1, \ldots, k$. To simplify the notation, let $k = 2$ (in the general case one can adopt the same line of reasoning or use induction on k). We can then write

$$\Phi(t, 0) = e^{s_{p_2} t_2} e^{s_{p_1} t_1} e^{-s_{p_1} t_1} B_{p_2}(t_2) e^{s_{p_1} t_1} B_{p_1}(t_1) = e^{s_{p_2} t_2} e^{s_{p_1} t_1} \tilde{B}_{p_2}(t_2) B_{p_1}(t_1)$$

where $\tilde{B}_{p_2}(t) := e^{-s_{p_1}t_1}B_{p_2}(t)e^{s_{p_1}t_1}$. We have

$$\begin{aligned}\frac{d}{dt}\tilde{B}_{p_2}(t) &= e^{-s_{p_1}t_1}e^{-s_{p_2}t}r_{p_2}e^{s_{p_2}t}B_{p_2}(t)e^{s_{p_1}t_1} = e^{-s_{p_1}t_1}e^{-s_{p_2}t}r_{p_2}e^{s_{p_2}t}e^{s_{p_1}t_1}e^{-s_{p_1}t_1}B_{p_2}(t)e^{s_{p_1}t_1}\\ &= e^{-s_{p_1}t_1}e^{-s_{p_2}t}r_{p_2}e^{s_{p_2}t}e^{s_{p_1}t_1}\tilde{B}_{p_2}(t)\end{aligned}$$

Thus we see that

$$\Phi(t,0) = e^{s_{p_2}t_2}e^{s_{p_1}t_1}\cdot \bar{B}(t) \tag{10}$$

where $\bar{B}(t)$ is the transition matrix of a switched/time-varying system generated by matrices in $\bar{\mathfrak{r}}$, i.e., $\frac{d}{dt}\bar{B}(t) = \bar{A}(t)\bar{B}(t)$ with $\bar{A}(t) \in \bar{\mathfrak{r}}\ \forall t \geq 0$. The norm of the first term in the above product is bounded by compactness, while the norm of the second goes to zero exponentially by Theorem 1 (see also Remark 1), and the statement of the theorem follows. □

REMARK 2. The fact that $\mathfrak{r}$ is the radical, implying that $\mathfrak{s}$ is semisimple, was not used in the proof. The statement of Theorem 2 remains valid for any decomposition of $\mathfrak{g}$ into the sum of a solvable ideal $\mathfrak{r}$ and a subalgebra $\mathfrak{s}$. Among all possible decompositions of this kind, the one considered above gives the strongest result. If $\mathfrak{g}$ is solvable, then $\mathfrak{s} = 0$ is of course compact, and we recover Theorem 1 as a special case.

EXAMPLE 1. Suppose that the matrices A_p, $p \in \mathcal{P}$ take the form $A_p = -\lambda_p I + S_p$ where $\lambda_p > 0$ and $S_p^T = -S_p$ for all $p \in \mathcal{P}$. These are automatically stable matrices. Suppose also that $\text{span}\{A_p, p \in \mathcal{P}\} \ni I$. Then the condition of Theorem 2 is satisfied. Indeed, take $\mathfrak{r} = \{\lambda I : \lambda \in \mathbb{R}\}$ (scalar multiples of the identity matrix) and observe that the Lie algebra $\{S_p : p \in \mathcal{P}\}_{LA}$ is compact because skew-symmetric matrices have purely imaginary eigenvalues.

In [12] the global uniform exponential stability property was deduced from the existence of a quadratic common Lyapunov function. In the present case we found it more convenient to obtain the desired result directly. However, under the hypothesis of Theorem 2 a quadratic common Lyapunov function for the family of linear systems (3) can also be constructed, as we now show. Let $\bar{V}(x) = x^TQx$ be a quadratic common Lyapunov function for the family of linear systems generated by matrices in $\bar{\mathfrak{r}}$ (which exists according to [12]). Define the function

$$V(x) := \int_{\mathcal{S}} \bar{V}(Sx)dS = x^T \cdot \int_{\mathcal{S}} S^TQSdS \cdot x$$

where $\mathcal{S}$ is the Lie group corresponding to $\mathfrak{s}$ and the integral is taken with respect to the Haar measure invariant under right translation on $\mathcal{S}$ (see Section A.4). Using (10), it is straightforward to show that the derivative of V along solutions of the switched linear system (2) satisfies

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= \frac{d}{dt}\int_{\mathcal{S}} \bar{V}(S\bar{B}(t)x(0))dS\\ &= \int_{\mathcal{S}} x^T(0)\bar{B}^T(t)S^T((S\bar{A}(t)S^{-1})^TQ + QS\bar{A}(t)S^{-1})S\bar{B}(t)x(0)dS < 0.\end{aligned}$$

The first equality in the above formula follows from the invariance of the measure, and the last inequality holds because $S\bar{A}(t)S^{-1} \in \bar{\mathfrak{r}}$ for all $t \geq 0$ and all $S \in \mathcal{S}$.

REMARK 3. It is now clear that the above results remain valid if piecewise constant switching signals are replaced by arbitrary measurable functions (cf. Remark 1).

The existence of a quadratic common Lyapunov function will be used to prove Corollary 3 below. It is also an interesting fact in its own right because, although the converse Lyapunov theorem proved in [15] implies that global uniform exponential stability always leads to the existence of a common Lyapunov function, in some cases it is not possible to find a quadratic one [4]. Incidentally, this clearly shows that the condition of Theorem 2 is not necessary for uniform exponential stability of the switched linear system (2).

Another way to see this is to note that the property of uniform exponential stability is robust with respect to small perturbations of the parameters of the system, whereas the condition of Theorem 2 is not. In fact, no Lie-algebraic condition of the type considered here can possess the indicated robustness property. This follows from the fact, proved in Section A.6, that in an arbitrarily small neighborhood of any pair of $n \times n$ matrices there exists a pair of matrices that generate the entire Lie algebra $gl(n, \mathbb{R})$.

We conclude this section with a local stability result for the nonlinear switched system (1). Let $f_p : D \to \mathbb{R}^n$ be continuously differentiable with $f_p(0) = 0$ for each $p \in \mathcal{P}$, where D is a neighborhood of the origin in $\mathbb{R}^n$. Consider the linearization matrices

$$F_p := \frac{\partial f_p}{\partial x}(0), \qquad p \in \mathcal{P}.$$

Assume that the matrices F_p are stable, that $\mathcal{P}$ is a compact subset of some topological space, and that $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p for each $x \in D$. Consider the Lie algebra $\tilde{\mathfrak{g}} := \{F_p : p \in \mathcal{P}\}_{LA}$ and its Levi decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{r}} \oplus \tilde{\mathfrak{s}}$. The following statement is a generalization of [12, Corollary 5].

Corollary 3 *If $\tilde{\mathfrak{s}}$ is a compact Lie algebra, then the switched system (1) is uniformly exponentially stable.*

Proof. This is a relatively straightforward application of Lyapunov's first method (see, e.g., [11]). For each $p \in \mathcal{P}$ we can write $f_p(x) = F_p x + g_p(x)x$. Here $g_p(x) = \frac{\partial f_p}{\partial x}(z) - \frac{\partial f_p}{\partial x}(0)$ where z is a point on the line segment connecting x to the origin. We have $g_p(x) \to 0$ as $x \to 0$. Under the present assumptions, the family of linear systems $\dot{x} = F_p x$, $p \in \mathcal{P}$ has a quadratic common Lyapunov function. Because of compactness of $\mathcal{P}$ and continuity of $\frac{\partial f_p}{\partial x}$ with respect to p, it is not difficult to verify that this function is a common Lyapunov function for the family of systems $\dot{x} = f_p(x)$, $p \in \mathcal{P}$ on a certain neighborhood $\bar{D}$ of the origin. Thus the switched system (1) is uniformly exponentially stable on $\bar{D}$. $\square$

An important problem for future research is to investigate how the structure of the Lie algebra generated by the original nonlinear vector fields f_p, $p \in \mathcal{P}$ is related to stability properties of the switched system (1). Taking higher-order terms into account, one may hope to obtain conditions that guarantee stability of nonlinear switched systems when the above linearization test fails. A first step in this direction is the observation made in [21] that a finite family of commuting nonlinear vector fields giving rise to exponentially stable systems has a local common Lyapunov function. Imposing certain additional assumptions, it is possible to obtain analogues of Lie's Theorem which yield triangular structure for families of nonlinear systems generating nilpotent or solvable Lie algebras (see [3, 10, 14]). However, the methods described in these papers require that the Lie algebra have full rank, and so typically they do not apply to families of systems with common equilibria of the type treated here.

3 A converse result

We already remarked that the condition of Theorem 2 is not necessary for uniform exponential stability of the switched linear system (2). It is natural to ask whether this condition can be improved. A more general question that arises is to what extent the structure of the Lie algebra can be used to distinguish between stable and unstable switched systems. The findings of this section will shed some light on these issues.

We find it useful to introduce a possibly larger Lie algebra $\hat{\mathfrak{g}}$ by adding to $\mathfrak{g}$ the scalar multiples of the identity matrix if necessary. In other words, define $\hat{\mathfrak{g}} := \{I, A_p : p \in \mathcal{P}\}_{LA}$. The Levi decomposition of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{g}} = \hat{\mathfrak{r}} \oplus \mathfrak{s}$ with $\hat{\mathfrak{r}} \supseteq \mathfrak{r}$ (because the subspace $\mathbb{R}I$ belongs to the radical of $\hat{\mathfrak{g}}$). Thus $\hat{\mathfrak{g}}$ satisfies the hypothesis of Theorem 2 if and only if $\mathfrak{g}$ does.

Our goal in this section is to show that if this hypothesis is not satisfied, then $\hat{\mathfrak{g}}$ can be generated by a family of stable matrices (which might in principle be different from $\{A_p : p \in \mathcal{P}\}$) with the property that the corresponding switched linear system is not stable. Such a statement could in some sense be interpreted as a converse of Theorem 2. It would imply that by working just with $\hat{\mathfrak{g}}$ it is not possible to obtain a stronger result than the one given in the previous section.

We will also see that there exists another set of stable generators for $\hat{\mathfrak{g}}$ which does give rise to a uniformly exponentially stable switched linear system. In fact, we will show that both generator sets can always be chosen in such a way that they contain the same number of elements as the original set that was used to generate $\hat{\mathfrak{g}}$. Thus, if the Lie algebra does not satisfy the hypothesis of Theorem 2, this Lie algebra alone (even together with the knowledge of how many stable matrices were used to generate it) does not provide enough information to determine whether or not the original switched linear system is stable.

Let $\{A_1, A_2, \ldots, A_m\}$ be any finite set of stable generators for $\hat{\mathfrak{g}}$ (if the index set $\mathcal{P}$ is infinite, a suitable finite subset can always be extracted from it). Then the following holds.

Theorem 4 *Suppose that $\mathfrak{s}$ is not a compact Lie algebra. Then there exists a set of m stable generators for $\hat{\mathfrak{g}}$ such that the corresponding switched linear system is not uniformly exponentially stable. There also exists another set of m stable generators for $\hat{\mathfrak{g}}$ such that the corresponding switched linear system is globally uniformly exponentially stable.*

PROOF. To prove the second statement of the theorem, we simply subtract λI from each of the generators A_1, A_2, $\ldots$, A_m, where $\lambda > 0$ is large enough. Namely, take λ to be any number larger than the largest eigenvalue of $(A_i + A_i^T)/2$ for all $i = 1, \ldots, m$. Then it is easy to check that the linear systems defined by the matrices $A_1 - \lambda I$, $A_2 - \lambda I$, $\ldots$, $A_m - \lambda I$ all share the common Lyapunov function $V(x) = x^T x$. To prove that these matrices indeed generate $\hat{\mathfrak{g}}$, it is enough to show that the span of these matrices and their iterated Lie brackets contains the identity matrix I. We know that I can be written as a linear combination of the matrices A_1, A_2, $\ldots$, A_m, and their suitable Lie brackets. Replacing each A_i in this linear combination by $A_i - \lambda I$, we obtain a scalar multiple of I. If it is nonzero, we are done; otherwise, we just have to increase λ by an arbitrary amount.

We now turn to the first statement of the theorem. Since $\mathfrak{s}$ is not compact, it contains a subalgebra that is isomorphic to $sl(2, \mathbb{R})$. Such a subalgebra can be constructed as shown in Section A.5. The existence of this subalgebra is the key property that we will explore.

It follows from basic properties of solutions to differential inclusions that if a family of matrices gives rise to a uniformly exponentially stable switched linear system, then all convex linear combinations of these matrices are stable (this fact is easily seen to be true from the converse Lyapunov theorems of [15, 4], although in [15] it was actually used to prove the result; see also Remark 5 below). To prove the theorem, we will first find a pair of stable matrices B_1, B_2 that lie in the subalgebra isomorphic to $sl(2, \mathbb{R})$ and have an unstable convex combination, and then use them to construct a desired set of generators for $\hat{\mathfrak{g}}$. (An alternative method of proof will be presented in the next section.)

Since every matrix representation of $sl(2, \mathbb{R})$ is a direct sum of irreducible ones, there is no loss of generality in considering only irreducible representations. Their complete classification in all dimensions (up to equivalence induced by linear coordinate transformations) is available. In particular, it is known that any irreducible representation of $sl(2, \mathbb{R})$ contains two matrices of the following form:

$$\tilde{B}_1 = \begin{pmatrix} 0 & \mu_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mu_r \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B}_2 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

(cf. Section A.2). The matrix $\tilde{B}_1$ has positive entries $\mu_1, \dots, \mu_r$ immediately above the main diagonal and zeros elsewhere, and the matrix $\tilde{B}_2$ has ones immediately below the main diagonal and zeros elsewhere.

It is not hard to check that the nonnegative matrix $\tilde{B} := (\tilde{B}_1 + \tilde{B}_2)/2$ is irreducible[1], and as such satisfies the assumptions of the Perron-Frobenius Theorem (see, e.g., [6, Chapter XIII]). According to that theorem, $\tilde{B}$ has a positive eigenvalue. Then for a small enough $\epsilon > 0$ the matrix $B := \tilde{B} - \epsilon I$ also has a positive eigenvalue. We have $B = (\tilde{B}_1 - \epsilon I + \tilde{B}_2 - \epsilon I)/2$. This implies that a desired pair of matrices in the given irreducible matrix representation of $sl(2,\mathbb{R})$ can be defined by $B_1 := \tilde{B}_1 - \epsilon I$ and $B_2 := \tilde{B}_2 - \epsilon I$. Indeed, these matrices are stable, but their average B is not.

For $\alpha \geq 0$, define $A_1(\alpha) := B_1 + \alpha A_1$ and $A_2(\alpha) := B_2 + \alpha A_2$. If α is small enough, then $A_1(\alpha)$ and $A_2(\alpha)$ are stable matrices, while $(A_1(\alpha)+A_2(\alpha))/2$ is unstable. Thus the matrices $A_1(\alpha), A_2(\alpha), A_3, \dots, A_m$ yield a switched system that is not uniformly exponentially stable. Moreover, it is not hard to show that for α small enough these matrices generate $\hat{\mathfrak{g}}$. Indeed, consider a basis for $\hat{\mathfrak{g}}$ formed by $A_1, \dots, A_m$, and their suitable Lie brackets. Replacing A_1 and A_2 in these expressions by $A_1(\alpha)$ and $A_2(\alpha)$ and writing the coordinates of the resulting elements relative to this basis, we obtain a square matrix $\Delta(\alpha)$. Its determinant is a polynomial in α whose value tends to ∞ as $\alpha \to \infty$, and therefore it is not identically zero. Thus $\Delta(\alpha)$ is nondegenerate for all but finitely many values of α; in particular, we will have a basis for $\hat{\mathfrak{g}}$ if we take α sufficiently small. This completes the proof. $\square$

REMARK 4. Given the matrices B_1 and B_2 as in the above proof, it is of course quite easy to construct a set of stable generators for $\hat{\mathfrak{g}}$ giving rise to a switched linear system that is not uniformly exponentially stable: just take any set of generators for $\hat{\mathfrak{g}}$ containing $-I$, B_1 and B_2, and make them into stable ones by means of subtracting positive multiples of the identity if necessary. The above more careful construction has the advantage of producing a set of generators with the same number of elements as in the original generating set for $\hat{\mathfrak{g}}$.

REMARK 5. The existence of an unstable convex combination actually leads to more specific conclusions than simply lack of uniform exponential stability. Namely, one can find a sequence of solutions of the switched system that converges in a suitable sense to a trajectory of the unstable linear system associated with such a convex combination. This is a consequence of the so-called *relaxation theorem* which in our case says that the set of solutions to the differential inclusion $\dot{x} \in \{A_p x : p \in \mathcal{P}\}$ is dense in the set of solutions to the differential inclusion $\dot{x} \in \text{co}\{A_p x : p \in \mathcal{P}\}$, where $\text{co}(K)$ denotes the convex hull of a set $K \subset \mathbb{R}^n$. For details, see [2, 5].

The results that we have obtained so far reveal the following important fact: the property of $\hat{\mathfrak{g}}$ which is being investigated here, namely, global uniform exponential stability of any switched system whose associated Lie algebra is $\hat{\mathfrak{g}}$, depends only on the structure of $\hat{\mathfrak{g}}$ (i.e., on the commutation relations between its matrices) and is independent of the choice of a particular representation.

4 Switched linear systems with low-dimensional Lie algebras

In the proof of Theorem 4 in the previous section, we needed to construct a pair of stable matrices in a representation of $sl(2,\mathbb{R})$ which give rise to an unstable switched system. To achieve this, we relied on the fact that a switched system defined by two matrices is not stable if these matrices have an unstable convex combination. However, even if all convex combinations are stable, stability of the switched system is not guaranteed. As a simple example that illustrates this, consider the switched system in $\mathbb{R}^2$ defined by the matrices $A_1 := \tilde{A}_1 - \epsilon I$ and $A_2 := \tilde{A}_2 - \epsilon I$, where

$$\tilde{A}_1 := \begin{pmatrix} 0 & k \\ -1 & 0 \end{pmatrix}, \qquad \tilde{A}_2 := \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}$$

[1] A matrix is called *irreducible* if it has no proper invariant subspaces spanned by coordinate vectors.

with $\epsilon > 0$ and $k > 1$. It is easy to check that all convex combinations of A_1 and A_2 are stable. When $\epsilon = 0$, the trajectories of the corresponding individual systems look as shown in Figure 1 (left) and Figure 1 (center), respectively. It is not hard to find a switching signal $\sigma : [0, \infty) \to \{1, 2\}$ that makes the switched system $\dot{x} = \tilde{A}_\sigma x$ unstable: simply let $\sigma = 1$ when $xy > 0$ and $\sigma = 2$ otherwise. For an arbitrary initial state, this results in the switched system $\dot{x} = \tilde{A}_{\sigma(t)} x$ whose solutions grow exponentially. Therefore, the original switched system $\dot{x} = A_\sigma x$ will also be destabilized by the same switching signal, provided that ϵ is sufficiently small.

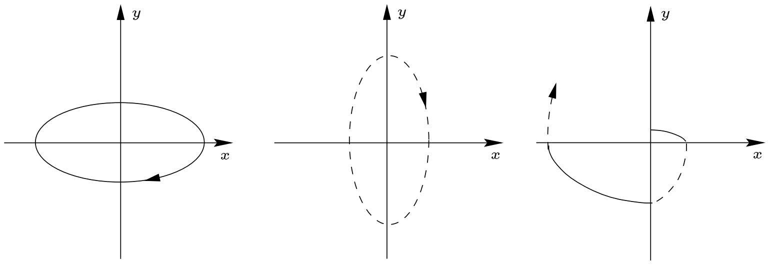

Figure 1: Unstable switched system in the plane

As a step towards understanding the behavior of switched systems in higher dimensions, in view of the findings of this paper it is natural to investigate the case when given matrices generate a Lie algebra that is isomorphic to the one generated by 2×2 matrices. This is the goal of the present section.

Consider the Lie algebra $\mathfrak{g} := \{A_p : p \in \mathcal{P}\}_{LA}$, and assume that $\mathfrak{g} = \mathbb{R}I_{n\times n} \oplus sl(2, \mathbb{R})$. Here $sl(2, \mathbb{R})$ means an n-dimensional matrix representation, which we take to be irreducible (as before, this will not introduce a loss of generality because every matrix representation of $sl(2, \mathbb{R})$ is a direct sum of irreducible ones). Then for each $p \in \mathcal{P}$ we can write

$$A_p = (n-1)\alpha_p I_{n\times n} + \beta_p \phi(e) + \gamma_p \phi(f) + \delta_p \phi(h) \tag{11}$$

where $\beta_p, \gamma_p, \delta_p$ are constants, ϕ is the standard representation of $sl(2, \mathbb{R})$ constructed in Section A.2 (n here corresponds to $k+1$ there), $\{e, h, f\}$ is the canonical basis for $sl(2, \mathbb{R})$, and $\alpha_p = \frac{1}{n(n-1)}\text{trace}(A_p)$. For each $p \in \mathcal{P}$, define the following 2×2 matrix:

$$\hat{A}_p := \alpha_p I_{2\times 2} - \beta_p e - \gamma_p f - \delta_p h. \tag{12}$$

We now demonstrate that the task of investigating stability of the switched system generated by the matrices A_p, $p \in \mathcal{P}$ reduces to that of investigating stability of the two-dimensional switched system generated by the matrices $\hat{A}_p$, $p \in \mathcal{P}$.

Proposition 5 *The switched linear system* (2) *with A_p given by* (11) *is globally uniformly exponentially stable if and only if the switched linear system $\dot{x} = \hat{A}_\sigma x$ with $\hat{A}_p$ given by* (12) *is globally uniformly exponentially stable.*

PROOF. The transition matrix of the switched system (2) for any particular switching signal takes the form

$$\Phi(t, 0) = e^{(n-1)(\alpha_{p_k} t_k + \cdots + \alpha_{p_1} t_1)I} e^{(\beta_{p_k}\phi(e) + \gamma_{p_k}\phi(f) + \delta_{p_k}\phi(h))t_k} \cdots e^{(\beta_{p_1}\phi(e) + \gamma_{p_1}\phi(f) + \delta_{p_1}\phi(h))t_1}.$$

Consider the (n-dimensional) linear space $P^{n-1}[x, y]$ of polynomials in x and y, homogeneous of degree $n-1$, with the basis chosen as in Section A.2. Denote the elements of this basis by $p_1, \ldots, p_n$ (these are monomials in x and y). Fix an arbitrary polynomial $p \in P^{n-1}[x, y]$, and let $a_1, \ldots, a_n$ be its coordinates relative to the above basis. As an immediate consequence of the calculations given in Section A.2, for any values of x and y we have

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \Phi(t,0) \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} = p\Big(\hat{\Phi}(t,0)\begin{pmatrix} x \\ y \end{pmatrix}\Big)$$

where

$$\hat{\Phi}(t,0) = e^{(\alpha_{p_k}t_k+\cdots+\alpha_{p_1}t_1)I}e^{(-\beta_{p_1}e-\gamma_{p_1}f-\delta_{p_1}h)t_1}\cdots e^{(-\beta_{p_k}e-\gamma_{p_k}f-\delta_{p_k}h)t_k}.$$

Since the polynomial p was arbitrary, it is clear that $\Phi(t,0)$ approaches the zero matrix as $t\to\infty$, uniformly over the set of all switching signals, if and only if so does $\hat{\Phi}(t,0)$. But $\hat{\Phi}(t,0)$ is the transition matrix of the switched system $\dot{x} = \hat{A}_\sigma x$, corresponding to the "reversed" switching signal on $[0,t]$. We conclude that this switched system is globally asymptotically stable, uniformly over σ, if and only if the same property holds for the original system (2). The statement of the proposition now follows from the fact that for switched linear systems, uniform asymptotic stability is equivalent to uniform exponential stability. □

We are now in position to justify the reduction procedure outlined in the Introduction. Assume that $\hat{\mathfrak{g}}$ has dimension at most 4. We know from Section A.5 that any noncompact semisimple Lie algebra contains a subalgebra isomorphic to $sl(2,\mathbb{R})$. Thus $\hat{\mathfrak{g}}$ contains a noncompact semisimple subalgebra if only if its dimension exactly equals 4 and the Killing form is nondegenerate and sign-indefinite on $\tilde{\mathfrak{g}} = \{\tilde{A}_1,\tilde{A}_2\}_{LA} = \hat{\mathfrak{g}} \bmod \mathbb{R}I$ (see Section A.4). In this case $\tilde{g}$ is isomorphic to $sl(2,\mathbb{R})$. An $sl(2)$-triple $\{h,e,f\}$ can be constructed as explained in Section A.5 (the procedure given there for a general noncompact semisimple Lie algebra certainly applies to $sl(2,\mathbb{R})$ itself). Specifically, as h we can take any element of the subspace on which the Killing form is positive definite, normalized in such a way that the eigenvalues of $\mathrm{ad}h$ equal 2 and -2. The corresponding eigenvectors yield e and f. The resulting representation of $sl(2,\mathbb{R})$ is not necessarily irreducible; the dimension of the largest invariant subspace is equal to $k+1$, where k is the largest eigenvalue of h. If the switched linear system restricted to this invariant subspace is globally uniformly exponentially stable, then the same property holds for the switched linear system restricted to any other invariant subspace. This is true because, in view of the role of the scalar $k=n-1$ in the context of Proposition 5, the matrices of the reduced (second-order) system associated with the system evolving on the largest invariant subspace are obtained from those of the reduced system associated with the system evolving on another invariant subspace by subtracting positive multiples of the identity matrix, and this cannot introduce instability (to see why this last statement is true, one can appeal to the existence of a convex common Lyapunov function [15]). Note that we do not need to identify the invariant subspaces; we only need to know the dimension of the largest one. Thus the outcome of the algorithm depends on the matrix representation of $\hat{\mathfrak{g}}$ and not just on the structure of $\hat{\mathfrak{g}}$ as a Lie algebra, but it does so in a rather weak way.

As another application of Proposition 5, we can obtain an alternative proof of Theorem 4. Indeed, let the matrices $\tilde{B}_1$ and $\tilde{B}_2$ be as in the proof of Theorem 4 given in the previous section (the existence of a subalgebra isomorphic to $sl(2,\mathbb{R})$ remains crucial). Define the matrices $B_1 := -k\tilde{B}_1 + \tilde{B}_2 - \epsilon I$ and $B_2 := -\tilde{B}_1 + k\tilde{B}_2 - \epsilon I$, where $\epsilon > 0$ and $k > 1$. Then the switched system

$$\dot{x} = B_\sigma x, \qquad \sigma : [0,\infty) \to \{1,2\} \tag{13}$$

is not stable for ϵ small enough (even though all convex combinations of B_1 and B_2 are stable). This follows from Proposition 5 and from the example presented at the beginning of this section; in fact, a specific (periodic) destabilizing switching signal for the system (13) can be constructed with the help of that example. Interestingly, it appears to be difficult to establish the same result by a direct analysis of (13). The rest of the proof of Theorem 4 can now proceed exactly as before.

It was shown by Shorten and Narendra in [22] that two stable two-dimensional linear systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ possess a quadratic common Lyapunov function if and only if all pairwise convex combinations of matrices from the set $\{A_1, A_2, A_1^{-1}, A_2^{-1}\}$ are stable. Combined with Proposition 5, this yields the following result.

Corollary 6 *Let $\mathcal{P} = \{1, 2\}$. Suppose that all pairwise convex combinations of matrices from the set $\{\hat{A}_1, \hat{A}_2, \hat{A}_1^{-1}, \hat{A}_2^{-1}\}$, with A_1 and A_2 given by* (12)*, are stable. Then the switched linear system* (2)*, with A_p given by* (11)*, is globally uniformly exponentially stable.*

The above corollary only provides sufficient and not necessary conditions for global uniform exponential stability of (2). This is due to the fact that, as we already mentioned earlier, it may happen that a switched linear system is globally uniformly exponentially stable while there is no quadratic common Lyapunov function for the individual subsystems (see the example in [4]).

A Basic facts about Lie algebras

In this appendix we give an informal overview of basic properties of Lie algebras. Only those facts that directly play a role in the developments of the previous sections are discussed. Most of the material is adopted from [8, 20], and the reader is referred to these and other standard references for more details.

A.1 Lie algebras and their representations

A *Lie algebra* $\mathfrak{g}$ is a finite-dimensional vector space equipped with a *Lie bracket*, i.e., a bilinear, skew-symmetric map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$. Any Lie algebra $\mathfrak{g}$ can be identified with a tangent space at the identity of a Lie group $\mathcal{G}$ (an analytic manifold with a group structure). If $\mathfrak{g}$ is a matrix Lie algebra, then the elements of $\mathcal{G}$ are given by products of the exponentials of the matrices from $\mathfrak{g}$. In particular, each element $A \in \mathfrak{g}$ generates the *one-parameter subgroup* $\{e^{At}, t \in \mathbb{R}\}$ in $\mathcal{G}$. For example, if $\mathfrak{g}$ is the Lie algebra $gl(n, \mathbb{R})$ of all real $n \times n$ matrices with the standard Lie bracket $[A, B] = AB - BA$, then the corresponding Lie group is given by the invertible matrices.

Given an abstract Lie algebra $\mathfrak{g}$, one can consider its (matrix) representations. A *representation* of $\mathfrak{g}$ on an n-dimensional vector space V is a homomorphism (i.e., a linear map that preserves the Lie bracket) $\phi : \mathfrak{g} \to gl(V)$. It assigns to each element $g \in \mathfrak{g}$ a linear operator $\phi(g)$ on V, which can be described by an $n \times n$ matrix. A representation ϕ is called *irreducible* if V contains no nontrivial subspaces invariant under the action of all $\phi(g)$, $g \in \mathfrak{g}$. A particularly useful representation is the *adjoint* one, denoted by 'ad'. The vector space V in this case is $\mathfrak{g}$ itself, and for $g \in \mathfrak{g}$ the operator $\mathrm{ad}g$ is defined by $\mathrm{ad}g(a) := [g, a]$, $a \in \mathfrak{g}$. There is also *Ado's Theorem* which says that every Lie algebra is isomorphic to a subalgebra of $gl(V)$ for some finite-dimensional vector space V (compare this with the adjoint representation which is in general not injective).

A.2 Example: $sl(2, \mathbb{R})$ and $gl(2, \mathbb{R})$

The *special linear Lie algebra* $sl(2, \mathbb{R})$ consists of all real 2×2 matrices of trace 0. A canonical basis for this Lie algebra is given by the matrices

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{14}$$

They satisfy the relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, and form what is sometimes called an $sl(2)$-*triple*. One can also consider other representations of $sl(2, \mathbb{R})$. Although all irreducible representations of $sl(2, \mathbb{R})$ can be classified by working with the Lie algebra directly (see [20, p. 27–30]), for our purposes it is more useful to exploit the corresponding Lie group $SL(2,\mathbb{R}) = \{S \in \mathbb{R}^{n\times n} : \det S = 1\}$. Let $P^k[x,y]$ denote the space of polynomials in two indeterminates x and y that are homogeneous of degree k (where k is a positive integer). A homomorphism ϕ that makes $SL(2,\mathbb{R})$ act on $P^k[x,y]$ can be defined as follows:

$$\phi(S)p\Big(\begin{pmatrix} x \\ y \end{pmatrix}\Big) = p\Big(S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\Big)$$

where $S \in SL(2,\mathbb{R})$ and $p \in P^k[x,y]$. The corresponding representation of the Lie algebra $sl(2,\mathbb{R})$, which we denote also by ϕ with slight abuse of notation, is obtained by considering the one-parameter subgroups of $SL(2,\mathbb{R})$ and differentiating the action defined above at $t=0$. For example, for e as in (14) we have

$$\phi(e)p\Big(\begin{pmatrix} x \\ y \end{pmatrix}\Big) = \frac{d}{dt}\Big|_{t=0} p\Big(e^{-et}\begin{pmatrix} x \\ y \end{pmatrix}\Big) = \frac{d}{dt}\Big|_{t=0} p\Big(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\Big) = -y\frac{\partial}{\partial x}p\Big(\begin{pmatrix} x \\ y \end{pmatrix}\Big)$$

Similarly, $\phi(f)p = -x\frac{\partial}{\partial y}p$ and $\phi(h)p = (-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})p$. With respect to the basis in $P^k[x,y]$ given by the monomials $y^k, -ky^{k-1}x, k(k-1)y^{k-2}x^2, \ldots, (-1)^k k! x^k$, the corresponding differential operators are realized by the matrices

$$h \mapsto \begin{pmatrix} k & \cdots & \cdots & 0 \\ \vdots & k-2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & -k \end{pmatrix}, \; e \mapsto \begin{pmatrix} 0 & \mu_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mu_k \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \; f \mapsto \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

where $\mu_i = i(k-i+1)$, $i = 1, \ldots, k$. It turns out that any irreducible representation of $sl(2,\mathbb{R})$ of dimension $k+1$ is equivalent (under a linear change of coordinates) to the one just described. An arbitrary representation of $sl(2,\mathbb{R})$ is a direct sum of irreducible ones.

When working with $gl(2,\mathbb{R})$ rather than $sl(2,\mathbb{R})$, one also has the 2×2 identity matrix $I_{2\times 2}$. It corresponds to the operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $P^k[x,y]$, whose associated matrix is $kI_{(k+1)\times(k+1)}$. One can thus naturally extend the above representation to $gl(2,\mathbb{R})$. The complementary subalgebras $\mathbb{R}I$ and $sl(2,\mathbb{R})$ are invariant under the resulting action.

A.3 Nilpotent and solvable Lie algebras

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are linear subspaces of a Lie algebra $\mathfrak{g}$, one writes $[\mathfrak{g}_1, \mathfrak{g}_2]$ for the linear space spanned by all the products $[g_1, g_2]$ with $g_1 \in \mathfrak{g}_1$ and $g_2 \in \mathfrak{g}_2$. Given a Lie algebra $\mathfrak{g}$, the sequence $\mathfrak{g}^{(k)}$ is defined inductively as follows: $\mathfrak{g}^{(1)} := \mathfrak{g}$, $\mathfrak{g}^{(k+1)} := [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(k)}$. If $\mathfrak{g}^{(k)} = 0$ for k sufficiently large, then $\mathfrak{g}$ is called *solvable*. Similarly, one defines the sequence $\mathfrak{g}^k$ by $\mathfrak{g}^1 := \mathfrak{g}$, $\mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k] \subset \mathfrak{g}^k$, and calls $\mathfrak{g}$ *nilpotent* if $\mathfrak{g}^k = 0$ for k sufficiently large. For example, if $\mathfrak{g}$ is a Lie algebra generated by two matrices A and B, we have: $\mathfrak{g}^{(1)} = \mathfrak{g}^1 = \mathfrak{g} = \mathrm{span}\{A, B, [A,B], [A,[A,B]], \ldots\}$, $\mathfrak{g}^{(2)} = \mathfrak{g}^2 = \mathrm{span}\{[A,B], [A,[A,B]], \ldots\}$, $\mathfrak{g}^{(3)} = \mathrm{span}\{[[A,B],[A,[A,B]]], \ldots\} \subset \mathfrak{g}^3 = \mathrm{span}\{[A,[A,B]], [B,[A,B]], \ldots\}$, and so on. Every nilpotent Lie algebra is solvable, but the converse is not true.

The *Killing form* on a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form K given by $K(a,b) := \mathrm{tr}(\mathrm{ad}a \circ \mathrm{ad}b)$ for $a, b \in \mathfrak{g}$. *Cartan's 1st criterion* says that $\mathfrak{g}$ is solvable if and only if its Killing form vanishes identically on $[\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field, and let ϕ be a representation of $\mathfrak{g}$ on a vector space V. *Lie's Theorem* states that there exists a basis for V with respect to which all the matrices $\phi(g)$, $g \in \mathfrak{g}$ are upper-triangular.

A.4 Semisimple and compact Lie algebras

A subalgebra $\bar{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is called an *ideal* if $[g, \bar{g}] \in \bar{\mathfrak{g}}$ for all $g \in \mathfrak{g}$ and $\bar{g} \in \bar{\mathfrak{g}}$. Any Lie algebra has a unique maximal solvable ideal $\mathfrak{r}$, the *radical*. A Lie algebra $\mathfrak{g}$ is called *semisimple* if its radical is 0.

Cartan's 2nd criterion says that $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate (meaning that if for some $g \in \mathfrak{g}$ we have $K(g,a) = 0\ \forall a \in \mathfrak{g}$, then g must be 0.)

A semisimple Lie algebra is called *compact* if its Killing form is negative definite. A general *compact Lie algebra* is a direct sum of a semisimple compact Lie algebra and a commutative Lie algebra (with the Killing form vanishing on the latter). This terminology is justified by the facts that the tangent algebra of any compact Lie group is compact according to this definition, and that for any compact Lie algebra $\mathfrak{g}$ there exists a connected compact Lie group $\mathcal{G}$ with tangent algebra $\mathfrak{g}$. Compactness of a semisimple matrix Lie algebra $\mathfrak{g}$ amounts to the property that the eigenvalues of all matrices in $\mathfrak{g}$ lie on the imaginary axis. If $\mathcal{G}$ is a compact Lie group, one can associate to any continuous function $f : \mathcal{G} \to \mathbb{R}$ a real number $\int_{\mathcal{G}} f(G)dG$ so as to have $\int_{\mathcal{G}} 1dG = 1$ and $\int_{\mathcal{G}} f(AGB)dG = \int_{\mathcal{G}} f(G)dG \quad \forall A, B \in \mathcal{G}$ (left and right invariance). The measure dG is called the *Haar measure.*

An arbitrary Lie algebra $\mathfrak{g}$ can be decomposed into the semidirect sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{r}$ is the radical, $\mathfrak{s}$ is a semisimple subalgebra, and $[\mathfrak{s}, \mathfrak{r}] \subseteq \mathfrak{r}$ because $\mathfrak{r}$ is an ideal. This is known as a *Levi decomposition.* To compute $\mathfrak{r}$ and $\mathfrak{s}$, switch to a basis in which the Killing form K is diagonalized. The subspace on which K is not identically zero corresponds to $\mathfrak{s} \oplus (\mathfrak{r} \bmod \mathfrak{n})$, where $\mathfrak{n}$ is the maximal nilpotent subalgebra of $\mathfrak{r}$. Construct the Killing form $\bar{K}$ for the factor algebra $\mathfrak{s} \oplus (\mathfrak{r} \bmod \mathfrak{n})$. This form will vanish identically on $(\mathfrak{r} \bmod \mathfrak{n})$ and will be nondegenerate on $\mathfrak{s}$. The subalgebra $\mathfrak{s}$ identified in this way is compact if and only if $\bar{K}$ is negative definite on it. For more details on this construction and examples, see [7, pp. 256–258].

A.5 Subalgebras isomorphic to $sl(2, \mathbb{R})$

Let $\mathfrak{g}$ be a real, noncompact, semisimple Lie algebra. Our goal here is to show that $\mathfrak{g}$ has a subalgebra isomorphic to $sl(2, \mathbb{R})$. To this end, consider a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is its orthogonal complement with respect to K. The Killing form K is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal commuting subalgebra of $\mathfrak{p}$. Then it is easy to check using the Jacobi identity that the operators $\mathrm{ad}a$, $a \in \mathfrak{a}$ are commuting. These operators are also symmetric with respect to a suitable inner product on $\mathfrak{g}$ (for $a, b \in \mathfrak{g}$ this inner product is given by $-K(a, \Theta b)$, where Θ is the map sending $k+p$, with $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$, to $k-p$), hence they are simultaneously diagonalizable. Thus $\mathfrak{g}$ can be decomposed into a direct sum of subspaces invariant under $\mathrm{ad}a$, $a \in \mathfrak{a}$, on each of which every operator $\mathrm{ad}a$ has exactly one eigenvalue. The unique eigenvalue of $\mathrm{ad}a$ on each of these invariant subspaces is given by a linear function λ on $\mathfrak{a}$, and accordingly the corresponding subspace is denoted by $\mathfrak{g}_\lambda$. Since $\mathfrak{p} \neq 0$ (because $\mathfrak{g}$ is not compact) and since K is positive definite on $\mathfrak{p}$, the subspace $\mathfrak{g}_0$ associated with λ being identically zero cannot be the entire $\mathfrak{g}$. Summarizing, we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \big(\textstyle\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \big)$$

where Σ is a finite set of nonzero linear functions on $\mathfrak{a}$ (which are called the *roots*) and $\mathfrak{g}_\lambda = \{g \in \mathfrak{g} : \mathrm{ad}a(g) = \lambda(a)g\ \forall a \in \mathfrak{a}\}$. Using the Jacobi identity, one can show that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu]$ is a subspace of $\mathfrak{g}_{\lambda+\mu}$ if $\lambda + \mu \in \Sigma \cup \{0\}$, and equals 0 otherwise. This implies that the subspaces $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ are orthogonal with respect to K unless $\lambda + \mu = 0$ (cf. [20, p. 38]). Since K is nondegenerate on $\mathfrak{g}$, it follows that if λ is a root, then so is $-\lambda$. Moreover, the subspace $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}]$ of $\mathfrak{g}_0$ has dimension 1, and λ is not identically zero on it (cf. [20, pp. 39–40]). This means that there exist some elements $e \in \mathfrak{g}_\lambda$ and $f \in \mathfrak{g}_{-\lambda}$ such that $h := [e, f] \neq 0$. It is now easy to see that, multiplying e, f and h by constants if necessary, we obtain an $sl(2)$-triple. Alternatively, we could finish the argument by noting that if $g \in \mathfrak{g}_\lambda$ for some $\lambda \in \Sigma$, then the operator $\mathrm{ad}g$ is nilpotent (because it maps each $\mathfrak{g}_\mu$ to $\mathfrak{g}_{\mu+\lambda}$ to $\mathfrak{g}_{\mu+2\lambda}$ and eventually to 0 since Σ is a finite set), and the existence of a subalgebra isomorphic to $sl(2, \mathbb{R})$ is guaranteed by the Jacobson-Morozov Theorem.

A.6 Generators for $gl(2,\mathbb{R})$

This subsection is devoted to showing that in an arbitrarily small neighborhood of any pair of $n\times n$ matrices one can find another pair of matrices that generate the entire Lie algebra $gl(n,\mathbb{R})$. This fact demonstrates that Lie-algebraic stability conditions considered in the previous sections are never robust with respect to small perturbations of the matrices that define the switched system. Constructions like the one presented here have certainly appeared in the literature, but we are not aware of a specific reference.

We begin by finding some matrices B_1, B_2 that generate $gl(n,\mathbb{R})$. Let B_1 be a diagonal matrix $B_1 = \mathrm{diag}(b_1, b_2, \dots, b_n)$ satisfying the following two properties:

1. $b_i - b_j \neq b_k - b_l$ if $(i,j) \neq (k,l)$
2. $\sum_{i=1}^{n} b_i \neq 0$

Denote by $od(n,\mathbb{R})$ the space of matrices with zero elements on the main diagonal. Let B_2 be any matrix in $od(n,\mathbb{R})$ such that all its off-diagonal elements are nonzero. It is easy to check that if $E_{i,j}$ is a matrix whose ij-th element is 1 and all other elements are 0, where $i \neq j$, then $[B_1, E_{i,j}] = (b_i - b_j)E_{i,j}$. Thus it follows from property 1 above that B_2 does not belong to any proper subspace of $od(n,\mathbb{R})$ that is invariant with respect to the operator $\mathrm{ad}B_1$. Therefore, the linear space spanned by the iterated brackets $ad^k B_1(B_2)$ is the entire $od(n,\mathbb{R})$. Taking brackets of the form $[E_{i,j}, E_{-i,-j}]$, we generate all traceless diagonal matrices (cf. the example $[e,f]=h$ in Section A.2). Since B_1 has a nonzero trace by property 2 above, we conclude that $\{B_1, B_2\}_{LA} = gl(n,\mathbb{R})$.

Now, let A_1 and A_2 be two arbitrary $n \times n$ matrices. Using the matrices B_1 and B_2 just constructed, we can define $A_1(\alpha) := A_1 + \alpha B_1$ and $A_2(\alpha) := A_2 + \alpha B_2$, where $\alpha \geq 0$. The two matrices $A_1(\alpha)$ and $A_2(\alpha)$ generate $gl(n,\mathbb{R})$ for any sufficiently small α, as can be shown by using the same argument as the one employed at the end of the proof of Theorem 4. Thus one can take $(A_1(\alpha), A_2(\alpha))$ as a desired pair of matrices in a neighborhood of (A_1, A_2).

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References

[1] D. Angeli, A note on stability of arbitrarily switched homogeneous systems, *Systems Control Lett.*, to appear.

[2] J. P. Aubin, A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[3] P. E. Crouch, Dynamical realizations of finite Volterra series, *SIAM J. Control Optim.*, vol. 19, 1981, pp. 177–202.

[4] W. P. Dayawansa, C. F. Martin, A converse Lyapunov theorem for a class of dynamical systems which undergo switching, *IEEE Trans. Automat. Control*, vol. 44, 1999, pp. 751–760.

[5] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer, Dordrecht, 1988.

[6] F. R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing, New York, 1960.

[7] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, John Wiley, New York, 1974.

[8] V. V. Gorbatsevich, A. L. Onishchik, E. B. Vinberg, *Structure of Lie Groups and Lie Algebras*, vol. 41 of *Encyclopaedia of Mathematical Sciences*, Springer-Verlag, Berlin, 1994.

[9] L. Gurvits, Stability of discrete linear inclusion, *Lin. Alg. Appl.*, vol. 231, 1995, pp. 47–85.

[10] M. Kawski, Nilpotent Lie algebras of vectorfields, *J. Reine Angew. Math.*, vol. 388, 1988, pp. 1–17.

[11] H. K. Khalil, *Nonlinear Systems*, Macmillan, New York, 1992.

[12] D. Liberzon, J. P. Hespanha, A. S. Morse, Stability of switched systems: a Lie-algebraic condition, *Systems Control Lett.*, vol. 37, 1999, pp. 117–122.

[13] D. Liberzon, A. S. Morse, Basic problems in stability and design of switched systems, *IEEE Control Systems Magazine*, vol. 19, 1999, pp. 59–70.

[14] A. Marigo, Constructive necessary and sufficient conditions for strict triangularizability of driftless nonholonomic systems, in *Proc. 38th Conf. on Decision and Control*, 1999, pp. 2138–2143.

[15] A. P. Molchanov, Y. S. Pyatnitskiy, Criteria of absolute stability of differential and difference inclusions encountered in control theory, *Systems Control Lett.*, vol. 13, 1989, pp. 59–64.

[16] Y. Mori, T. Mori, Y. Kuroe, A solution to the common Lyapunov function problem for continuous-time systems, in *Proc. 36th Conf. on Decision and Control*, 1997, pp. 3530–3531.

[17] A. S. Morse, Control using logic-based switching, in *Trends in Control: a European Perspective* (A. Isidori, Ed.), Springer-Verlag, 1995, pp. 69–113.

[18] K. S. Narendra, J. Balakrishnan, A common Lyapunov function for stable LTI systems with commuting A-matrices, *IEEE Trans. Automat. Control*, vol. 39, 1994, pp. 2469–2471.

[19] T. Ooba, Y. Funahashi, On a common quadratic Lyapunov function for widely distant systems, *IEEE Trans. Automat. Control*, vol. 42, 1997, pp. 1697–1699.

[20] H. Samelson, *Notes on Lie Algebras*, Van Nostrand Reinhold, New York, 1969.

[21] H. Shim, D. J. Noh, J. H. Seo, Common Lyapunov function for exponentially stable nonlinear systems, presented at the *4th SIAM Conf. Control and its Applications*, 1998.

[22] R. N. Shorten, K. S. Narendra, Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for two stable second order linear time-invariant systems, Technical Report 9806, Yale University, 1998.