QUANTITATIVE TIGHTNESS FOR THREE-DIMENSIONAL CONTACT MANIFOLDS: A SUB-RIEMANNIAN APPROACH

ANDREI A. AGRACHEV[1], STEFANO BARANZINI[2], EUGENIO BELLINI[3], AND LUCA RIZZI[1]

ABSTRACT. Through the use of sub-Riemannian metrics we provide quantitative estimates for the maximal tight neighbourhood of a Reeb orbit on a three-dimensional contact manifold. Under appropriate geometric conditions we show how to construct closed curves which are boundaries of overtwisted disks. We introduce the concept of *contact* Jacobi curve, and prove sharp lower bounds of the so-called tightness radius (from a Reeb orbit) in terms of Schwarzian derivative bounds. We also prove similar, but non-sharp, comparison theorems in terms of sub-Riemannian canonical curvature bounds. We apply our results to K-contact sub-Riemannian manifolds. In this setting, we prove a contact analogue of the celebrated Cartan-Hadamard theorem.

CONTENTS

[1]SISSA, VIA BONOMEA 265, 34136 TRIESTE, ITALY

[2]DIPARTIMENTO DI MATEMATICA "GIUSEPPE PEANO", UNIVERSITÀ DEGLI STUDI DI TORINO, TORINO, ITALY

[3]DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA, MILANO, ITALY.

Date: June 28, 2024.

1. Introduction

A (co-oriented) contact structure Δ on a 3-manifold M is a totally non-integrable plane field, i.e. $\Delta = \ker \omega$ for some one-form ω satisfying the non-integrability condition $\omega \wedge d\omega \neq 0$. A contact manifold is called *overtwisted* if it contains an embedded disk such that Δ is tangent to the disk in a unique interior point and along the boundary (see Definition 2.4). A contact manifold is called *tight* if it is not overtwisted.

The two categories defined by the tight/overtwisted dichotomy have demonstrated surprisingly different properties. Even tough overtwisted contact structures on 3-manifolds have been classified [Eli89] and tight contact structures have been extensively studied, it is not easy to determine whether a given contact structure is tight or overtwisted.

The well-known Darboux theorem states that contact structures can be locally normalized: any point in a contact manifold has a neighbourhood diffeomorphic to the standard contact structure $(\mathbb{R}^3, \omega_{\text{st}})$, where $\omega_{\text{st}} = dz + \frac{1}{2}(xdy - ydx)$. It is well-known that the latter structure is tight [Ben83]. Combining this fact and Darboux theorem we deduce that every contact structure is locally tight. If we endow the contact manifold with a metric structure, then it makes sense to ask the following question: what is the size of the maximal tight neighbourhood of a given point? In [EKM12, EKM16] the authors estimated the answer in the context of Riemannian geometry, unveiling surprising connections between different notions of convexity, Riemannian curvature, and tightness.

In this paper, instead, we investigate tightness criteria and geometric detection of overtwisted disks via sub-Riemannian geometry. The fact that the sub-Riemannian geodesics are tangent to Δ makes this framework natural and well-adapted to contact geometry. In this regard, recent works [BBC22, BB23] have linked sub-Riemannian geometry to contact topology.

In order to introduce our results, we anticipate some definitions. A (three-dimensional) contact sub-Riemannian manifold (M, ω, g) is a co-orientable contact manifold (M, Δ), with $\Delta = \ker \omega$, endowed with a bundle metric g on Δ. Any positive rescaling of ω defines the same contact structure, which can be then normalized via the sub-Riemannian metric. This choice, in turn, yields a canonical complement of Δ: the so-called *Reeb field*, denoted by f_0.

To every (M, ω, g) there is an associated *sub-Riemannian distance* d_{SR}. For any $p, q \in M$, $d_{SR}(p, q)$ is defined as the infimum of the lengths of the curves tangent to Δ and joining the two points. Thanks to the contact condition, (M, d_{SR}) is a metric space with the same topology of M. In contact sub-Riemannian manifolds, geodesics (i.e., locally length-minimizing curves

parametrized by constant speed) are projections of solutions of a Hamiltonian system on T^*M. To introduce it, we define the sub-Riemannian Hamiltonian $H : T^*M \to \mathbb{R}$ as

$$H(\lambda) := \frac{1}{2} \sup \left\{ \langle \lambda, v \rangle \,\Big|\, v \in \Delta_{\pi(\lambda)},\, \|v\|_g = 1 \right\}, \qquad \lambda \in T^*M,$$

which is a smooth and fiber-wise non-negative quadratic form (albeit degenerate and with non-compact level sets). We denote with $\vec{H}$ the corresponding Hamiltonian vector field, i.e. the vector field on T^*M defined as the symplectic gradient of H. Thus, every contact sub-Riemannian geodesics are precisely the projections of integral curves of $\vec{H}$. In the following, we tacitly assume that (M, d_{SR}) is a complete metric space, in which case $\vec{H}$ is a complete vector field.

1.1. Tightness radius from a Reeb orbit.
Let Γ be an embedded piece of Reeb orbit (which can also be closed, see Section 3). The annihilator bundle of Γ is

$$A\Gamma := \{\lambda \in T^*M \mid \pi(\lambda) \in \Gamma,\ \langle \lambda, f_0 \rangle = 0\}.$$

Given $\rho > 0$ we also define the following bundles:

$$A^{<\rho}\Gamma := A\Gamma \cap \{\sqrt{2H} < \rho\}, \qquad A^1\Gamma := A\Gamma \cap \{\sqrt{2H} = 1\}.$$

$A\Gamma$ (resp. $A^1\Gamma$) plays the same role of the normal bundle (resp. unit normal bundle) in Riemannian geometry. The injectivity radius of Γ is defined as

$$r_{\text{inj}}(\Gamma) = \sup\{\rho > 0 \mid E : A^{<\rho}\Gamma \to M \text{ is a diffeomorphism onto its image}\},$$

where E is the restriction ot $A^1\Gamma$ of the sub-Riemannian Hamiltonian flow, more precisely

$$E : A^1\Gamma \to M, \qquad E(\lambda) := \pi \circ e^{\vec{H}}(\lambda).$$

It turns out that, for $\lambda \in A^{<\rho}\Gamma$, for $\rho = r_{\text{inj}}(\Gamma)$, we have $d_{SR}(\Gamma, E(\lambda)) = 2H(\lambda)$ so that sub-Riemannian geodesic involved in this construction are precisely those realizing the minimal distance from Γ. We are interested in the size of the largest such neighbourhood on which the contact structure is tight.

Definition (Tightness radius). *Let (M, ω, g) be a contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\text{inj}}(\Gamma) > 0$. The tightness radius is*

$$r_{\text{tight}}(\Gamma) := \sup \left\{ 0 < r < r_{\text{inj}}(\Gamma) \mid (A^{<r}\Gamma, E^*\omega) \text{ is tight} \right\}.$$

1.2. Contact Jacobi curves.
The main tool that we introduce to study the tightness radius is the *contact Jacobi curve*. These are inspired by Jacobi curves in (sub-)Riemannian geometry and geometric control theory, which are curves in the Lagrange Grassmannian whose dynamics is intertwined with the presence of conjugate points and curvature (see [AG97, AZ02a, AZ02b, ABR18] and references within). The dynamics of contact Jacobi curves, instead, is related with the presence of overtwisted disks.

Definition (Contact Jacobi curve). *Let (M, ω, g) be a contact sub-Riemannian manifold, and let Γ be an embedded piece of a Reeb orbit. Consider the following one-parameter family of one-forms on $A^1\Gamma$:*

$$\omega_r := \left((\pi \circ e^{r\vec{H}})^* \omega \right)\Big|_{A^1\Gamma}, \qquad r \in [0, +\infty).$$

The contact Jacobi curve at $\lambda \in A^1\Gamma$ is the projectivization of $\omega_r|_\lambda$:

$$\Omega_\lambda : [0, +\infty) \to P(T^*_\lambda A^1\Gamma) \simeq \mathbb{RP}^1, \qquad \Omega_\lambda(r) := P(\omega_r|_\lambda).$$

The first singular radius of the contact Jacobi curve is

$$r_o(\lambda) := \inf\{r > 0 \mid \Omega_\lambda(r) = \Omega_\lambda(0)\}.$$

A first important result is a structural theorem for the contact Jacobi curve, namely Theorem 5.12, where we also prove that the definition above is well-posed for all $r \in \mathbb{R}$. Theorem 5.12 is technical, and for this reason it is omitted from this introduction.

The first singular time of contact Jacobi curves detects the presence of overtwisted disks. This is our second main result, corresponding to Theorem 5.16 in the body of the paper.

Theorem A (Geometric tightness radius estimates)**.** *Let* (M, ω, g) *be a contact sub-Riemannian manifold and let* Γ *be an embedded piece of a Reeb orbit with* $r_{\mathrm{inj}}(\Gamma) > 0$*. Then, letting*

$$r_o^-(\Gamma) := \inf\{r_o(\lambda) \mid \lambda \in A^1\Gamma\}, \qquad r_o^+(\Gamma) := \sup\{r_o(\lambda) \mid \lambda \in A^1\Gamma\},$$

the following estimates hold:

$$\min\left\{r_{\mathrm{inj}}(\Gamma), r_o^-(\Gamma)\right\} \le r_{\mathrm{tight}}(\Gamma) \le \min\left\{r_{\mathrm{inj}}(\Gamma), r_o^+(\Gamma)\right\}.$$

Moreover, if $r_o^+(\Gamma) < r_{\mathrm{inj}}(\Gamma)$*, then for any* $q \in \Gamma$*, the set*

$$D_q := \{E(r\lambda) \mid \lambda \in A_q^1\Gamma, \ r \le r_o(\lambda)\}$$

is an overtwisted disk, and thus (M, ω) *is an overtwisted contact manifold.*

1.3. **Comparison theory via Schwarzian derivative.** We apply Theorem A by estimating the first singular radius in two conceptually different ways. The first one, which is natural in view of the definition of the contact Jacobi curve as curve in $\mathbb{RP}^1$, is through its Schwarzian derivative (see Section 5.1). This yields a sharp comparison theorem for the tightness radius.

Theorem B (Tightness radius estimates with Schwarzian derivative bounded above)**.** *Let* (M, ω, g) *be a contact sub-Riemannian manifold and let* Γ *be an embedded piece of a Reeb orbit with* $r_{\mathrm{inj}}(\Gamma) > 0$*. Assume that there exist* $k_1, k_2 \in \mathbb{R}$ *such that the Schwarzian derivative of the contact Jacobi curves are bounded above by:*

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) \le -\frac{3}{4r^2} + k_1 r + k_2 r^2, \qquad \forall r \in (0, r_{\mathrm{inj}}(\Gamma)], \quad \forall \lambda \in A^1\Gamma. \tag{1}$$

Then for the tightness radius of Γ *it holds*

$$r_{\mathrm{tight}}(\Gamma) \ge \min\left\{r_*(k_1, k_2), r_{\mathrm{inj}}(\Gamma)\right\},$$

where

$$r_*(k_1, k_2) := \begin{cases} \frac{-k_1 + \sqrt{8\pi k_2^{3/2} + k_1^2}}{2k_2}, & \text{if } k_1, k_2 > 0, \\ \frac{\sqrt{2\pi}}{k_2^{1/4}}, & \text{if } k_1 \le 0, \ k_2 > 0, \\ \left(\frac{3j_{2/3}}{2}\right)^{2/3} \frac{1}{k_1^{1/3}}, & \text{if } k_1 > 0, \ k_2 \le 0, \\ +\infty & \text{if } k_1, k_2 \le 0, \end{cases}$$

and $j_{2/3} \sim 3.37$ *is the first positive root of the Bessel function of first kind* $J_{2/3}$*.*

The form of the upper bound (1) is motivated by the structure theorem for contact Jacobi curves Theorem 5.12. All three terms in the r.h.s. of (1) are needed in order to get sharp and general results. We refer to Remark 5.18 and Remark 5.20 for details.

Furthermore, Theorem B is sharp for the standard sub-Riemannian contact structure (resp. overtwisted structure) of Example 2.9 (resp. Example 2.10). See Remark 5.19 for details.

1.4. **Comparison theory via canonical curvature.** A second way to estimate the singular radius of a contact Jacobi curve, and thus the tightness radius from a Reeb orbit, is via curvature-type invariants. Our result if formulated in terms of the so-called *canonical curvatures*. The canonical curvature plays the role of sectional curvature in sub-Riemannian comparison geometry. Historically, it was introduced in [AG97, AZ02a], formalized in [ZL09], and further studied and developed in [LZ11, LLZ16, AL14, AL15, ABR18, ABR17, BR16, BR20, BR17]. See Section 5.5 for details. In the contact case, there are only two such curvatures, associated to a given sub-Riemannian geodesic $\gamma^\lambda : [0, T] \to M$, with initial covector $\lambda \in T^*M$, and are denoted by $R_a^\lambda, R_c^\lambda : [0, T] \to \mathbb{R}$. The next result corresponds to Theorem 5.24.

Theorem C (Tightness radius estimates with curvature bounded above). *Let (M, ω, g) be a contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\mathrm{inj}}(\Gamma) > 0$. Assume that there exist $A, C > 0$ such that*

$$\sqrt{1 + R_a^\lambda(r)^2} \leq A, \quad \sqrt{1 + R_c^\lambda(r)^2} \leq C, \qquad \forall\, r \in [0, r_{\mathrm{inj}}(\Gamma)], \quad \forall\, \lambda \in A^1\Gamma,$$

where $R_a^\lambda(r), R_c^\lambda(r)$ are the canonical curvatures at λ. Then for the tightness radius of Γ it holds

$$r_{\mathrm{tight}}(\Gamma) \geq \min\left\{ \tau(A, C), r_{\mathrm{inj}}(\Gamma) \right\},$$

where

$$\tau(A, C) := \int_0^\infty \frac{1}{Au^2 + Cu + 1}\, \mathrm{d}u.$$

Theorem C provides a link between tightness and sub-Riemannian curvature invariants for general structures. However, it is our opinion that curvature is not as precise as contact Jacobi curves and their Schwarzian derivative in the detection of tightness, as it is shown by the (sharp) Theorem B. As it will be clear from the proof, Theorem C is non-sharp, even for the standard contact structure. Theorem C is in the same spirit of [EKM16, Thms. 1.6, 1.7], which are rather expressed in terms of the *Riemannian* (sectional) curvature of an associated metric.

1.5. **Hadamard theorem for K-contact structures.** A contact sub-Riemannian manifold (M, ω, g) is called K-contact if the Reeb flow acts on M by isometries. Denote with κ the Gaussian curvature of the surface obtained by locally quotienting M under the action of the Reeb flow, see Remark 2.8. It is a well-defined function on M, constant along the Reeb orbits. We prove a contact analogue of the celebrated Cartan-Hadamard theorem (see Theorem 6.5).

Theorem D (Contact Cartan-Hadamard). *Let (M, ω, g) be complete and simply connected K-contact sub-Riemannian manifold. If $\kappa \leq 0$ then $(M, \Delta = \ker \omega)$ is contactomorphic to the standard contact structure.*

It follows from [Nie06, NP09] that every K-contact structure is tight (see [EKM16, Rmk. 1.3]). Under the non-positive curvature and simply connected assumptions, Theorem D tells us that the structure is contactomorphic to the standard one. The proof of Theorem D actually tells us something more on the fundamental group of M, as stated in the next result (see Corollary 6.8).

Corollary E. *Let (M, ω, g) be complete K-contact sub-Riemannian manifold. If $\kappa \leq 0$ then any closed orbit of the Reeb field is the generator of an infinite cyclic subgroup of the fundamental group $\pi_1(M)$.*

In Section 6.1 we show how to apply Theorem D to the contact structures coming from the Boothby-Wang construction. Such a construction yields K-contact structures on principal circle bundles over a Riemannian surface (B, η) with prescribed curvature κ_η (see Theorem 6.9). We show that any such contact manifold, provided that $\kappa_\eta \leq 0$, has tight universal cover.

In Section 6.2 we show that, analogously to the classical Cartan-Hadamard theorem, the assumption of non-positive κ in Theorem D (and Corollary E) can be weakened. In particular, we show an example of K-contact sub-Riemannian manifold with $\kappa > 0$ and for which the proof of Theorem D and their thesis holds unchanged.

1.6. **Structure of the paper.** In Section 2 we review some preliminaries in contact sub-Riemannian geometry. In Section 3 we introduce special coordinates on the annihilator of a Reeb orbit, instrumental for our proofs. The tightness radius is defined in Section 4, and it is computed for a class of models (Section 4.1). In the same section we study the relation between the tightness radius and the so-called singular locus of a Reeb orbit (Section 4.2). The core of the paper is Section 5, where we introduce contact Jacobi curves, and we prove Theorems A, B and C. Finally, in Section 6 we focus on K-contact structures, proving Theorem D and Corollary E. The core examples, namely the standard contact structure (resp. the standard overtwisted structure) on $\mathbb{R}^3$, are presented throughout the paper, in Examples 2.2 and 2.9 and Corollary 4.3 (resp. Examples 2.3 and 2.10 and Corollary 4.4).

2. Preliminaries

2.1. **Three-dimensional contact manifolds.** We review some preliminaries on contact manifolds. We refer e.g. to [Mas14, Gei08] for more details. A three dimensional contact manifold (M, Δ) is a smooth, connected 3-manifold M endowed with a contact distribution, i.e., a plane field $\Delta \subset TM$ satisfying the non-integrability condition $\Delta + [\Delta, \Delta] = TM$. The latter equation is called the contact condition. We say that two contact manifolds (M, Δ) and (M', Δ') are *contactomorphic* if there exists a diffeomorphism $\psi : M \to M'$ such that $\psi_* \Delta = \Delta'$.

In the present paper we assume contact manifolds to be co-orientable, i.e., we assume the existence of a smooth one-form ω such that $\Delta = \ker \omega$. Such a differential form is called a *contact form*. Any positive rescaling of ω determines the same contact structure. The contact condition can be expressed in terms of the contact form:

$$\Delta + [\Delta, \Delta] = TM \iff \omega \wedge d\omega \text{ is a volume form} \iff d\omega|_\Delta \text{ is symplectic.}$$

The geometric meaning of such condition is that the plane field Δ twists monotonically along horizontal foliations, i.e., foliations by curves which are tangent to Δ. This prevents Δ from having any integral surface. A fundamental notion in the study of contact structures is that of a characteristic foliation.

Definition 2.1. *Let (M, Δ) be a contact manifold, and $\Sigma \subset M$ be a embedded surface. The characteristic foliation is the singular line field $\Delta \cap T\Sigma$ i.e., a rank 1 distribution with singularities. We say that $q \in \Sigma$ is* singular *if*

$$\Delta_q = T_q \Sigma.$$

Let us introduce some important examples, which are needed in the following.

Example 2.2 (Standard contact structure). The standard contact structure on $\mathbb{R}^3$ is defined, in cylindrical coordinates $x = r \cos \theta$, $y = r \sin \theta$, z, by the contact form

$$\omega_{\mathrm{st}} := dz + \frac{r^2}{2} d\theta, \qquad \Delta_{\mathrm{st}} := \ker \omega_{\mathrm{st}}.$$

Let $\Sigma = \{z = 0\}$. Its characteristic foliation $\Delta_{\mathrm{st}} \cap T\Sigma$ consists of radial lines and has a unique singularity at the origin. The distribution twists monotonically along radial lines: the angle between Δ_{st} and Σ is the monotone function

$$\phi_{\mathrm{st}}(r) := \arctan\left(\frac{r^2}{2}\right),$$

which reaches $\pi/2$ asymptotically as $r \to \infty$, namely:

$$\lim_{r \to \infty} \phi_{\mathrm{st}}(r) = \frac{\pi}{2}.$$

Example 2.3 (Standard overtwisted structure). The standard overtwisted structure on $\mathbb{R}^3$ is defined, in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, z, by the contact form

$$\omega_{\mathrm{ot}} := \cos\left(\frac{r^2}{2}\right) dz + \sin\left(\frac{r^2}{2}\right) d\theta, \qquad \Delta_{\mathrm{ot}} := \ker\omega_{\mathrm{ot}}.$$

Let $\Sigma := \{z = 0\}$. Its characteristic foliation $\Delta_{\mathrm{st}} \cap T\Sigma$ consists of radial lines and the singularities are the origin and the circles $C_k := \{(x, y, 0) \mid x^2 + y^2 = 2k\pi\}$, $k \in \mathbb{N}$. The distribution twists monotonically along radial lines: the angle between Δ_{ot} and Σ is the monotone function

$$\phi_{\mathrm{ot}}(r) := \frac{r^2}{2},$$

which rotates infinitely many times as $r \to \infty$, namely:

$$\lim_{r \to \infty} \phi_{\mathrm{st}}(r) = +\infty.$$

The disk delimited by the first singular circle,

$$D_{\mathrm{ot}} := \{(x, y, 0) \mid x^2 + y^2 \leq 2\pi\},$$

together with its characteristic foliation, is called the standard overtwisted disk.

Definition 2.4. *A contact manifold (M, Δ) is* overtwisted *if it contains an overtwisted disk. Namely, there exists an embedded disk $D \subset M$ and a diffeomorphism $\psi : D \to D_{\mathrm{ot}}$, such that*

$$\psi_* \left(\Delta \cap TD\right) = \left(\Delta_{\mathrm{ot}} \cap TD_{\mathrm{ot}}\right).$$

A contact manifold is called tight *if it is not overtwisted.*

Remark 2.5. Definition 2.4 is equivalent to the existence of neighbourhoods $U \subset \mathbb{R}^3$ of D_{ot}, $V \subset M$, and a diffeomorpshism $\Psi : V \to U$ such that $\Psi(D) = D_{\mathrm{ot}}$ and $\Psi_*\Delta = \Delta_{\mathrm{ot}}$. See [Mas14, Lemma 13 and Exercise p. 48].

Remark 2.6 (Universal tightness). Given a contact manifold (M, Δ) with universal cover $p : \tilde{M} \to M$, there is a natural way to endow $\tilde{M}$ with a contact structure, which at each $q \in \tilde{M}$ is determined by the equation $p_*\tilde{\Delta}_q = \Delta_{p(q)}$. If $(\tilde{M}, \tilde{\Delta})$ is tight then (M, Δ) is called *universally tight*. If (M, Δ) is overtwisted, then any overtwisted disk can be lifted to the universal cover. Therefore universal tightness implies tightness.

The standard overtwisted structure is overtwisted, while the standard contact structure Δ_{st}, as proven by Bennequin [Ben83], is tight. A well-known property of contact structures is that they can be locally normalized: according to the Darboux theorem every contact structure is locally contactomorphic to the standard one $(\mathbb{R}^3, \Delta_{\mathrm{st}})$. Therefore every contact structure is locally tight. If a contact manifold is endowed with a metric structure, it makes sense to ask what is the size of the maximal tight neighbourhood of a point. In this work the selected metric structure is a sub-Riemannian one. This choice is natural in view of the fact that length-minimizing curves are horizontal and the distribution twists monotonically along them, so that we can expect to detect overtwisted disks in terms of the sub-Riemannian distance.

2.2. **Contact sub-Riemannian geometry.** A (three-dimensional) contact sub-Riemannian manifold (M, ω, g) is a co-orientable contact manifold (M, Δ), with $\Delta = \ker \omega$, endowed with a bundle metric g on Δ, called sub-Riemannian metric. Note that as a consequence of the definition, M is oriented by the non-vanishing top-form $\omega \wedge d\omega$. The *Reeb field* f_0 is the vector field transverse to Δ defined by the conditions

$$\omega(f_0) = 1, \qquad d\omega(f_0, \cdot) = 0.$$

We extend the sub-Riemannian metric g to a Riemannian one by declaring f_0 to be orthonormal to Δ. We say that ω and g are *compatible* if

$$\omega \wedge d\omega = \mathrm{Vol}_g, \tag{2}$$

where Vol_g is the volume form induced by the extended metric on the oriented manifold M.

Note that for any contact sub-Riemannian manifold (M, ω, g) one can multiply ω by a suitable positive function $f : M \to \mathbb{R}$ so that the compatibility condition (2) is satisfied for $(M, f\omega, g)$. For this reason, in the following, we always assume that (2) is satisfied.

The sub-Riemannian structure induces a almost complex structure, i.e. a smooth bundle morphism $J : \Delta \to \Delta$ such that $J^2 = -\mathbb{1}$, which is defined by

$$d\omega(X, JY) = g(X, Y), \qquad \forall X, Y \in \Delta_p, \quad \forall p \in M. \tag{3}$$

Let f_0, f_1, f_2 be a local oriented orthonormal frame (in particular f_1, f_2 are horizontal). Let ν_0, ν_1, ν_2 the corresponding dual co-frame. Then $\omega = \nu_0$ and (2)-(3) read

$$d\nu_0 = \nu_1 \wedge \nu_2, \qquad Jf_1 = f_2, \qquad Jf_2 = -f_1.$$

The structural coefficients of the frame f_1, f_2, f_0 are the locally defined functions

$$c_{ij}^k := g([f_i, f_j], f_k), \qquad i, j, k = 0, 1, 2,$$

or equivalently

$$d\nu_k = - \sum_{0 \leq i < j = 2} c_{ij}^k \nu_i \wedge \nu_j, \qquad k = 0, 1, 2.$$

The next result is taken from [ABB20, Ch. 17] where a different convention is used for the sign of structure coefficient.

Proposition 2.7. *Let (M, ω, g) be a sub-Riemannian structure. Let f_0 be its Reeb vector field, and let f_0, f_1, f_2 be a local oriented orthonormal frame, then*

$$[f_1, f_2] = c_{12}^1 f_1 + c_{12}^2 f_2 - f_0,$$
$$[f_1, f_0] = c_{10}^1 f_1 + c_{10}^2 f_2,$$
$$[f_2, f_0] = c_{20}^1 f_1 + c_{20}^2 f_2.$$

Moreover, it holds

$$c_{01}^1 + c_{02}^2 = 0.$$

Furthermore, the following quantities

$$\chi := \sqrt{\left(c_{01}^1\right)^2 + \frac{1}{4}\left(c_{01}^2 + c_{02}^1\right)^2},$$

$$\kappa := f_1\left(c_{12}^2\right) - f_2\left(c_{12}^1\right) - \left(c_{12}^1\right)^2 - \left(c_{12}^2\right)^2 + \frac{c_{02}^1 - c_{01}^2}{2},$$

do not depend on the choice of f_1, f_2 and therefore are globally defined functions $\chi, \kappa : M \to \mathbb{R}$.

Remark 2.8. The functions χ and κ are invariant under the action of smooth sub-Riemannian isometries of (M, ω, g) (diffeomorphisms that preserve the sub-Riemannian distribution and its metric). The invariant χ vanishes if and only if the flow of the Reeb flow acts by sub-Riemannian isometries. The invariant κ has a less obvious geometric interpretation. If $\chi = 0$, then κ is the Gaussian curvature of the surface obtained by locally quotienting M under the action of the Reeb flow. More precisely, if $N = M/f_0$, i.e. the quotient of M by the Reeb flow, is a smooth surface, then it inherits a Riemannian structure with Gaussian curvature κ. More generally let $U \subset M$ be an open subset. Then $(U, \omega|_U, g|_U)$, is a smooth sub-Riemannian manifold. If the quotient manifold $U/(f_0|_U)$ is well-defined (which is the case when U is sufficiently small), then $U/(f_0|_U)$ has the natural structure of Riemannian surface with Gaussian curvature κ. For more details see [ABB20, Ch. 17].

Example 2.9 (Standard sub-Riemannian contact structure). Let $(\mathbb{R}^3, \omega_{\mathrm{st}})$ be the standard contact structure of Example 2.2. We define a sub-Riemannian metric g on $\Delta_{\mathrm{st}} = \ker \omega_{\mathrm{st}}$ such that the following global vector fields are orthonormal:

$$f_1 := \partial_x + \frac{y}{2}\partial_z, \qquad f_2 := \partial_y - \frac{x}{2}\partial_z.$$

The compatible contact form and the Reeb vector field are, respectively

$$\omega_{\mathrm{st}} = dz + \frac{1}{2}(xdy - ydx), \qquad f_0 = \partial_z.$$

The contact sub-Riemannian manifold $(\mathbb{R}^3, \omega_{\mathrm{st}}, g)$ is called the *Heisenberg group*, and its metric invariants vanish: $\chi = \kappa = 0$.

We introduce in this paper a sub-Riemannian metric on the standard overtwisted structure, which will turn out to be natural for tightness radius estimates.

Example 2.10 (The standard sub-Riemannian overtwisted structure). Let $(\mathbb{R}^3, \omega_{\mathrm{ot}})$ be the standard overtwisted structure of Example 2.3. We define a sub-Riemannian metric g on $\Delta_{\mathrm{ot}} = \ker \omega_{\mathrm{ot}}$ such that the following vector fields (in cylindrical coordinates) are orthonormal:

$$f_1 := \partial_r, \qquad f_2 := \frac{1}{r}\cos\left(\frac{r^2}{2}\right)\partial_\theta - \frac{1}{r}\sin\left(\frac{r^2}{2}\right)\partial_z.$$

The compatible contact form and the Reeb vector field are, respectively

$$\omega_{\mathrm{ot}} = \cos\left(\frac{r^2}{2}\right)dz + \sin\left(\frac{r^2}{2}\right)d\theta, \qquad f_0 = \cos\left(\frac{r^2}{2}\right)\partial_z + \sin\left(\frac{r^2}{2}\right)\partial_\theta.$$

The metric invariants of $(\mathbb{R}^3, \omega_{\mathrm{ot}}, g)$ are $\chi = \frac{r}{2}$ and $\kappa = \frac{r^2}{2}$.

Remark 2.11. Note that f_1, f_2 of Example 2.10 are defined only out of the z axis, however, the corresponding sub-Riemannian metric is globally well-defined and smooth.

The sub-Riemannian metric induces a length structure on horizontal curves; given $T > 0$ and absolutely continuous $\gamma : [0, T] \to M$, with $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for a.e. $t \in [0, T]$, we denote

$$\ell(\gamma) := \int_0^T \|\dot{\gamma}(t)\|_g \, dt,$$

where $\|\cdot\|_g$ is the norm induced on Δ by g. We define the *sub-Riemannian distance* as

$$d_{SR}(q, p) := \inf\{\ell(\gamma) \mid \gamma \text{ is horizontal}, \ \gamma(0) = q, \ \gamma(T) = p\}.$$

The next result justifies the above definition, and it is a consequence of the Rashevskii-Chow theorem, see e.g. [ABB20, Ch. 3].

Theorem 2.12. *Let (M, ω, g) be a contact sub-Riemannian manifold, then (M, d_{SR}) is a metric space and the metric topology coincides with the manifold topology.*

A *sub-Riemannian geodesic* is a horizontal curve $\gamma : [0, T] \to M$, parametrized with constant speed $\|\dot{\gamma}(t)\|_g = C$ such that $d_{SR}(\gamma(t), \gamma(s)) = C|t - s|$ for all $t, s \in [0, T]$. It is well-known that, for contact structures, sub-Riemannian geodesics are smooth and are projections of solutions of a Hamiltonian system on T^*M, which we now describe.

Let f_1, f_2 be a local orthonormal frame for the sub-Riemannian structure, f_0 be the Reeb field and let $h_0, h_1, h_2 : T^*M \to \mathbb{R}$ be the associated hamiltonian functions

$$h_i(\lambda) := \langle \lambda, f_i \rangle, \qquad \lambda \in T^*M, \qquad i = 0, 1, 2. \tag{4}$$

If c_{ij}^k are the structural coefficients of the frame f_1, f_2, f_0, i.e., $[f_i, f_j] = c_{ij}^k f_k$, then the Poisson brackets $\{\cdot, \cdot\} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$ satisfy

$$\{h_i, h_j\} = \sum_{k=0}^{2} c_{ij}^k h_k.$$

For any smooth function $h : T^*M \to \mathbb{R}$ recall that the symplectic gradient is defined as the unique smooth vector field $\vec{h} \in \Gamma(T^*M)$ such that $dh = \sigma(\cdot, \vec{h})$, where σ is the canonical symplectic form on T^*M. Equivalently, $\vec{h}(\cdot) = \{h, \cdot\}$. The *sub-Riemannian Hamiltonian* $H : T^*M \to \mathbb{R}$ is

$$H := \frac{1}{2} \left(h_1^2 + h_2^2 \right). \tag{5}$$

The right hand side of (5) is independent of the choice of f_1, f_2 and thus yields a well-defined smooth function on T^*M. The corresponding Hamiltonian vector field satisfies

$$\vec{H} = h_1 \vec{h}_1 + h_2 \vec{h}_2. \tag{6}$$

Every sub-Riemannian geodesic $\gamma : [0, T] \to M$ is the projection of an integral curve of $\vec{H}$, namely there exists $\lambda \in T^*M$ such that

$$\gamma(t) = \pi \circ e^{t\vec{H}}(\lambda), \qquad \forall t \in [0, T],$$

and conversely, any sufficiently short segment of such projections is a sub-Riemannian geodesic. The following rescaling property holds, as a consequence of the homogeneity of the Hamiltonian:

$$e^{\vec{H}}(\alpha\lambda) = \alpha e^{\alpha\vec{H}}(\lambda), \qquad \forall \alpha > 0, \quad \lambda \in T^*M. \tag{7}$$

We recall the following sub-Riemannian version of the Hopf-Rinow theorem, see e.g. [ABB20].

Theorem 2.13. *Let (M, ω, g) be a contact sub-Riemannian manifold, then (M, d_{SR}) is a complete metric space if and only there exists $q \in M$ such that the restriction of the Hamiltonian flow $e^{t\vec{H}}|_{T_q^*M} : T_q^*M \to T^*M$ is well-defined for all $t \in \mathbb{R}$.*

As a consequence the following definition is well-posed.

Definition 2.14. *Let (M, ω, g) be a complete contact sub-Riemannian manifold. For $q \in M$, we define the exponential map $\exp_q : T_q^*M \to M$ as*

$$\exp_q(\lambda) := \pi \circ e^{\vec{H}}(\lambda),$$

where $\pi : T^*M \to M$ is the bundle projection.

3. Exponential coordinates

We describe the contact structure in a special coordinate neighbourhood of a Reeb orbit. For the following, let (M, ω, g) a contact sub-Riemannian manifold. Let $\Gamma \subset M$ be an embedded piece of Reeb orbit, which we always assume to be connected, without boundary, and non-trivial. We can have two cases: either Γ is diffeomorphic to an open interval or to a circle.

3.1. **Tubular neighbourhood of a Reeb orbit.** We first study the annihilator bundle of Γ.

Definition 3.1. *The annihilator bundle of Γ is*

$$A\Gamma := \{\lambda \in T^*M \mid \pi(\lambda) \in \Gamma, \ \langle \lambda, f_0 \rangle = 0\}.$$

Given $\rho > 0$ we also define the following bundles:

$$A^{<\rho}\Gamma := A\Gamma \cap \{\sqrt{2H} < \rho\}, \qquad A^1\Gamma := A\Gamma \cap \{\sqrt{2H} = 1\}.$$

Remark 3.2. The restriction $2H|_{A\Gamma} : A\Gamma \to \mathbb{R}$ is a fiber-wise positive definite quadratic form. In particular $A\Gamma$ is an Euclidean bundle.

The restriction of the Hamiltonian flow to $A\Gamma$ yields coordinates in a neighbourhood of Γ such that the sub-Riemannian distance $\delta : M \to \mathbb{R}$ from Γ, given by

$$\delta(q) := \inf_{p \in \Gamma} d_{SR}(p, q), \qquad \forall\, q \in M,$$

has a simple expression. The next theorem is a particular case of a sub-Riemannian version of the tubular neighbourhood theorem which holds for general non-characteristic embedded submanifolds, see e.g. [FPR20, Prop. 3.1]. See also [ACEAG98, Sec. 2.4] for an earlier version for three-dimensional contact structures.

Theorem 3.3 (Tubular neighbourhood)**.** *There exists a neighbourhood $U \subset A\Gamma$ of the zero section of $A\Gamma$ such that the map $E : U \to M$ given by*

$$E(\lambda) := \pi \circ e^{\vec{H}}(\lambda),$$

is a diffeomorphism onto its image, and for the distance δ from Γ it holds

$$\delta \circ E = \sqrt{2H}. \tag{8}$$

If Γ has compact closure, then we can choose $U = A^{<\rho}\Gamma$ for some $\rho > 0$. Furthermore if $q \in M$ and $\sigma : [0, T] \to M$ is a geodesic realizing the distance between q and Γ, i.e. $\sigma(0) \in \Gamma$, $\sigma(T) = q$ and $\delta(q) = \ell(\sigma) = T$, then there exists $\lambda \in A^1\Gamma$ such that

$$\sigma(t) = E(t\lambda), \quad \forall\, t \in [0, T].$$

Definition 3.4. *The injectivity radius of Γ is defined as*

$$r_{\mathrm{inj}}(\Gamma) := \sup\{\rho > 0 \mid E : A^{<\rho}\Gamma \to M \text{ is a diffeomorphism onto its image}\},$$

with the understanding that $r_{\mathrm{inj}}(\Gamma) = 0$ if the above set is empty.

3.2. **Cylindrical coordinates on the annihilator bundle.** The following property is crucial for the construction coordinates on $A\Gamma$.

Lemma 3.5. *The restriction $\Delta|_\Gamma$ is a trivial vector bundle over Γ. In other words there exist a horizontal orthonormal frame f_1, f_2 defined in a neighbourhood of Γ.*

Proof. If Γ is diffeomorphic to an interval, then there is nothing to prove. Assume then that Γ is diffeomorphic to a circle. This means that we can find $\tau > 0$ and $q \in \Gamma$ such that the curve $\gamma : [0, \tau] \to M$ given by

$$\gamma(z) = e^{z f_0}(q),$$

satisfies $\gamma(0) = \gamma(\tau)$ and $\Gamma = \gamma([0, \tau])$. Let X_0, Y_0 be a basis for Δ_q. Then

$$X_z := e_*^{z f_0} X_0, \qquad Y_z := e_*^{z f_0} Y_0, \qquad z \in [0, \tau],$$

is a basis for $\Delta_{\gamma(z)}$, since the Reeb flow preserves ω. Notice that $e_*^{\tau f_0} : \Delta_q \to \Delta_q$ is an orientation preserving endomorphism, the orientation being the one induced on Δ_q by $d\omega$. In particular $e_*^{\tau f_0}$ is homotopic to the identity: there exists a smooth curve $G : [0, \tau] \to \mathrm{GL}(2)$ such that

$$G_0 = \mathrm{Id}, \qquad \begin{pmatrix} X_\tau \\ Y_\tau \end{pmatrix} = G_\tau \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}.$$

We obtain a smooth trivializing frame for $\bar{X}, \bar{Y}$ for $\Delta|_\Gamma$ by setting

$$\begin{pmatrix} \bar{X}_z \\ \bar{Y}_z \end{pmatrix} = G_z^{-1} \begin{pmatrix} X_z \\ Y_z \end{pmatrix}, \qquad \forall\, z \in [0, \tau].$$

We can extend $\bar{X}$ and $\bar{Y}$ to an horizontal frame in a neighbourhood of Γ, which in turn yields a orthonormal horizontal frame f_1, f_2 by Gram-Schmidt. $\qquad\square$

Let $\Gamma \subset M$ be an embedded piece of a Reeb orbit. We recall that either Γ is diffeomorphic to an open interval or to a circle. Fix $q \in \Gamma$ and let $\gamma : I \to \Gamma$ be the diffeomorphism

$$\gamma(z) = e^{z f_0}(q), \qquad \forall\, z \in I,$$

where I is either an open interval or $I = \mathbb{R}/\tau\mathbb{N} \simeq \mathbb{S}^1$ for some $\tau > 0$ (the period of the orbit).

Since $\Delta|_\Gamma$ is a trivial bundle over Γ, there exists an oriented orthonormal frame f_0, f_1, f_2 on Γ and a corresponding dual frame $\nu_0 = \omega, \nu_1, \nu_2$, which we fix for the next proposition.

Proposition 3.6. *The map* $\Psi : I \times \mathbb{R}^2 \to A\Gamma$ *defined by*

$$\Psi(z, x, y) := (x\nu_1 + y\nu_2)|_{\gamma(z)}, \tag{9}$$

is an isomorphism of Euclidean bundles satisfying

$$2H \circ \Psi = \| \cdot \|_{\mathbb{R}^2}^2.$$

In particular, all the bundles of Definition 3.1 are trivial, with

$$A\Gamma \simeq I \times \mathbb{R}^2, \qquad A^{<\rho}\Gamma \simeq I \times B_\rho, \qquad A^1\Gamma \simeq I \times \partial B_1,$$

where B_ρ is the 2-dimensional Euclidean open ball of radius $\rho > 0$.

Proof. Any covector $\lambda \in T^*M|_\Gamma$ on Γ can be written as

$$\lambda = \sum_{i=0}^{2} \langle \lambda, f_i \rangle \nu_i = \sum_{i=0}^{2} h_i(\lambda) \nu_i,$$

where the $h_i : T^*M \to \mathbb{R}$ are defined in (4). Notice that

$$A\Gamma = \{\lambda \in T^*M \mid \pi(\lambda) \in \Gamma, \ h_0(\lambda) = 0\},$$

thus any $\lambda \in A\Gamma$ can be written in a unique way as $\lambda = x\nu_1 + y\nu_2$, with $x, y \in \mathbb{R}^2$. It follows that (9) is a bundle isomorphism Since $2H = h_1^2 + h_2^2$, then it holds $2H \circ \Psi = \| \cdot \|_{\mathbb{R}^2}^2$ as claimed. $\quad\square$

Proposition 3.6 yields a global set of coordinates on $A\Gamma$. The corresponding cylindrical coordinates $(z, r, \theta) \in I \times (0, +\infty) \times [0, 2\pi)$ defined outside of the zero section of $A\Gamma$, given by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z,$$

will be instrumental in our analysis, and referred to as *cylindrical coordinates* on $A\Gamma$.

4. THE TIGHTNESS RADIUS FROM A REEB ORBIT

In this section we introduce the tightness radius and prove our first estimates for it.

Definition 4.1. *Let (M, ω, g) be a contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\mathrm{inj}}(\Gamma) > 0$. The tightness radius is*

$$r_{\mathrm{tight}}(\Gamma) := \sup\left\{0 < r < r_{\mathrm{inj}}(\Gamma) \mid (A^{<r}\Gamma, E^*\omega) \text{ is tight}\right\}.$$

4.1. **A class of models and their tightness radius.** We compute the tightness radius for a class of examples. For the next statement we say that a smooth function $\alpha : \mathbb{R}^3 \to \mathbb{R}$ is *radial* if, in cylindrical coordinates (r, θ, z) it holds $\partial_\theta \alpha = \partial_z \alpha = 0$.

Theorem 4.2. *Let ω be a contact form on $\mathbb{R}^3$, such that in cylindrical coordinates on $\mathbb{R}^3$*

$$\omega = \alpha dz + \beta d\theta,$$

for some smooth radial functions α, β. Let $\gamma := \alpha\partial_r\beta - \beta\partial_r\alpha$. Then

$$g := \left(dr \otimes dr + \frac{\gamma^2}{\alpha^2 + \beta^2}(d\theta \otimes d\theta + dz \otimes dz)\right)\Big|_{\ker\omega}$$

is a sub-Riemannian metric compatible with ω. Thus $(\mathbb{R}^3, \omega, g)$ is a contact sub-Riemannian structure. The curve

$$\Gamma = \{(0, 0, z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}$$

is the orbit of the Reeb field starting passing through the origin. In cylindrical coordinates on $A\Gamma$ of Section 3.2 the map $E : A\Gamma \to \mathbb{R}^3$ is the identity:

$$E(z, r\cos\theta, r\sin\theta) = (r\cos\theta, r\sin\theta, z).$$

Moreover, the frame $\{f_0, N, JN\}$ is orthonormal and oriented, where

$$N = \partial_r, \qquad JN = \frac{\alpha}{\gamma}\partial_\theta - \frac{\beta}{\gamma}\partial_z, \qquad f_0 = \frac{\partial_r\beta}{\gamma}\partial_z - \frac{\partial_r\alpha}{\gamma}\partial_\theta. \tag{10}$$

Finally, the tightness and injectivity radii are:

$$r_{\mathrm{inj}}(\Gamma) = +\infty, \qquad r_{\mathrm{tight}}(\Gamma) = \inf\{r > 0 \mid \beta(r) = 0\}.$$

Proof. Since $\alpha, \beta \in C^\infty(\mathbb{R}^3)$ are smooth radial functions, the maps

$$\mathbb{R} \ni r \mapsto \alpha(r\cos\theta, r\sin\theta, z), \qquad \beta(r\cos\theta, r\sin\theta, z),$$

are smooth as functions of one variable and furthermore $\partial_r^{2j+1}\alpha(0, z) = \partial_r^{2j+1}\beta(0, z) = 0$ for all $j \in \mathbb{N}$ and $z \in \mathbb{R}$, where here and in the following we employ the shorthand $(0, z) = (0, 0, z) \in \mathbb{R}^3$. The fact that ω is a smooth contact form poses some additional constraints. First, notice that

$$\omega = \alpha dz + \frac{\beta}{r^2}(xdy - ydx).$$

Therefore $\beta(0, z) = 0$ and β/r^2 must be a smooth function. Furthermore, ω is a contact form and thus it cannot vanish: from $\omega|_{r=0} = \alpha(0, z)dz$, we deduce that the radial function α satisfies $\alpha_0 := \alpha(0, z) = \alpha(0, 0) \neq 0$. Additionally by the contact condition

$$\omega \wedge d\omega = \frac{\gamma}{r}dx \wedge dy \wedge dz \qquad \Rightarrow \qquad \frac{\gamma}{r} \text{ is smooth and never vanishing.} \tag{11}$$

Combining (11) with the fact that β/r^2 is smooth we deduce

$$\lim_{r\to 0}\frac{\gamma}{r} = \lim_{r\to 0}\frac{\alpha\partial_r\beta - \beta\partial_r\alpha}{r} = \lim_{r\to 0}\frac{\alpha\partial_r\beta}{r} = \alpha_0\lim_{r\to 0}\frac{\partial_r\beta}{r} \neq 0. \tag{12}$$

One can check that the vector fields N, JN of (10) are orthonormal for g. Moreover

$$\omega(f_0) = 1, \qquad d\omega(f_0, \cdot) = 0, \qquad d\omega(N, JN) = 1.$$

Thus g is a sub-Riemannian metric, ω is compatible with g, and f_0 is the Reeb field of ω.

Using (11) and (12) we obtain that

$$\lim_{r\to 0} f_0 = \lim_{r\to 0}\frac{\partial_r\beta/r}{\gamma/r}\partial_z - \lim_{r\to 0}\frac{\partial_r\alpha/r}{\gamma/r}\partial_\theta = \frac{1}{\alpha_0}\partial_z.$$

In particular $\Gamma = \{(0, 0, z) \mid z \in \mathbb{R}\}$ is the orbit of f_0 passing through the origin.

The Hamiltonian functions of the vector fields (10) are

$$h_N = p_r, \qquad h_{JN} = \frac{\alpha}{\gamma}p_\theta - \frac{\beta}{\gamma}p_z, \qquad h_0 = \frac{\partial_r\beta}{\gamma}p_z - \frac{\partial_r\alpha}{\gamma}p_\theta,$$

where the p's are the canonical momenta associated to (r, θ, z). Observe that

$$A\Gamma = \{\lambda \in T^*\mathbb{R}^3 \mid \pi(\lambda) \in \Gamma, \ p_z(\lambda) = 0\} = \{\lambda \in T^*\mathbb{R}^3 \mid \lambda = (p_x dx + p_y dy)\,|_{(0,0,z)}\}.$$

Therefore, the sub-Riemannian Hamiltonian satisfies

$$2H = p_r^2 + h_{JN}^2 = p_r^2 + \left(\frac{\alpha}{\gamma}p_\theta - \frac{\beta}{\gamma}p_z\right)^2.$$

Hamilton's equation in coordinates are then

$$\begin{cases} \dot{p}_r = -h_{JN}\partial_r h_{JN}, \\ \dot{p}_\theta = 0, \\ \dot{p}_z = 0, \end{cases} \qquad \begin{cases} \dot{r} = p_r, \\ \dot{\theta} = h_{JN}\partial_{p_\theta}h_{JN}, \\ \dot{z} = h_{JN}\partial_{p_z}h_{JN}. \end{cases}$$

Since $h_{JN}|_{p_\theta, p_z = 0} = 0$, then for each $p_x, p_y, z \in \mathbb{R}$ the curve

$$\mathbb{R} \ni t \mapsto (tp_x, tp_y, z, p_x, p_y, 0) \in T^*\mathbb{R}^3,$$

is an integral curve of $\vec{H}$ with initial condition $(0, 0, z, p_x, p_y, 0) \in A\Gamma$. Recall that $E : A\Gamma \to \mathbb{R}^3$ is given by $E = \pi \circ e^{\vec{H}}|_{A\Gamma}$. Therefore in cylindrical coordinates on $A\Gamma$ (see Section 3.2) we have

$$E(z, r\cos\theta, r\sin\theta) = (r\cos\theta, r\sin\theta, z),$$

as claimed. It follows that $r_{\mathrm{inj}}(\Gamma) = +\infty$.

To conclude, we compute the tightness radius. For $R > 0$ we denote with $C_R = \{(x, y, x) \in \mathbb{R}^3 \mid x^2 + y^2 < R^2\}$ the sub-Riemannian cylinder around Γ of radius R. Let us denote

$$R_0 = \inf\{r > 0 \mid \beta(r) = 0\}.$$

If $R > R_0$ then, according to Definition 2.4, for each $z \in \mathbb{R}$ the disk

$$D_0 = \{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 \mid r \le R_0, \ \theta \in [0, 2\pi)\},$$

is diffeomorphic to the standard overtwisted disk, thus $r_{\mathrm{tight}}(\Gamma) \le R_0$. To prove the opposite inequality, we will build a diffeomorphism $\varphi : C_{R_0} \to C_{R_0}$ such that $\varphi_*\Delta_{\mathrm{st}} = \Delta$, where Δ_{st} is the standard contact structure of Example 2.2, which is tight.

To motivate the construction, note that for any such diffeomorphism there is a non-vanishing function $f \in C^\infty(C_{R_0})$ such that

$$\varphi^*(f\omega) = \omega_{\mathrm{st}}, \qquad \text{on } C_{R_0},$$

where $\omega_{\text{st}} = dz + \frac{r^2}{2}d\theta$. In particular, φ maps orbits of the Reeb field of ω_{st} to orbits of the Reeb field of $\bar{\omega} := f\omega$. Since the Reeb field of ω_{st}, which is ∂_z, does not have closed orbits, we select f in such a way that the Reeb field $\bar{f}_0$ of $\bar{\omega}$ has none either, and use it to build the claimed diffeomorphism.

We now proceed with the construction of φ. By definition of R_0, the smooth function β/r^2 is non-vanishing for $r \in [0, R_0)$, thus we set

$$\bar{\omega} := \left(\frac{r^2}{2\beta}\right)\omega = \bar{\alpha}dz + \bar{\beta}d\theta, \qquad \text{where} \qquad \bar{\alpha} := \frac{r^2}{2}\frac{\alpha}{\beta}, \quad \bar{\beta} := \frac{r^2}{2}.$$

For the contact form $\bar{\omega}$ all previous computation holds, replacing α, β and γ with their barred counterparts, and in particular from (10) the Reeb field is

$$\bar{f}_0 = \frac{\partial_r \bar{\beta}}{\bar{\gamma}}\partial_z - \frac{\partial_r \bar{\alpha}}{\bar{\gamma}}\partial_\theta.$$

In particular, using (11), we observe that

$$dz(\bar{f}_0) = \frac{\partial_r \bar{\beta}}{\bar{\gamma}} = \left(\frac{\bar{\gamma}}{r}\right)^{-1} \qquad \text{is non-vanishing,}$$

so that $\bar{f}_0$ has no closed orbits. Notice that $\bar{f}_0$ is a vector field tangent to the cylinders C_R and its components are radial functions, hence it is complete. We claim that the map $\varphi : C_{R_0} \to C_{R_0}$

$$\varphi(x, y, z) := e^{z\bar{f}_o}(x, y, 0),$$

is a diffeomorphism such that $\varphi^*\bar{\omega} = \omega_{\text{st}}$. We prove that latter property first, in fact:

$$\varphi^*\bar{\omega} = \bar{\omega}(\varphi_*\partial_r)dr + \bar{\omega}(\varphi_*\partial_\theta)d\theta + \bar{\omega}(\varphi_*\partial_z)dz$$

$$= \bar{\omega}(e^{z\bar{f}_0}_*\partial_r)dr + \bar{\omega}(e^{z\bar{f}_0}_*\partial_\theta)d\theta + \bar{\omega}(\bar{f}_0)dz$$

$$= \bar{\omega}(\partial_r)dr + \bar{\omega}(\partial_\theta)d\theta + dz = dz + \frac{r^2}{2}d\theta = \omega_{\text{st}}.$$

In particular φ is an immersion, since it sends a contact form to a contact form. Moreover φ is injective: since $dz(\bar{f}_0) > 0$, then the equality $\varphi(x, y, z) = \varphi(x', y', z')$ implies $z = z'$, which in turns implies $x = x'$, $y = y'$. The map is also surjective since the integral curves of $\bar{f}_0$ foliate the cylinders C_R, $R \in (0, R_0)$. This concludes the proof of the claim. $\square$

As a corollary of Theorem 4.2 we compute the tightness radius for Examples 2.9 and 2.10.

Corollary 4.3. *The curve $\Gamma = \{(0, 0, z) \mid z \in \mathbb{R}\}$ is an orbit of the Reeb field for the standard sub-Riemannian contact structure of Example 2.9. In cylindrical coordinates:*

$$\omega_{\text{st}} = dz + \frac{r^2}{2}d\theta.$$

It holds:

$$r_{\text{inj}}(\Gamma) = +\infty, \qquad r_{\text{tight}}(\Gamma) = +\infty.$$

Corollary 4.4. *The curve $\Gamma = \{(0, 0, z) \mid z \in \mathbb{R}\}$ is an orbit of the Reeb field for the standard sub-Riemannian overtwisted structure of Example 2.10. In cylindrical coordinates:*

$$\omega_{\text{ot}} = \cos\left(\frac{r^2}{2}\right)dz + \sin\left(\frac{r^2}{2}\right)d\theta.$$

It holds:

$$r_{\text{inj}}(\Gamma) = +\infty, \qquad r_{\text{tight}}(\Gamma) = \sqrt{2\pi}.$$

4.2. **Tightness radius and singular locus.** The tightness radius can be estimated studying the characteristic foliation of the fibers of the annihilator bundle.

Definition 4.5 (Singular locus of Reeb orbits). *Let (M, ω, g) be a contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $\rho := r_{\mathrm{inj}}(\Gamma) > 0$. Consider the embedded surfaces*

$$\Sigma_q := \{E(\lambda) \mid \lambda \in A^{<\rho}\Gamma, \ \pi(\lambda) = q, \ \lambda \neq 0\}, \qquad q \in \Gamma.$$

The singular locus of Γ is the union, for $q \in \Gamma$, of the singularities of the characteristic foliations of Σ_q in the contact manifold $(M, \Delta = \ker \omega)$, namely

$$\mathrm{Sing}(\Gamma) := \bigcup_{q \in \Gamma} \left\{ p \in \Sigma_q \,\middle|\, T_p \Sigma_q = \Delta_p \right\}.$$

Theorem 4.6. *Let (M, ω, g) be a contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\mathrm{inj}}(\Gamma) > 0$. Then $\mathrm{Sing}(\Gamma) \subset M$ is an embedded surface and*

$$r_{\mathrm{tight}}(\Gamma) \geq \min\{d_{SR}(\Gamma, \mathrm{Sing}(\Gamma)), r_{\mathrm{inj}}(\Gamma)\}, \tag{13}$$

with the understanding that, if $\mathrm{Sing}(\Gamma) = \emptyset$ then $r_{\mathrm{tight}}(\Gamma) = r_{\mathrm{inj}}(\Gamma)$.

The equality holds in (13) *for the models of Theorem 4.2, and in particular of the standard sub-Riemannian overtwisted structure of Example 2.10.*

Proof. Let $\rho = r_{\mathrm{inj}}(\Gamma)$, and recall that $E : A^{<\rho}\Gamma \to M$ is a diffeomorphism onto its image. On $A\Gamma$ we fix a set of cylindrical coordinates $(r \cos \theta, r \sin \theta, z)$ as defined in Section 3.2.

Since $E_* \partial_r = \pi_* \vec{H}$ is horizontal, there exist $\phi, N \in C^\infty(A^{<\rho}\Gamma)$, with $N > 0$, such that

$$E^* \omega = N(\cos \phi \, dz + \sin \phi \, d\theta).$$

Combining Theorem 3.3 and Proposition 3.6 it holds

$$\delta(E(r \cos \theta, r \sin \theta, z)) = r, \qquad \forall \, r < \rho. \tag{14}$$

Furthermore, if $q = E(0, 0, z)$, then in these coordinates

$$E^{-1}(\Sigma_q) = \{(r \cos \theta, r \sin \theta, z) \mid r \in (0, \rho), \ \theta \in [0, 2\pi)\}.$$

Hence, singularities of the characteristic foliation are characterized by $E^* \omega \propto dz$, and

$$E^{-1}(\mathrm{Sing}(\Gamma)) = \phi^{-1}(\pi \mathbb{N}). \tag{15}$$

Computing the contact condition $E^*(\omega \wedge d\omega) > 0$ in cylindrical coordinates yields

$$E^*(\omega \wedge d\omega) = \frac{N^2 \partial_r \phi}{r} r \, dr \wedge d\theta \wedge dz.$$

It follows that $f := \partial_r \phi / r$ is smooth and positive on $A^{<\rho}\Gamma$. In particular $\phi : A^{<\rho}\Gamma \to \mathbb{R}$ is monotone increasing along radial lines $r \mapsto (r \cos \theta, r \sin \theta, z)$. Therefore, from equality (15), we deduce that $\mathrm{Sing}(\Gamma)$ is an embedded surface. Moreover, by (14), we deduce that

$$d_{SR}(\Gamma, \mathrm{Sing}(\Gamma)) = \inf\{r > 0 \mid (r \cos \theta, r \sin \theta, z) \in A^{<\rho}\Gamma, \ \phi(r \cos \theta, r \sin \theta, z) = \pi\}. \tag{16}$$

To conclude, observe that

$$\frac{\phi}{H} = \frac{2\phi}{r^2} = \frac{2}{r^2} \int_0^r s f(s \cos \theta, s \sin \theta, z) \, \mathrm{d}s = 2 \int_0^1 s f(rs \cos \theta, rs \sin \theta, z) \, \mathrm{d}s.$$

It follows that the function ϕ/H is well-defined on $A^{<\rho}\Gamma$, smooth and positive. Using this fact, define the smooth function $\varphi : A^{<\rho}\Gamma \to \mathbb{R}^3$

$$\varphi(x, y, z) = \left(\sqrt{\frac{\phi}{H}} x, \sqrt{\frac{\phi}{H}} y, z \right).$$

By the monotonicity of ϕ we have that φ is injective. Furthermore, letting $\omega_{\mathrm{ot}} = \cos(r^2/2)dz + \sin(r^2/2)d\theta$ be the standard contact overtwisted form of Example 2.3, we have

$$\varphi^*\omega_{\mathrm{ot}} = \cos\phi\, dz + \sin\phi\, d\theta = \frac{E^*\omega}{N}. \tag{17}$$

It follows that φ is an immersion, and hence a diffeomorphism on its image. Denoting $R := \min\{d_{SR}(\Gamma, \mathrm{Sing}(\Gamma)), \rho\}$, by equation (16), monotonicity of the angle ϕ and the second equality of (17), the image of the restricted map

$$\varphi : A^{<R}\Gamma \to \mathbb{R}^3$$

is contained in the cylinder $C_{\sqrt{2\pi}} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 < 2\pi\} \subset \mathbb{R}^3$, which, by Corollary 4.4 is a tight contact manifold when endowed with the restriction of $\ker\omega_{\mathrm{ot}}$. Thus in view of (17) the contact manifold $(A^{<R}\Gamma, \ker E^*\omega)$ contains no overtwisted disks. The inequality (13) is proven in the case in which Γ is not a closed orbit.

If Γ is closed, let us denote with $N := E(A^{<\rho}\Gamma)$ its tubular neighbourhood and consider the universal cover, $\tilde{N}$. The latter inherits the natural structure of a contact sub-Riemannian manifold locally isometric to $(A^{<\rho}\Gamma, E^*\omega, E^*g)$. According to Proposition 3.6 $N \simeq \Gamma \times B_\rho$, therefore $\tilde{N} = \tilde{\Gamma} \times B_\rho \simeq \mathbb{R} \times B_\rho$, where $\tilde{\Gamma}$ is the unique lift of Γ going through the point $(0,0,0)$, i.e. $\tilde{\Gamma} = \{(z,0,0) \in \tilde{N} \mid z \in \mathbb{R}\}$. Since $\tilde{\Gamma}$ is not closed, the already proved part of the theorem applies, i.e.

$$r_{\mathrm{tight}}\left(\tilde{\Gamma}\right) \geq \tilde{d}_{SR}\left(\tilde{\Gamma}, \mathrm{Sing}\left(\tilde{\Gamma}\right)\right).$$

Let $p : \tilde{N} \to N$ be the canonical projection. Observe that, for any $q \in \tilde{\Gamma}$, $A_q^{<\rho}\tilde{\Gamma}$ has neighbourhood $\mathcal{U}$ such that the map $p : U \to p(U)$ is a sub-Riemannian isometry. By Definition 4.5 of the singular locus, it follows that

$$\tilde{d}_{SR}\left(\tilde{\Gamma}, \mathrm{Sing}\left(\tilde{\Gamma}\right)\right) = d_{SR}(\Gamma, \mathrm{Sing}(\Gamma)).$$

Moreover, Remark 2.6 implies that $r_{\mathrm{tight}}(\Gamma) \geq r_{\mathrm{tight}}\left(\tilde{\Gamma}\right)$. Therefore

$$r_{\mathrm{tight}}(\Gamma) \geq r_{\mathrm{tight}}\left(\tilde{\Gamma}\right) \geq \tilde{d}_{SR}\left(\tilde{\Gamma}, \mathrm{Sing}\left(\tilde{\Gamma}\right)\right) = d_{SR}(\Gamma, \mathrm{Sing}(\Gamma)),$$

concluding the proof. $\qquad\square$

5. Contact Jacobi curves and quantitative tightness

In (sub-)Riemannian geometry and geometric control theory, Jacobi curves are curves in the Lagrange Grassmannian whose dynamics is intertwined with the presence of conjugate points and curvature (see [AG97, AZ02a, AZ02b, ABR18] and references within). Inspired by that approach, we introduce the concept of *contact* Jacobi curves, whose dynamics is related with the presence of overtwisted disks. As a preparation, in Section 5.1 we recall the concept of Schwarzian derivative and some comparison results adapted to our setting.

5.1. **The Schwarzian derivative.** In this section we develop a general comparison theory for curves in $\mathbb{RP}^1$, adapted to our setting, which will then be applied to contact Jacobi curves.

Definition 5.1. *Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{RP}^1$ be a smooth immersion, which in homogeneous coordinates reads*

$$f(t) = [f_0(t) : f_1(t)].$$

The Schwarzian derivative *of f, denoted $\mathcal{S}(f) : I \to \mathbb{R}$, is defined as*

$$\mathcal{S}(f) := \frac{\dddot{v}}{\dot{v}} - \frac{3}{2}\left(\frac{\ddot{v}}{\dot{v}}\right)^2, \qquad where \qquad v = \frac{f_0}{f_1}, \quad or \quad \frac{f_1}{f_0}. \tag{18}$$

The Schwarzian derivative is independent of the choice of the homogeneous coordinate v. In particular if $A : \mathbb{P}^1 \to \mathbb{P}^1$ is any projective transformation then $\mathcal{S}(A \circ f) = \mathcal{S}(f)$.

Proposition 5.2. *Consider the second order linear ODE on an open interval $I \subset \mathbb{R}$*

$$\ddot{u}(t) + q(t)u(t) = 0, \qquad (\star)$$

where $q \in C(I)$. There is a one-to-one correspondence between pairs of linearly independent solutions of $(\star)$ (up to constant rescaling) and smooth immersions $f : I \to \mathbb{RP}^1$ with Schwarzian derivative $\mathcal{S}(f) = 2q$.

Proof. The space of solutions of $(\star)$ is a vector space of dimension 2. Choose two linearly independent solutions u_1, u_2. Their Wronskian is

$$W(u_1, u_2) := \det \begin{pmatrix} u_1 & u_2 \\ \dot{u}_1 & \dot{u}_2 \end{pmatrix} = u_1 \dot{u}_2 - u_2 \dot{u}_1.$$

From $(\star)$ it follows that $W(u_1, u_2)$ is a non-zero constant. In particular, u_1, u_2 cannot vanish simultaneously on I. Thus, the smooth curve $f : I \to \mathbb{RP}^1$ given by

$$f(t) = [u_1(t) : u_2(t)],$$

is well-defined. One can check that it is an immersion and that $2q = \mathcal{S}(f)$.

We show that this correspondence is injective. Let $\tilde{u}_1, \tilde{u}_2$ be another choice of linearly independent solutions of $(\star)$ yielding the same curve in $\mathbb{RP}^1$. Then there is $a(t) \neq 0$ such that $\tilde{u}_i(t) = a(t)u_i(t)$ for all $t \in I$ and $i = 1, 2$. It follows that $a : I \to \mathbb{R}$ is smooth (since solutions cannot vanish simultaneously) and by $(\star)$ that

$$\ddot{a}(t)u_i(t) + 2\dot{a}(t)\dot{u}_i(t) = 0, \qquad i = 1, 2.$$

Multiply the first equation by $u_2(t)$, the second one by $u_1(t)$, and take the difference. We get

$$\dot{a}(t)W(u_1, u_2) = 0 \qquad \implies \qquad \dot{a}(t) = 0,$$

so that $a(t) = a$ must be a non-zero constant. Thus $(\tilde{u}_1, \tilde{u}_2) = a(u_1, u_2)$.

We show that the correspondence is surjective. Let $f : I \to \mathbb{RP}^1$ be an immersion, with

$$f(t) = [f_1(t) : f_2(t)],$$

for some smooth $f_i : I \to \mathbb{R}$. The immersion condition implies that the function

$$\beta : I \to \mathbb{R}, \qquad \beta(t) := \left(\frac{1}{|\dot{f}_1(t)f_2(t) - \dot{f}_2(t)f_1(t)|} \right)^{1/2},$$

is well-defined. We note that $u_i := \beta f_i : I \to \mathbb{R}$ satisfy

$$\dot{u}_1 u_2 - \dot{u}_2 u_1 = \pm 1.$$

Therefore, the u_i are linearly independent solutions of $(\star)$, for some $q : I \to \mathbb{R}$. By the first part of the proof, since $f(t) = [u_1(t) : u_2(t)]$ for $t \in I$, we necessarily have $2q = \mathcal{S}(f)$. $\square$

We will apply the characterization of Proposition 5.2 to contact Jacobi curves. An important fact is that the contact Jacobi curves *always* have singular Schwarzian derivative at the initial time. The corresponding ODE $(\star)$ is singular, and so are their solutions. Suitable assumptions on the singularity are then necessary. More precisely, we assume that $2q = \mathcal{S}(f)$ satisfies

$$q = -\frac{3}{4t^2} + \text{regular part}, \qquad t \in (0, T).$$

This assumption is used for the following asymptotic properties of solutions at the singularity.

Lemma 5.3. *Consider the ODE* $(\star)$, *and assume that*

$$q(t) = -\frac{3}{4t^2} + \tilde{q}(t), \qquad \forall\, t \in (0, T),$$

where $\tilde{q} \in C([0, T])$. *Then* $(\star)$ *has linearly independent solutions* $g_1, g_2 : (0, T) \to \mathbb{R}$ *such that*

$$g_1(t) = t^{3/2} \tilde{g}_1(t),$$
$$g_2(t) = t^{-1/2} \tilde{g}_2(t),$$

where $\tilde{g}_j \in C^0([0, T])$, $\tilde{g}_j(0) = 1$ *for* $j = 1, 2$.

Proof. If $\tilde{q}(t)$ is analytic the result is well-known from the theory second order ODEs with regular singularities, see e.g. [CL55, Ch. 5]. The case under investigation is treated (in greater generality) in [LV11, Thm. 2.1], where it is attributed to M. Bôcher. $\qquad\square$

Proposition 5.4. *Let* $f : [0, T) \to \mathbb{RP}^1$ *be a continuous curve, such that its restriction to* $(0, T)$ *is a smooth immersion. Consider the ODE* $(\star)$ *with* $2q = \mathcal{S}(f)$. *Assume that*

$$q(t) = -\frac{3}{4t^2} + \tilde{q}(t), \qquad \forall\, t \in (0, T),$$

where $\tilde{q} \in C([0, T])$. *Let* $t_* \in (0, T)$ *such that the curve* f *has a self-intersection:* $f(0) = f(t_*)$. *Then there exists a non-trivial solution* $u : (0, T) \to \mathbb{R}$ *of* $(\star)$ *such that*

$$u(0) := \lim_{t \to 0} u(t) = u(t_*) = 0.$$

Proof. By Proposition 5.2 there are two independent solutions u_1, u_2 of $(\star)$ such that $f(t) = [u_1(t) : u_2(t)]$ for $t \in (0, T)$. Since $f(0) = f(t_*)$ we can assume without loss of generality that the following limits exist and are finite

$$\alpha := \lim_{t \to 0} \frac{u_1(t)}{u_2(t)} = \lim_{t \to t_*} \frac{u_1(t)}{u_2(t)}. \tag{19}$$

In particular since u_1, u_2 cannot vanish simultaneously on $(0, T)$ we must have $u_2(t_*) \neq 0$. We let then $u := u_1 - \alpha u_2$. By construction, $u(t_*) = 0$. To show that $u(0) = 0$ we use Lemma 5.3. There exist $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$ such that

$$u_1 = ag_1 - bg_2,$$
$$u_2 = cg_1 - dg_2,$$

where g_1, g_2 are independent solutions. By (19) we have

$$\alpha = \lim_{t \to 0} \frac{at^{3/2} - bt^{-1/2}}{ct^{3/2} - dt^{-1/2}} = \lim_{t \to 0} \frac{at^2 - b}{ct^2 - d}.$$

By the existence of the limit and since b, d cannot vanish at the same time, we see that $d \neq 0$ and $\alpha = b/d$. It follows that

$$u = u_1 - \alpha u_2 = ag_1 - \frac{b}{d}cg_1 = \frac{ad - bc}{d} g_1.$$

Since $g_1(t) = t^{3/2} \tilde{g}_1(t)$, with $\tilde{g}_1(0) = 1$ we have that $\lim_{t \to 0} u(t) = 0$. $\qquad\square$

We have the following version of the Sturm-Picone comparison theorem with singularities.

Proposition 5.5. *Let $u, \bar{u} : (0, T) \to \mathbb{R}$ be non-trivial solutions of*

$$\ddot{u}(t) + q(t)u(t) = 0, \tag{20}$$

$$\ddot{\bar{u}}(t) + \bar{q}(t)\bar{u}(t) = 0, \tag{21}$$

where $q : (0, T) \to \mathbb{R}$ satisfies $q(t) \le \bar{q}(t)$ for $t \in (0, T)$. Then between any two consecutive zeroes $t_1 < t_2 \in [0, T)$ of u there exists at least one zero of $\bar{u}$, unless $q \equiv \bar{q}$ on (t_1, t_2).

Proof. Assume without loss of generality that $u > 0$ on (t_1, t_2). We claim that there exists a sequence $\varepsilon_n \to 0$ such that $\dot{u}(t_1 + \varepsilon_n) > 0$. In fact, if for all small enough $\varepsilon > 0$ we had $\dot{u} \le 0$ on $(0, \varepsilon)$, then using that u is continuous on $[t_1, T)$ with $u(t_1) = 0$ we have

$$u(t) = \int_{t_1}^{t} \dot{u}(\tau)\,\mathrm{d}\tau \le 0, \qquad \forall t \in (t_1, t_1 + \varepsilon),$$

which is a contradiction. Similarly one shows that $\dot{u}(t_2) < 0$.

Assume by contradiction that $\bar{u} > 0$ on (t_1, t_2). Therefore by multiplying (20)-(21) by $\bar{u}, u$, respectively, subtracting, and integrating on $(t_1 + \varepsilon_n, t_2)$, we obtain

$$[\dot{u}\bar{u} - u\dot{\bar{u}}]_{t_1+\varepsilon_n}^{t_2} = \int_{t_1+\varepsilon_n}^{t_2} (\bar{q}(t) - q(t))u(t)\bar{u}(t)\,\mathrm{d}t.$$

Now we take the limit of the above for $\varepsilon_n \to 0$. By the properties $\dot{u}(t_1 + \varepsilon_n) > 0$ and $\dot{u}(t_2) < 0$, the limit of the left hand side exists and is non-positive. On the other hand, observe that the limit of the right hand side exists (possibly $+\infty$) since it is a non-decreasing sequence, and it must be ≥ 0. Furthermore the limit is zero if and only if $\bar{q} \equiv q$ on (t_1, t_2). If this is not the case then we reach a contradiction, so that $\bar{u}$ must have a zero in the interval (t_1, t_2). $\qquad\square$

Theorem 5.6. *Let $f : [0, T) \to \mathbb{RP}^1$ be a continuous curve, such that its restriction to $(0, T)$ is a smooth immersion. Assume that its Schwarzian derivative $2q = \mathcal{S}(f)$ satisfies*

$$q(t) + \frac{3}{4t^2} \in C([0, T)). \tag{22}$$

Assume also that there exists $\bar{q} \in C((0, T))$ such that

$$q \le \bar{q}, \qquad on\ (0, T).$$

Let $t_* \in (0, T)$ such that $f(0) = f(t_*)$. Let $\bar{u}$ be a non-trivial solution of $\ddot{\bar{u}} + \bar{q}\bar{u} = 0$. Then

$$t_* \ge \sup_{\bar{u}} \inf\{t > 0 \mid \bar{u}(t) = 0\}, \tag{23}$$

where the sup is over all non-trivial solutions of $\ddot{\bar{u}} + \bar{q}\bar{u} = 0$.

If all solutions of $\ddot{\bar{u}} + \bar{q}\bar{u} = 0$ on $(0, T)$ have a (possibly infinite) limit at $t = 0$, and there are solutions with $\bar{u}(0) = 0$ then the sup is attained on this set.

Remark 5.7. In particular, when $\bar{q}$ has the form of Lemma 5.3, then all solutions have (possibly infinite) limit at $t = 0$, and there is only one solution (up to rescaling) with $\bar{u}(0) = 0$. Then such a solution yields the optimal estimate in (23). This is the case in which we use Theorem 5.6.

Proof. By Proposition 5.4 there is a non-trivial solution $u : (0, T) \to \mathbb{R}$ of $\ddot{u} + qu = 0$ such that

$$u(0) := \lim_{t \to 0} u(t) = u(t_*) = 0.$$

By Proposition 5.5, the non-trivial solution $\bar{u} : (0, T) \to \mathbb{R}$ of $\ddot{\bar{u}} + \bar{q}\bar{u} = 0$ must have a zero in $(0, t_*)$. Optimizing over all possible solutions we obtain (23).

To prove the final part of the statement, it is sufficient to show that under the additional assumptions the separation theorem holds: if $\bar{u}, \bar{v} : (0, T) \to \mathbb{R}$ are linearly-independent solution,

with $\bar{v}(0) = \lim_{t \to 0} v(t) \neq 0$, and $0 = t_1 < t_2 < T$ are consecutive zeroes of $\bar{u}$, then $\bar{v}$ has exactly one zero in (t_1, t_2). To prove this claim, assume by contradiction that $\bar{v}$ has no zeroes on (t_1, t_2). It follows that $\bar{v}$ is non-vanishing on $[t_1, t_2]$. In fact, if $t_1, t_2 > 0$ this is true since $\bar{u}, \bar{v}$ are linearly independent, while but $\bar{v}(t_1) = \bar{v}(0) \neq 0$ by assumption. Then the function

$$\psi(t) := \frac{\bar{u}(t)}{\bar{v}(t)}, \qquad t \in [t_1, t_2],$$

is continuous on $[t_1, t_2]$, $\dot{\psi}$ exists on (t_1, t_2) and $\psi(t_1) = \psi(t_2) = 0$. By Rolle's theorem there is $c \in (t_1, t_2)$ with $\psi'(c) = 0$. Hence, the Wronskian of $\bar{u}, \bar{v}$ is zero at c which is a contradiction. $\square$

5.2. **Contact Jacobi curves.** We introduce the contact Jacobi curve, and study its properties. Without further mention, in this section we assume that the sub-Riemannian manifold (M, ω, g) is complete, so that the Hamiltonian flow is well-defined for all times.

Definition 5.8. *Let (M, ω, g) be a contact sub-Riemannian manifold, and let Γ be an embedded piece of a Reeb orbit. Consider the following one-parameter family of one-forms on $A^1\Gamma$:*

$$\omega_r := \left((\pi \circ e^{r\vec{H}})^* \omega \right)\Big|_{A^1\Gamma}, \qquad r \in [0, +\infty). \tag{24}$$

The contact Jacobi curve at $\lambda \in A^1\Gamma$ is the projectivization of $\omega_r|_\lambda$:

$$\Omega_\lambda : [0, +\infty) \to P(T_\lambda^* A^1\Gamma) \simeq \mathbb{RP}^1, \qquad \Omega_\lambda(r) := P(\omega_r|_\lambda). \tag{25}$$

The first singular radius *of the contact Jacobi curve is*

$$r_o(\lambda) := \inf\{r > 0 \mid \Omega_\lambda(r) = \Omega_\lambda(0)\}.$$

Remark 5.9. The contact Jacobi curve Ω_λ is defined for any $\lambda \in T^*M$ such that $\langle \lambda, f_0 \rangle = 0$ and $2H(\lambda) = 1$. In fact, the definition does not depend the choice of Γ: if Γ' is any other piece of Reeb orbit containing $\pi(\lambda)$, then $T_\lambda(A^1\Gamma) = T_\lambda(A^1\Gamma')$, yielding the same contact Jacobi curve Ω_λ. The same observation holds for Definition 5.10.

Definition 5.10. *Let (M, ω, g) be a contact sub-Riemannian manifold, and let Γ be an embedded piece of a Reeb orbit. A real number $r > 0$ is called a* focal radius *for $\lambda \in A^1\Gamma$ if*

$$d_{r\lambda} E : T_{r\lambda}(A^1\Gamma) \to T_{E(r\lambda)}M, \qquad \text{is not an isomorphism.}$$

Remark 5.11. Note that $r_{\text{foc}}(\lambda) > 0$ since all points on the zero section of $A\Gamma$ are regular points for E. Furthermore, by definition of $r_{\text{inj}}(\Gamma)$ (see Definition 3.4), it holds $r_{\text{foc}}(\lambda) \geq r_{\text{inj}}(\Gamma)$.

The next result is a crucial structure theorem for the contact Jacobi curve. In particular it is well-defined, is an immersion outside of focal radii (which is a discrete set of points), and its Schwarzian satisfies the assumptions for the general comparison theory developed in Section 5.1.

Theorem 5.12. *Let (M, ω, g) be a contact sub-Riemannian manifold, and let Γ be an embedded piece of a Reeb orbit. Let $\lambda \in A^1\Gamma$.*

  (i) *The set of focal radii for λ is discrete, separated from zero;*
  (ii) *The contact Jacobi curve $\Omega_\lambda : [0, +\infty) \to \mathbb{RP}^1$ is well-defined and smooth;*
  (iii) *The contact Jacobi curve is an immersion outside of zero and the focal radius;*
  (iv) *For any focal radius $\bar{r} > 0$, the following asymptotic formula for the Schwarzian derivative holds: there exists $k = k(\bar{r}) \in \{0, 1, 2, 3, 4\}$ such that*

$$\mathcal{S}(\Omega_\lambda)(r) = -\frac{k(k+2)}{2(r-\bar{r})^2} + k\, O\left(\frac{1}{r-\bar{r}}\right) + O(1); \tag{26}$$

(v) *The following asymptotic formula for the Schwarzian derivative holds at $r = 0$:*

$$\mathcal{S}(\Omega_\lambda)(r) = -\frac{3}{2r^2} + O(r). \tag{27}$$

The term $O(r^n)$ denotes a smooth remainder. Namely there exist $\varepsilon > 0$ and a smooth function $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$, both depending on λ, such that $O(r^n) = r^n f(r)$ for all $r \in (-\varepsilon, \varepsilon)$.

Remark 5.13. A case-by-case analysis shows that only the values $k = 0, 1, 2$ can occur in (26), see [Bel24]. This also shows that the contact Jacobi curve can be an immersion at focal radii, in which case $k = 0$. As it will be clear from the proof, it is never an immersion at $t = 0$.

Proof. Let $(r \cos\theta, r \sin\theta, z)$ be cylindrical coordinates on $A\Gamma$ as in Section 3.2. Note that, out of the zero section, it holds

$$T_\lambda(A^1\Gamma) \simeq \mathrm{span}\{\partial_\theta|_\lambda, \partial_z|_\lambda\}.$$

Writing $E^*\omega$ in these coordinates, we obtain

$$
\begin{aligned}
(E^*\omega)|_{r\lambda} &= \omega(E_*\partial_r|_{r\lambda})dr + \omega(E_*\partial_\theta|_{r\lambda})d\theta + \omega(E_*\partial_z|_{r\lambda})dz \\
&= \omega\left(\pi_*\vec{H}|_{e^{\vec{H}}(r\lambda)}\right)dr + \omega\left((\pi \circ e^{r\vec{H}})_*\partial_\theta|_\lambda\right)d\theta + \omega\left((\pi \circ e^{r\vec{H}})_*\partial_z|_\lambda\right)dz \\
&= \omega_r(\partial_\theta|_\lambda)d\theta + \omega_r(\partial_z|_\lambda)dz,
\end{aligned}
$$

where in the second line we used that fact that $\pi_*\vec{H}|_\lambda$ is horizontal, and we also used the homogeneity property of the Hamiltonian (7). Consequently, we obtain

$$E^*(\omega \wedge d\omega)|_{r\lambda} = \frac{\omega_r \wedge \dot{\omega}_r(\partial_\theta|_\lambda, \partial_z|_\lambda)}{r} r\, dr \wedge d\theta \wedge dz, \tag{28}$$

where the dot denotes the derivative w.r.t. parameter r.

To proceed, we need the following claims: for any $\lambda \in A^1\Gamma$ and $\bar{r} \geq 0$ the one-parameter family of one-forms ω_r, evaluated at λ, has finite order at $\bar{r}$, namely there exists $m \in \{0, 1, 2, 3\}$ and a non-zero one-form β on $A^1\Gamma$ such that

$$\omega_r|_\lambda = (r - \bar{r})^m \beta|_\lambda + O\left((r - \bar{r})^{m+1}\right), \tag{29}$$

where $O(r - \bar{r})$ denotes a smooth remainder. Furthermore, the one-parameter family of two-forms has a similar property: there exist $n \in \{0, 1, 2, 3, 4\}$ (depending on $\lambda, \bar{r}$), and a non-zero two-form α on $A^1\Gamma$ such that

$$\omega_r \wedge \dot{\omega}_r|_\lambda = (r - \bar{r})^n \alpha|_\lambda + O\left((r - \bar{r})^{n+1}\right). \tag{30}$$

The proof of (29)-(30) is postponed to Appendix A (see Lemma A.2).

In the following we will often omit the evaluation of forms at the fixed $\lambda \in A^1\Gamma$, for brevity.

Proof of (i)**.** By (28), and since $\omega \wedge d\omega$ is a volume form on M, we note that $\bar{r} > 0$ is a focal radius if and only if $\omega_{\bar{r}} \wedge \dot{\omega}_{\bar{r}} = 0$. In particular it follows from (30) that the set of focal radii is discrete. Furthermore, by definition, $r = 0$ cannot be a focal time since E is always a diffeomorphism in a neighbourhood of the zero section of $A\Gamma$.

Proof of (ii) **and** (iii)**.** If $r > 0$ is non-focal, $\omega_r \wedge \dot{\omega}_r \neq 0$. In particular $\omega_r \neq 0$. It follows that the contact Jacobi curve

$$\Omega_\lambda(r) = P(\omega_r|_\lambda) \simeq [\omega_r(\partial_\theta) : \omega_r(\partial_z)],$$

is well-defined and smooth in a neighbourhood of r. Furthermore, assuming $\omega_r(\partial_z) \neq 0$ we have

$$\dot{v}(r) = -\frac{\omega_r \wedge \dot{\omega}_r(\partial_\theta, \partial_z)}{\omega_r(\partial_z)^2} \neq 0,$$

and similarly if $\omega_r(\partial_\theta|_\lambda) \neq 0$. It follows that the contact Jacobi curve is an immersion at all non-focal radii $r > 0$. Note that by (28), since $r\, dr \wedge d\theta \wedge dz \neq 0$ and smooth at $r = 0$, we must have $\omega_0 \wedge \dot\omega_0 = 0$ so that at $r = 0$ the contact Jacobi curve is not an immersion.

We now prove how the contact Jacobi curve is well-defined and smooth also in a neighbourhood of the focal radii. Let $\bar r > 0$ be a focal radius. By (29) it follows that in a neighbourhood of $\bar r$ the one-parameter family of one-forms on $A^1\Gamma$

$$\bar\omega_r := \frac{\omega_r}{(r - \bar r)^m}, \tag{31}$$

is smooth as a function of r and non-zero (when evaluated at λ). Thus, for all $r \neq \bar r$ in a neighbourhood of $\bar r$, it holds

$$\Omega_\lambda(r) = P(\omega_r|_\lambda) = P(\bar\omega_r|_\lambda) \simeq [\bar\omega_r(\partial_\theta) : \bar\omega_r(\partial_z)],$$

and so the contact Jacobi curve extends smoothly through $\bar r$.

Proof of (iv). Let $\bar r > 0$ be a focal radius. Let $\bar\omega_r$ be the one-parameter family of one-form in (31), which is smooth and non-zero in a neighbourhood of $\bar r$. Assume without loss of generality that $\bar\omega_{\bar r}(\partial_z) \neq 0$. Then $v(r) := \bar\omega_r(\partial_\theta)/\bar\omega_r(\partial_z)$ defines a smooth coordinate for the contact Jacobi curve in a neighbourhood of $\bar r$, and furthermore for all $r \neq \bar r$ in a neighbourhood of $\bar r$ it holds

$$\dot v(r) = \frac{d}{dr}\frac{\bar\omega_r(\partial_\theta)}{\bar\omega_r(\partial_z)} = \frac{d}{dr}\frac{\omega_r(\partial_\theta)}{\omega_r(\partial_z)} = -\frac{\omega_r \wedge \dot\omega_r(\partial_\theta, \partial_z)}{\omega_r(\partial_z)^2}. \tag{32}$$

By (30) and the last term of (32) it follows that there exists $k \in \mathbb{N}$ and $c \neq 0$ such that

$$\dot v(r) = c(r - \bar r)^k + O(r - \bar r)^{k+1}. \tag{33}$$

On one hand, $k \geq 0$ since from the first equality of (32) $\dot v$ is smooth around $\bar r$. On the other hand $k \leq 4$ by (30). By plugging (33) in the formula for the Schwarzian derivative (18), we obtain the following asymptotic expansion as $r \to \bar r$:

$$\begin{aligned}
\mathcal{S}(\Omega_\lambda)(r) &= \frac{\dddot v}{\dot v} - \frac{3}{2}\left(\frac{\ddot v}{\dot v}\right)^2 \\
&= \frac{k(k-1)}{(r - \bar r)^2} - \frac{3k^2}{2(r - \bar r)^2} + k\, O\left(\frac{1}{r - \bar r}\right) + O(1) \\
&= -\frac{k(k+2)}{2(r - \bar r)^2} + k\, O\left(\frac{1}{r - \bar r}\right) + O(1),
\end{aligned}$$

concluding the proof of (26).

Proof of (v). The proof of (27) is similar, but we can exploit more precise computations at $r = 0$. Assume without loss of generality that $\omega_0(\partial_z) \neq 0$. Consider the smooth coordinate $v(r) = \omega_r(\partial_\theta)/\omega_r(\partial_z)$, in a neighbourhood of $r = 0$. We have:

$$\dot v(r) = -\frac{\omega_r \wedge \dot\omega_r(\partial_\theta, \partial_z)}{\omega_r(\partial_z)^2} = -\frac{\omega_r(\partial_\theta)\dot\omega_r(\partial_z) - \omega_r(\partial_z)\dot\omega_r(\partial_\theta)}{\omega_r(\partial_z)^2}. \tag{34}$$

Routine computations, which are presented in Appendix A (see Lemma A.2), show that

$$\begin{aligned}
\omega_0(\partial_\theta) &= 0, & \ddot\omega_0(\partial_\theta) &= 1, & \omega_0(\partial_z) &= 1, \\
\dot\omega_0(\partial_\theta) &= 0, & \dddot\omega_0(\partial_\theta) &= 0, & \dot\omega_0(\partial_z) &= 0.
\end{aligned}$$

We deduce then from (34) that $\dot v(r) = r + r^3 f(r)$, for some $f \in C^\infty([0, \varepsilon))$. From the definition of Schwarzian derivative (18) and elementary computation we obtain

$$\mathcal{S}(\Omega_\lambda)(r) = -\frac{3}{2r^2} + 3\dot f(0)r + O(r^2), \tag{35}$$

which corresponds to (27). □

We continue the discussion on the standard contact (resp. overtwisted) structure of Examples 2.9 and 2.10, computing their contact Jacobi curve and Schwarzian derivative.

Example 5.14 (Contact Jacobi curve for the standard contact structure). Let $\Gamma = \{(0, 0, z) \mid z \in \mathbb{R}\}$, which is an embedded Reeb orbit of the standard sub-Riemannian contact structure on $\mathbb{R}^3$ (see Corollary 4.3). For all $\lambda \in A^1\Gamma$ the contact Jacobi curve is

$$\Omega_\lambda(r) = \left[\frac{r^2}{2} : 1\right], \qquad \forall\, r \in \mathbb{R}.$$

A homogeneous coordinate is $v = r^2/2$ and the Schwarzian derivative is

$$\mathcal{S}(\Omega_\lambda)(r) = -\frac{3}{2r^2}.$$

Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions. In the following corollary we exploit the following useful formulas for the Schwarzian derivative

$$\mathcal{S}(f \circ g) = (\mathcal{S}(f) \circ g)\,(g')^2 + \mathcal{S}(g).$$

Example 5.15 (Contact Jacobi curve for the standard overtwisted structure). Let $\Gamma = \{(0, 0, z) \mid z \in \mathbb{R}\}$, which is an embedded Reeb orbit of the standard sub-Riemannian contact structure on $\mathbb{R}^3$ (see Corollary 4.4). For all $\lambda \in A^1\Gamma$ the contact Jacobi curve is

$$\Omega_\lambda(r) = \left[\sin\left(\frac{r^2}{2}\right) : \cos\left(\frac{r^2}{2}\right)\right], \qquad \forall\, r \in \mathbb{R}.$$

A homogeneous coordinate is $v = \tan(r^2/2)$. Since $\mathcal{S}(\tan(r)) = 2$, the Schwarzian derivative is

$$\mathcal{S}(\Omega_\lambda)(r) = -\frac{3}{2r^2} + 2r^2.$$

5.3. **A sharp tightness radius estimate via contact Jacobi curves.** The first singular radius of a contact Jacobi curve detects the presence of overtwisted disks.

Theorem 5.16. *Let (M, ω, g) be a contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\mathrm{inj}}(\Gamma) > 0$. Then, letting*

$$r_o^-(\Gamma) := \inf\{r_o(\lambda) \mid \lambda \in A^1\Gamma\}, \qquad r_o^+(\Gamma) := \sup\{r_o(\lambda) \mid \lambda \in A^1\Gamma\},$$

the following estimates hold:

$$\min\left\{r_{\mathrm{inj}}(\Gamma), r_o^-(\Gamma)\right\} \leq r_{\mathrm{tight}}(\Gamma) \leq \min\left\{r_{\mathrm{inj}}(\Gamma), r_o^+(\Gamma)\right\}. \tag{36}$$

Moreover, if $r_o^+(\Gamma) < r_{\mathrm{inj}}(\Gamma)$, then for any $q \in \Gamma$, the set

$$D_q := \{E(r\lambda) \mid \lambda \in A_q^1\Gamma,\ r \leq r_o(\lambda)\} \tag{37}$$

is an overtwisted disk, and thus (M, ω) is an overtwisted contact manifold.

Proof. We first prove the lower bound in (36). We characterize the singular locus of the Reeb orbit and employ Theorem 4.6. Let $\rho = r_{\mathrm{inj}}(\Gamma)$, and recall that $E : A^{<\rho}\Gamma \to M$ is a diffeomorphism on its image by Theorem 3.3. Let $(r\cos\theta, r\sin\theta, z)$ be cylindrical coordinates on $A\Gamma$ as explained in Section 3.2. By the same computation in the proof of Theorem 5.12 we have

$$E^*\omega|_{r\lambda} = \omega_r(\partial_\theta|_\lambda)d\theta + \omega_r(\partial_z|_\lambda)dz, \tag{38}$$

for all $(r, \lambda) \in (0, +\infty) \times A^1\Gamma$. By definition of singular locus (see Definition 4.5) we see that

$$\mathrm{Sing}(\Gamma) = E\Big(\{r\lambda \in A\Gamma \mid (r, \lambda) \in (0, r_{\mathrm{inj}}(\Gamma)) \times A^1\Gamma,\ \omega_r(\partial_\theta|_\lambda) = 0\}\Big).$$

Note that $\omega_0(\partial_\theta) = \omega(\pi_*\partial_\theta) = 0$ by construction. For all $(r, \lambda) \in (0, r_{\mathrm{inj}}(\Gamma)) \times A^1\Gamma$ we have then the following characterization in terms of the contact Jacobi curve $\Omega_\lambda(\cdot)$:

$$E(r\lambda) \in \mathrm{Sing}(\Gamma) \quad \Longleftrightarrow \quad \Omega_\lambda(r) = \Omega_\lambda(0). \tag{39}$$

Using characterization (39) we obtain that, for all $\lambda \in A^1\Gamma$, it holds

$$\begin{aligned} r_0(\lambda) &= \inf\{r > 0 \mid \Omega_\lambda(r) = \Omega_\lambda(0)\} \\ &\leq \inf\{0 < r < r_{\mathrm{inj}}(\Gamma) \mid \Omega_\lambda(r) = \Omega_\lambda(0)\} \\ &= \inf\{\delta(E(r\lambda)) \mid E(r\lambda) \in \mathrm{Sing}(\Gamma)\}, \end{aligned} \tag{40}$$

where δ is the sub-Riemannian distance from Γ.

Assume now that $\mathrm{Sing}(\Gamma) \neq \emptyset$. This means that there exists $(r, \lambda) \in (0, r_{\mathrm{inj}}(\Gamma)) \times A^1\Gamma$ such that $\Omega_\lambda(r) = \Omega_\lambda(0)$ and thus inequality (40) is actually an equality. Therefore, taking the infimum over all $\lambda \in A^1\Gamma$ we obtain

$$r_0^-(\Gamma) = d_{SR}(\Gamma, \mathrm{Sing}(\Gamma)) \leq r_{\mathrm{tight}}(\Gamma),$$

where the last inequality is Theorem 4.6. If $\mathrm{Sing}(\Gamma) = \emptyset$ then again by Theorem 4.6 we have that $r_{\mathrm{tight}}(\Gamma) = r_{\mathrm{inj}}(\Gamma)$. Combining both cases we obtain the left inequality of (36).

We now prove the upper bound in (36). By definition $r_{\mathrm{tight}}(\Gamma) \leq r_{\mathrm{inj}}(\Gamma)$, so we proceed under the assumption $r_o^+(\Gamma) < r_{\mathrm{inj}}(\Gamma)$. It is sufficient to show that the sets (37) are overtwisted disks.

As explained in the proof of Theorem 4.6 there exist smooth functions $\phi, N \in C^\infty(A^{<\rho}\Gamma)$, with $N > 0$, such that

$$E^*\omega = N(\sin\phi d\theta + \cos\phi dz). \tag{41}$$

Comparing (41) with (38) yields, for all $(r, \lambda) \in (0, r_{\mathrm{inj}}(\Gamma)) \times A^1\Gamma$

$$N(r\lambda)\sin\phi(r\lambda) = \omega_r(\partial_\theta|_\lambda), \qquad N(r\lambda)\cos\phi(r\lambda) = \omega_r(\partial_z|_\lambda).$$

Since $\omega_0(\partial_\theta|_\lambda) = 0$, it follows that $r_o(\lambda)$ is solution to the equation

$$\phi(r_o(\lambda)\lambda) = \pi.$$

Since any positive value of ϕ is a regular one (see the proof of Theorem 4.6), $r_o : A^1\Gamma \to \mathbb{R}$ is smooth. It follows that, for $q \in \Gamma$, the set $D_q \subset M$ of (37) is a smooth embedded disk. Recall the map $\varphi : A^{<\rho}\Gamma \to \mathbb{R}^3$

$$\varphi(x, y, z) = \left(\sqrt{\frac{\phi}{H}} x, \sqrt{\frac{\phi}{H}} y, z \right).$$

As we already observed in the proof of Theorem 4.6, φ is a diffeomorphism on the image, and

$$\varphi^*\omega_{\mathrm{ot}} = \sin\phi d\theta + \cos\phi dz = \frac{1}{N}E^*\omega.$$

It follows that on the tubular neighbourhood $E(A^{<\rho}\Gamma) \subset M$ it holds $(\varphi \circ E^{-1})_*\Delta = \Delta_{\mathrm{ot}}$. Furthermore it is easy to check (recalling that $r \mapsto \phi(r\lambda)$ is monotone increasing for $r \in (0, r_{\mathrm{inj}}(\Gamma))$ and $\lambda \in A^1\Gamma$) that, if $q = E(0, 0, z)$, then

$$\varphi \circ E^{-1}(D_q) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 2\pi\},$$

which is (up to a translation along the z axis) the standard overtwisted disk of $(\mathbb{R}^3, \Delta_{\mathrm{ot}})$. It follows that D_q is a overtwisted disk. $\qquad\square$

5.4. **Comparison theorems under Schwarzian derivative bounds.** We can apply the comparison Theorem 5.6 to contact Jacobi curves, to estimate their singular radius. Combined with Theorem 5.16, we obtain a comparison theorem for the tightness radius.

Theorem 5.17. *Let (M, ω, g) be a contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\mathrm{inj}}(\Gamma) > 0$. Assume that there exist $k_1, k_2 \in \mathbb{R}$ such that the Schwarzian derivative of the contact Jacobi curves are bounded above by:*

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) \leq -\frac{3}{4r^2} + k_1 r + k_2 r^2, \qquad \forall\, r \in (0, r_{\mathrm{inj}}(\Gamma)], \quad \forall\, \lambda \in A^1\Gamma. \tag{42}$$

Then for the tightness radius of Γ it holds

$$r_{\mathrm{tight}}(\Gamma) \geq \min\left\{r_*(k_1, k_2), r_{\mathrm{inj}}(\Gamma)\right\}, \tag{43}$$

where

$$r_*(k_1, k_2) := \begin{cases} \frac{-k_1 + \sqrt{8\pi k_2^{3/2} + k_1^2}}{2k_2}, & \text{if } k_1, k_2 > 0, \\ \frac{\sqrt{2\pi}}{k_2^{1/4}}, & \text{if } k_1 \leq 0,\ k_2 > 0, \\ \left(\frac{3 j_{2/3}}{2}\right)^{2/3} \frac{1}{k_1^{1/3}}, & \text{if } k_1 > 0,\ k_2 \leq 0, \\ +\infty & \text{if } k_1, k_2 \leq 0, \end{cases}$$

and $j_{2/3} \sim 3.37$ is the first positive root of the Bessel function of first kind $J_{2/3}$.

Remark 5.18. Fix $\lambda \in A^1\Gamma$. At $r = 0$ and at focal radii $\bar{r}$, the following asymptotics hold

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) = -\frac{3}{4r^2} + O(r), \qquad \frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) = -\frac{k(k+2)}{4(r - \bar{r})^2} + k\, O\left(\frac{1}{r - \bar{r}}\right) + O(1), \tag{44}$$

with smooth remainders, and $k \in \{0, 1, 2, 3, 4\}$ (see Items (iv) and (v) of Theorem 5.12). In particular the Schwarzian derivative remains bounded *from above* in a neighbourhood of focal radii. Since focal radii are discrete, one can see that for any $\lambda \in A^1\Gamma$ there is a smooth function $R_\lambda : [0, +\infty) \to \mathbb{R}$ such that

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) \leq -\frac{3}{4r^2} + R_\lambda(r), \qquad \forall\, r \in (0, +\infty).$$

It follows that for any bounded interval I the bound (42) holds on that interval, for some $k_1, k_2 \in \mathbb{R}$, depending on λ, I. In particular the upper bound in (42) is always verified for a given λ if $r_{\mathrm{inj}}(\Gamma) < +\infty$ (e.g. when M is compact). The *uniformity* of this bound w.r.t. $\lambda \in A^1\Gamma$ is more delicate: this is due to the fact that the remainder terms at focal radii in (44), albeit smooth w.r.t. r, may not be upper semi-continuous w.r.t. λ.

Remark 5.19 (Sharpness). The tightness radius estimate is sharp for the standard sub-Riemannian contact structure (resp. overtwisted structure) of Example 2.9 (resp. Example 2.10). More precisely, for any embedded piece of a Reeb orbit Γ we have $r_{\mathrm{inj}}(\Gamma) = +\infty$, (see Corollaries 4.3 and 4.4) and as computed in Examples 5.14 and 5.15, for all $\lambda \in A^1\Gamma$ it holds

$$\mathcal{S}(\Omega_\lambda)(r) = -\frac{3}{4r^2} + k_2 r^2, \qquad \forall\, r \in (0, +\infty),$$

with $k_2 = 0$ (resp. $k_2 = 1$). For these structures, $r_{\mathrm{tight}}(\Gamma) = +\infty$ (resp. $r_{\mathrm{tight}}(\Gamma) = \sqrt{2\pi}$).

Remark 5.20 (On the necessity of k_1). The fact that $k_1 = 0$ in the models of Remark 5.19 is no coincidence. Indeed, for any radial model of Theorem 4.2 the Schwarzian derivative satisfies

$$\mathcal{S}(\Omega_\lambda(r)) = -\frac{3}{2r^2} + O(r^2). \tag{45}$$

This follows from (35) and the fact that, for smoothness reasons, the homogeneous coordinate $\dot{v}(r)$ is always an odd function of r. However (45) does not necessarily hold in less symmetric structures, as we illustrate with an example.

Let $\varepsilon > 0$ and let $f_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ be the following smooth function

$$f_\varepsilon(r\cos\theta, r\sin\theta, z) = f_{\varepsilon,\theta}(r) := \frac{r^2}{2} + \varepsilon \frac{r^5 \cos\theta}{1 + r^2},$$

where $r^2 = x^2 + y^2$. For $\varepsilon > 0$ small enough the form

$$\omega = \sin(f_{\varepsilon,\theta}(r))d\theta + \cos(f_{\varepsilon,\theta}(r))dz,$$

defines a smooth contact structure on $\mathbb{R}^3$, and the symmetric form

$$g = \left(dr \otimes dr + \left(\frac{\partial f_{\varepsilon,\theta}}{\partial r} \right)^2 (d\theta \otimes d\theta + dz \otimes dz) \right)\Bigg|_{\ker\omega},$$

defines a smooth sub-Riemannian metric compatible with ω. Arguing as in Theorem 4.2 one can show that $\Gamma = \{(0,0,z) \mid z \in \mathbb{R}\}$ is the orbit of the Reeb field through the origin, and that, in the sense of Theorem 4.2, $E : A\Gamma \cong \mathbb{R}^3 \to \mathbb{R}^3$ is the identity. It follows that for all $\lambda \in A^1\Gamma$

$$\Omega_\lambda(r) = [\sin(f_{\varepsilon,\theta}(r)) : \cos(f_{\varepsilon,\theta}(r))], \qquad \forall\, r \in \mathbb{R}.$$

Therefore, setting $v_{\varepsilon,\theta}(r) = \tan(f_{\varepsilon,\theta}(r))$, elementary computations show that

$$\mathcal{S}(\Omega_\lambda)(r) = \frac{\dddot{v}_{\varepsilon,\theta}}{\dot{v}_{\varepsilon,\theta}} - \frac{3}{2}\left(\frac{\ddot{v}_{\varepsilon,\theta}}{\dot{v}_{\varepsilon,\theta}} \right)^2 = -\frac{3}{2r^2} + 15\varepsilon \cos(\theta)r + O\left(r^2\right).$$

In this case, $k_1 \neq 0$ is necessary in order to apply Theorem 5.17.

Remark 5.21. If assumption (42) is verified only for $r \in (0, T]$, then the estimates hold only up to distance T. More precisely (43) holds by adding T to the arguments of the minimum.

Proof. By Theorem 5.16, it is sufficient to prove that, for any $\lambda \in A^1\Gamma$ with $r_o(\lambda) < r_{\mathrm{inj}}(\Gamma)$, it holds $r_o(\lambda) \geq r_*(k_1, k_2)$. Fix then such a λ.

By Definition 5.8, $r_o(\lambda)$ is the first self-intersection time of the contact Jacobi curve $\Omega_\lambda : [0, +\infty) \to \mathbb{RP}^1$, which we can restrict under our working assumption to the interval $[0, r_{\mathrm{inj}}(\Gamma))$, where (42) holds. We intend to apply Theorem 5.6 with $q = 2\mathcal{S}(\Omega_\lambda)$ and a suitable model $\bar{q}$ for comparison, given by the right hand side of (42). We note that the regularity assumption (22) is verified thanks to Item (v) of Theorem 5.12.

1. Case $k_1, k_2 > 0$. In this general case, the solutions of the model ODE

$$\ddot{\bar{u}}(r) + \left(-\frac{3}{4r^2} + k_1 r + k_2 r^2 \right)\bar{u}(r) = 0$$

cannot be expressed in terms of elementary functions. We choose then a larger $\bar{q}$, which has the advantage of having elementary solution. Let then

$$\bar{q}(r) := -\frac{3}{4\left(r + \frac{k_1}{2k_2} \right)^2} + k_2 \left(r + \frac{k_1}{2k_2} \right)^2, \qquad r \in (0, +\infty).$$

Observe that (42) and elementary estimates (with $k_1, k_2 > 0$) imply

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) \leq \bar{q}(r), \qquad \forall\, r \in (0, r_{\mathrm{inj}}(\Gamma)).$$

Furthermore, the solution of $\ddot{\bar{u}} + \bar{q}\bar{u} = 0$, such that $\bar{u}(0) = 0$ (see Remark 5.7), is

$$\bar{u}(r) = \frac{\sin\left(\frac{\sqrt{k_2}}{2}r(r+2w)\right)}{\sqrt{r+w}}, \qquad w := \frac{k_1}{2k_2}, \qquad r \in (0, +\infty).$$

Its first positive root is the non-zero solution of $\sqrt{k_2}r(r+2w) = 2\pi$. By Theorem 5.6 we obtain

$$r_o(\lambda) \geq \inf\{r > 0 \mid \bar{u}(r) = 0\} = \frac{-k_1 + \sqrt{8\pi k_2^{3/2} + k_1^2}}{2k_2}.$$

2. Case $k_1 \leq 0$, $k_2 > 0$. By (42) we have

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) \leq -\frac{3}{4r^2} + k_1 r + k_2 r^2 \leq -\frac{3}{4r^2} + k_2 r^2, \qquad \forall\, r \in (0, r_{\text{inj}}(\Gamma)).$$

Hence this case can be seen as a special case of the above case, by setting $k_1 = 0$.

3. Case $k_1 > 0$ and $k_2 \leq 0$. By (42) we have

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) \leq -\frac{3}{4r^2} + k_1 r := \bar{q}(r), \qquad \forall\, r \in (0, r_{\text{inj}}(\Gamma)).$$

In this case the non-trivial solutions of $\ddot{\bar{u}} + \bar{q}\bar{u} = 0$ is most easily obtained by the replacement $\bar{u}(t) = r^{1/2}\xi(2\sqrt{k_1}/3r^{3/2})$, where ξ is a solution of the Bessel equation:

$$z^2\ddot{\xi}(z) + z\dot{\xi}(z) + (z^2 - 4/9)\xi(z) = 0.$$

Therefore we get that a solution of $\ddot{\bar{u}} + \bar{q}\bar{u} = 0$ with $\bar{u}(0) = 0$ (see Remark 5.7) is

$$\bar{u}(r) = r^{1/2}J_{2/3}\left(\frac{2\sqrt{\kappa_1}r^{3/2}}{3}\right), \qquad r \in (0, +\infty),$$

where J_n is the Bessel function of first kind. Therefore applying Theorem 5.6 we obtain

$$r_o(\lambda) \geq \inf\{r > 0 \mid \bar{u}(r) = 0\} = \left(\frac{3j_{2/3}}{2}\right)^{2/3} \frac{1}{k_1^{1/3}},$$

where $j_{2/3} \sim 3.37$ is the first positive zero of the Bessel function $J_{2/3}$.

3. Case $k_1, k_2 \leq 0$. By (42) we have

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) \leq -\frac{3}{4r^2} := \bar{q}(r), \qquad \forall\, r \in (0, r_{\text{inj}}(\Gamma)).$$

A non-trivial solution of $\ddot{\bar{u}} + \bar{q}\bar{u} = 0$ with $\bar{u}(0) = 0$ (see Remark 5.7) is

$$\bar{u}(r) = t^{3/2}, \qquad r \in (0, +\infty).$$

Therefore by Theorem 5.6 we obtain $r_o(\lambda) \geq \inf\{r > 0 \mid \bar{u}(r) = 0\} = +\infty$. $\qquad\qquad \square$

5.5. **Comparison theorems under canonical curvature bounds.** In this section we develop a different type of tightness radius estimate, assuming a control on the so-called *canonical curvature*, rather than on the Schwarzian derivative of the contact Jacobi curve. The canonical curvature plays the role of sectional curvature in sub-Riemannian comparison geometry. Historically, it was introduced in [AG97, AZ02a] for the study of curves in the Lagrange Grassmannians. The concept of canonical curvature (and the related canonical frame) was then formalized in full generality in [ZL09], and was further studied and developed in [LZ11, LLZ16, AL14, AL15, ABR18, ABR17, BR16, BR20, BR17] in sub-Riemannian geometry and its comparison theory. We follow here the presentation that can be found in [ABR18].

Definition 5.22. *Let (M, ω, g) be a contact sub-Riemannian manifold. Let $\lambda \in T^*M \setminus \{H = 0\}$. The* canonical frame *is a one-parameter family of moving frames for $T_\lambda(T^*M)$, denoted by $\{E_i(t), F_i(t)\}_{i=a,b,c}$, which is determined up to sign by the following properties:*

- *It is a one-parameter family of Darboux frames:*

$$\sigma(E_i(t), E_j(t)) = \sigma(F_i(t), F_j(t)) = \sigma(E_i(t), F_j(t)) - \delta_{ij} = 0, \qquad \forall\, t \in \mathbb{R};$$

- *the "E" part of the frame satisfies:*

$$\ker \pi_* = e_*^{r\vec{H}} \operatorname{span}\{E_a(t), E_b(t), E_c(t)\}, \qquad \forall\, t \in \mathbb{R};$$

- *It satisfies the* structural equations*:*

$$
\begin{aligned}
\dot{E}_a(t) &= -F_a(t), & \dot{F}_a(t) &= R_a^\lambda(t) E_a(t) - F_c(t), \\
\dot{E}_b(t) &= -F_b(t), & \dot{F}_b(t) &= 0, \\
\dot{E}_c(t) &= E_a(t), & \dot{F}_c(t) &= R_c^\lambda(t) E_c(t),
\end{aligned}
$$

where the dot denotes the derivative with respect to t, and $R_a^\lambda(t), R_c^\lambda(t)$ are smooth functions called canonical curvatures *at λ.*

On three-dimensional contact sub-Riemannian manifolds, the canonical frame can be computed explicitly. Let us recall some notation. Let f_0, f_1, f_2 be a local oriented orthonormal frame and $\nu_0 = \omega, \nu_1, \nu_2$ be the corresponding dual frame. Let $h_1, h_2, h_0 : T^*M \to \mathbb{R}$ be the linear-on-fibers functions defined in (4). Note that, for any $q \in M$ such that f_1, f_2 are defined, the map $(h_0, h_1, h_2) : T_q^*M \to \mathbb{R}^3$ induces coordinates on the fiber T_q^*M and, in turn, vertical[1] vector fields $\partial_{h_1}, \partial_{h_2}, \partial_{h_0}$ locally defined on T^*M. They are characterized by the following property in terms of the symplectic form

$$\sigma(\partial_{h_i}, \cdot) = (\pi^*\nu_i)(\cdot), \qquad i = 0, 1, 2.$$

Correspondingly, we define the vertical vector fields

$$\partial_\theta = h_1 \partial_{h_2} - h_2 \partial_{h_1}, \qquad \mathfrak{e} = h_1 \partial_{h_1} + h_2 \partial_{h_2} + h_0 \partial_{h_0}.$$

Both vector fields are globally well-defined on T^*M independently on the choice of the f_0, f_1, f_2. Furthermore $\mathfrak{e}$ is the *Euler vector field*, the generator of fiber-wise dilations, namely

$$e^{(\ln \alpha)\mathfrak{e}}(\lambda) = \alpha\lambda, \qquad \forall \alpha > 0, \quad \lambda \in T^*M.$$

Finally, let also introduce the following globally-defined smooth vector field:

$$\vec{H}' := [\partial_\theta, \vec{H}] = h_2 \vec{h}_1 - h_1 \vec{h}_2 - \left(\sum_{j=1}^2 c_{12}^j \partial_\theta h_j \right) \partial_\theta + \left(\sum_{i,j=1}^2 h_i c_{i0}^j \partial_\theta h_j \right) \partial_{h_0}. \tag{46}$$

In terms of the above ingredients, the canonical frame has the following expression (see [ABR18, Sec. 7.5], to which we refer to for a proof and more details):

$$E_c(t) = \frac{1}{\sqrt{2H}} e_*^{-t\vec{H}} \partial_{h_0}, \qquad F_c(t) = \frac{1}{\sqrt{2H}} e_*^{-t\vec{H}} \left(-[\vec{H}, \vec{H}'] + R_a^\lambda(t) \partial_\theta \right), \tag{47}$$

$$E_a(t) = \frac{1}{\sqrt{2H}} e_*^{-t\vec{H}} \partial_\theta, \qquad F_a(t) = \frac{1}{\sqrt{2H}} e_*^{-t\vec{H}} \vec{H}', \tag{48}$$

$$E_b(t) = \frac{1}{\sqrt{2H}} e_*^{-t\vec{H}} \mathfrak{e}, \qquad F_b(t) = \frac{1}{\sqrt{2H}} e_*^{-t\vec{H}} \vec{H}. \tag{49}$$

---

[1]We call a vector (or a vector field) on T^*M *vertical* if $\pi_* V = 0$.

We have the following expressions for the canonical curvatures:

$$R_a^\lambda(t) = \frac{1}{2H}\sigma_{\lambda(t)}([\vec{H}, \vec{H}'], \vec{H}'), \tag{50}$$

$$R_c^\lambda(t) = \frac{1}{2H}\sigma_{\lambda(t)}([\vec{H}, [\vec{H}, \vec{H}']], [\vec{H}, \vec{H}']) - \frac{1}{(2H)^2}\sigma_{\lambda(t)}([\vec{H}, \vec{H}'], \vec{H}')^2, \tag{51}$$

where the right hand sides of (50)-(51) are evaluated at $\lambda(t) = e^{t\vec{H}}(\lambda)$. (In [ABR18, Prop. 7.13], from which the expressions (50)-(51) are taken, there is a typo in the formulas for the curvatures, where R_{bb} should be instead R_{cc}.)

The next statement links the canonical frame with the contact Jacobi curve.

Lemma 5.23. *Let (M, ω, g) be a contact sub-Riemannian manifold, and $\lambda \in T^*M \setminus \{H(0)\}$. Let $\{E_i(r), F_i(r)\}_{i=a,b,c}$ be the canonical frame and let $\{E^i(r), F^i(r)\}_{i=a,b,c}$ be its dual. Then*

$$F^c(r) = -\sqrt{2H}(\pi \circ e^{r\vec{H}})^*\omega, \qquad \forall\, r \in \mathbb{R},$$

where both sides are evaluated at the given λ.

Proof. For simplicity, assume $2H(\lambda) = 1$ It is sufficient to show that for $i = a, b, c$ it holds

$$\langle(\pi \circ e^{r\vec{H}})^*\omega, F_i(r)\rangle = -\delta_{ic}, \qquad \langle(\pi \circ e^{r\vec{H}})^*\omega, E_i(r)\rangle = 0.$$

We begin with E_i. It holds:

$$\langle(\pi \circ e^{r\vec{H}})^*\omega, E_i(r)\rangle = \langle\pi^*\omega, e^{r\vec{H}*}E_i(r)\rangle = 0.$$

Note that $\vec{H}$ pointwise projects to an horizontal vector, namely

$$\pi_*\vec{H}|_\lambda = h_1(\lambda)f_1|_{\pi(\lambda)} + h_2(\lambda)f_2|_{\pi(\lambda)}. \tag{52}$$

Therefore, using (49) and (52) it holds

$$\langle(\pi \circ e^{r\vec{H}})^*\omega, F_b(r)\rangle = \langle\pi^*\omega, \vec{H}\rangle = 0.$$

Similarly, for $\vec{H}'$ it follows from equation (46) that

$$\pi_*\vec{H}'|_\lambda = h_2(\lambda)f_1|_{\pi(\lambda)} - h_1(\lambda)f_2|_{\pi(\lambda)}. \tag{53}$$

Therefore

$$\langle(\pi \circ e^{r\vec{H}})^*\omega, F_a(r)\rangle = \langle\pi^*\omega, \vec{H}'\rangle = 0.$$

Finally, using (47) we obtain

$$\begin{aligned}
\langle(\pi \circ e^{r\vec{H}})^*\omega, F_c(r)\rangle &= \langle\pi^*\omega, [\vec{H}', \vec{H}]\rangle \\
&= \vec{H}'(\pi^*\omega(\vec{H})) - \vec{H}(\pi^*\omega(\vec{H}')) - d\omega(\pi_*\vec{H}', \pi_*\vec{H}) \\
&= \nu_1 \wedge \nu_2(h_1f_1 + h_2f_2, h_2f_1 - h_1f_2) = -1,
\end{aligned}$$

where we used $\pi^*\omega(\vec{H}) = \pi^*\omega(\vec{H}') = 0$, $d\omega = \nu_1 \wedge \nu_2$ and (52)–(53). $\qquad\square$

We now prove our comparison theorem for tightness radius under canonical curvature bounds. As it will be clear from the techniques in the proof, the result is non-sharp. However, it provides a connection between tightness and sub-Riemannian curvature invariants.

Theorem 5.24. *Let (M, ω, g) be a contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\mathrm{inj}}(\Gamma) > 0$. Assume that there exist $A, C > 0$ such that*

$$\sqrt{1 + R_a^\lambda(r)^2} \le A, \quad \sqrt{1 + R_c^\lambda(r)^2} \le C, \qquad \forall\, r \in [0, r_{\mathrm{inj}}(\Gamma)], \quad \forall\, \lambda \in A^1\Gamma,$$

where $R_a^\lambda(r), R_c^\lambda(r)$ are the canonical curvatures at λ. Then for the tightness radius of Γ it holds

$$r_{\text{tight}}(\Gamma) \geq \min\{\tau(A, C), r_{\text{inj}}(\Gamma)\},$$

where

$$\tau(A, C) := \int_0^\infty \frac{1}{Au^2 + Cu + 1}\, \mathrm{d}u. \tag{54}$$

Proof. By Theorem 5.16, we have to prove that, for all $\lambda \in A^1\Gamma$ with $r_o(\lambda) < r_{\text{inj}}(\Gamma)$, it holds

$$r_o(\lambda) > \int_0^\infty \frac{1}{Au^2 + Cu + 1}\, \mathrm{d}u.$$

Fix then such a λ. Recall that, by Definition 5.8, the first singular radius is

$$r_o(\lambda) = \inf\{r > 0 \mid \Omega_\lambda(r) = \Omega_\lambda(0)\}$$
$$= \inf\{r > 0 \mid \ker \omega_r|_\lambda = \ker \omega_0|_\lambda\},$$

where $\omega_r = (\pi \circ e^{r\vec{H}})^* \omega|_{A^1\Gamma}$ is the one-form (24). Observe that

$$\ker \omega_0|_\lambda = \ker \pi^* \omega \cap T_\lambda(A^1\Gamma) = \text{span}\{\partial_\theta\} = \{E_a(0)\},$$

where we used (48). Using Lemma 5.23 and we can write $r_o(\lambda)$ as

$$r_o(\lambda) = \inf\{r > 0 \mid \langle F^c(r), E_a(0) \rangle = 0\}.$$

From Definition 5.22 we deduce the structural equation for the dual of the canonical frame:

$$\dot{F}^a = E^a, \qquad\qquad\qquad \dot{E}^a = -R_a F^a - E^c,$$
$$\dot{F}^c = F^a, \qquad\qquad\qquad \dot{E}^c = -R_c F^c,$$

where we omit the dependence on r and λ. It follows that $x_0(r) := \langle F^c(r), E_a(0) \rangle$ satisfies the following Cauchy problem:

$$\begin{cases} \frac{d}{dr}(\dddot{x}_0 + R_a \dot{x}_0) - R_c x_0 = 0, \\ x_0(0) = 0,\ \dot{x}_0(0) = 0,\ \ddot{x}_0(0) = 1,\ \dddot{x}_0(0) = 0. \end{cases} \tag{55}$$

We rewrite it as a first-order Cauchy problem by defining $x := (x_0, x_1, x_2, x_3)$ with

$$x_1 := x^{(1)}, \quad x_2 := x^{(2)}, \quad x_3 := x^{(3)} + R_a x^{(1)},$$

so that (55) rewrites as

$$\begin{cases} \dot{x}_0 = x_1, \\ \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3 - R_a x_1, \\ \dot{x}_3 = R_c x_0, \\ x(0) = (0, 0, 1, 0). \end{cases}$$

Therefore we have $r_o(\lambda) = \inf\{r > 0 \mid x_0(r) = 0\}$. Notice that, since

$$x_0(r) = \int_0^r \left(\int_0^t x_2(s)\, \mathrm{d}s\right) \mathrm{d}t, \qquad x_2(0) = 1,$$

then $r_o(\lambda) > \tau := \inf\{r > 0 \mid x_2(r) = 0\}$. We define then

$$v := \begin{pmatrix} x_0/x_2 \\ x_1/x_2 \\ x_3/x_2 \end{pmatrix}, \qquad M = (0, R_a, -1) \cdot v\, \mathbb{1} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ R_c & 0 & 0 \end{pmatrix}, \qquad w := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where $\mathbb{1}$ is the identity matrix and $\cdot$ denotes the scalar product. We find that v is the solution of the following Cauchy problem

$$\begin{cases} \dot{v} = Mv + w, \\ v(0) = 0. \end{cases}$$

Thus, letting $u := |v|$ we have the inequality

$$\dot{u}(r) \leq \sqrt{1 + R_a^2(t)}u(r)^2 + \sqrt{1 + R_c^2(t)}u(r) + 1 \leq Au(t)^2 + Cu(r) + 1, \qquad \forall\, r \in [0, \tau).$$

We deduce that $r_o(\lambda)$ must be greater than the first blow-up time of the Cauchy problem

$$\begin{cases} \dot{u} = Au^2 + Cu + 1, \\ u(0) = 0, \end{cases}$$

which is given by (54). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

6. K-contact sub-Riemannian manifolds

A contact sub-Riemannian manifold (M, ω, g) is called K-contact if the associated Reeb field f_0 is a Killing vector field for the extended Riemannian metric g, or, equivalently, if the Reeb flow acts on M by sub-Riemannian isometries.

Remember the two metric invariants $\chi, \kappa : M \to \mathbb{R}$ introduced in Section 2.2. The first one can be given in terms of an orthonormal frame f_0, f_1, f_2 and its structural coefficients by:

$$\chi = \sqrt{\left(c_{01}^1\right)^2 + \frac{1}{4}\left(c_{01}^2 + c_{02}^1\right)^2}.$$

The next result is well-known, see [ABB20, Cor. 17.10].

Proposition 6.1. (M, ω, g) is K-contact if and only $\chi = 0$.

The second metric invariant is given by the formula:

$$\kappa = f_1\left(c_{12}^2\right) - f_2\left(c_{12}^1\right) - \left(c_{12}^1\right)^2 - \left(c_{12}^2\right)^2 + \frac{c_{02}^1 - c_{01}^2}{2}.$$

We recall that, in K-contact case, κ is the Gaussian curvature of the surface obtained by locally quotienting M under the action of the Reeb flow, see Remark 2.8.

Remark 6.2. Let (M, ω, g) be a contact sub-Riemannian manifold (not necessarily K-contact). Let $H_R : T^*M \to \mathbb{R}$ be the Hamiltonian of the extended Riemannian metric, then the Riemannian tubular neighbourhood map reads

$$E_R : A\Gamma \to M, \qquad E_R(\lambda) = \pi \circ e^{\vec{H}_R}(\lambda).$$

In general $E_R \neq E$. However in the K-contact case the two maps coincide. Indeed

$$H_R = H + \frac{1}{2}h_0^2 \qquad \Rightarrow \qquad \vec{H}_R = \vec{H} + h_0\vec{h}_0,$$

where $H : T^*M \to \mathbb{R}$ is the sub-Riemannian Hamiltonian. Since $[\vec{H}, \vec{h}_0] = 0$ and $h_0|_{A\Gamma} = 0$, then $\vec{H}_R|_{A\Gamma} = \vec{H}|_{A\Gamma}$ and both are tangent to $A\Gamma$. It follows that

$$E_R(\lambda) = \pi \circ e^{\vec{H}_R}(\lambda) = \pi \circ e^{\vec{H}}(\lambda) = E(\lambda), \qquad \forall\, \lambda \in A\Gamma.$$

For K-contact structures, the next results shows that there cannot be overtwisted disks within the injectivity radius from a Reeb orbit.

Theorem 6.3. *Let (M, ω, g) be a complete K-contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\mathrm{inj}}(\Gamma) > 0$. Then*

$$r_{\mathrm{tight}}(\Gamma) = r_{\mathrm{inj}}(\Gamma). \tag{56}$$

Proof. Let $q \in \Gamma$ and let $f_1(q), f_2(q) \in T_q M$ be an orthonormal basis for Δ_q. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a parametrization of Γ near q, namely $\gamma(z) = e^{z f_0}(q)$. Since M is K-contact, the frame

$$f_i(\gamma(z)) = e_*^{z f_0} f_i(q), \qquad i = 1, 2,$$

is an orthonormal frame for the sub-Riemannian metric, along $\gamma(-\varepsilon, \varepsilon)$. Consequently the dual frame ν_1, ν_2 satisfies

$$e^{z \vec{h}_0}(\nu_i|_q) = \left(e^{-z f_0}\right)^*(\nu_i|_q) = \nu_i|_{\gamma(z)}.$$

Let $(r \cos \theta, r \sin \theta, z)$ be cylindrical coordinates on $A\Gamma$ as in Section 3.2, built with the frame f_0, f_1, f_2, so that the generic $\lambda \in A^1 \Gamma$ reads

$$\lambda = (\cos \theta \nu_1 + \sin \theta \nu_2)|_{\gamma(z)} =: \nu_{\theta, z}.$$

Since M is K-contact, then $\{H, h_0\} = 0$, or equivalently $[\vec{H}, \vec{h}_0] = 0$. It follows that

$$\pi \circ e^{r \vec{H}}(\nu_{\theta, z}) = \pi \circ e^{r \vec{H}} \circ e^{z \vec{h}_0}(\nu_{\theta, 0}) = \pi \circ e^{z \vec{h}_0} \circ e^{r \vec{H}}(\nu_{\theta, 0}),$$

so that for $\partial_z \in A^1 \Gamma$ it holds $\omega_r(\partial_z) = \omega(\pi_* \circ e_*^{r \vec{H}} \partial_z) = \omega(f_0) = 1$ (cf. Definition 5.8).

Let $\lambda \in A^1 \Gamma$. By Theorem 5.12, the contact Jacobi curve is an immersion when restricted to the interval $(0, r_{\mathrm{foc}}(\lambda))$, hence

$$\Omega_\lambda : [0, r_{\mathrm{foc}}(\lambda)) \to P(T_\lambda^*(A^1 \Gamma)) \simeq \mathbb{S}^1$$

cannot be surjective: since $\omega_r(\partial_z) = 1$, we have

$$\Omega_\lambda(r) \neq \{\eta \in T_\lambda^* A^1 \Gamma \mid \eta(\partial_z) = 0\} \in P(T_\lambda^* A^1 \Gamma), \qquad \forall \, r \in [0, r_{\mathrm{foc}}(\lambda)).$$

It follows that $r_o(\lambda) > r_{\mathrm{foc}}(\lambda) \geq r_{\mathrm{inj}}(\Gamma)$. Using Theorem 5.16, we obtain (56). $\qquad \square$

We need the following comparison theory for focal radii from the Reeb orbit.

Lemma 6.4. *Let (M, ω, g) be a complete K-contact sub-Riemannian manifold and let Γ be an embedded piece of a Reeb orbit with $r_{\mathrm{inj}}(\Gamma) > 0$. Assume that the sub-Riemannian metric invariant $\kappa \leq \kappa_+$ for some $\kappa_+ \in \mathbb{R}$. Then for all $\lambda \in A^1 \Gamma$ it holds*

$$r_{\mathrm{foc}}(\lambda) \geq \begin{cases} \frac{\pi}{\sqrt{\kappa_+}} & \kappa_+ > 0, \\ +\infty & \kappa_+ \leq 0. \end{cases}$$

Similarly, with reversed inequality, if $\kappa \geq \kappa_-$ for some $\kappa_- \in \mathbb{R}$.

In particular, if $\kappa \leq 0$, then the tubular neighbourhood map $E : A\Gamma \to M$ is an immersion.

Proof. By Remark 6.2, the sub-Riemannian Hamiltonian flow from $A\Gamma$ coincides with the one of the Riemannian extension. In other words, $E : A\Gamma \to M$ coincides with the Riemannian normal exponential map from Γ, by identifying the normal bundle $N\Gamma$ with $A\Gamma$ through the Riemannian metric. With these identifications, $E_* \partial_z = f_0$. Consider then the geodesic $t \mapsto E(t\lambda)$. It follows that r is a focal radius for $\lambda \in A^1 \Gamma$ if and only if r is a conjugate radius for the Riemannian exponential map for the Riemannian manifold obtained by taking the quotient by the Reeb flow of a small neighbourhood of the geodesic $t \mapsto E(t\lambda)$. Such a quotient has Gauss curvature equal to the sub-Riemannian invariant κ (see Remark 2.8). By classical comparison theory for conjugate points (see e.g. [dC92, Prop. 2.4]) it follows that if $\kappa_- \leq \kappa \leq \kappa_+$ we have

$$\frac{\pi}{\sqrt{\kappa_+}} \leq r_{\mathrm{foc}}(\lambda) \leq \frac{\pi}{\sqrt{\kappa_-}},$$

with the convention that the bounds are $+\infty$ if $\kappa_\pm \leq 0$. $\square$

We can now prove the main result of this section: a Hadamard-type theorem for K-structures.

Theorem 6.5. *Let (M, ω, g) be complete and simply connected K-contact sub-Riemannian manifold. If $\kappa \leq 0$ then $(M, \Delta = \ker \omega)$ is contactomorphic to the standard contact structure (i.e., the Heisenberg group of Example 2.2).*

Proof. Our first claim is that the Reeb field f_0 is complete. Let $q \in M$ and $\gamma : I \to M$, $\gamma(z) = e^{zf_0}(q)$, be the maximal integral curve of f_0 going through q. If every Reeb orbit is periodic then there is nothing to prove. Assume by contradiction that $I \subsetneq \mathbb{R}$ is an open interval and that $\gamma : I \to M$ is not periodic. Without loss of generality assume $I = (0, 2)$. Then $z \in I$ can be written as $z = k + \delta$, with $k \in \{0, 1\}$ and $\delta \in (0, 1)$. By the triangle inequality and the fact that the Reeb flow is an isometry we have for all $z \in I$:

$$d_{SR}(q, \gamma(z)) \leq d_{SR}(q, e^{kf_0}(q)) + d_{SR}(e^{kf_0}(q), e^{(k+\delta)f_0}(q))$$
$$= d_{SR}(q, e^{f_0}(q)) + d_{SR}(q, e^{\delta f_0}(q))$$
$$\leq 2 \max_{|\tau| \leq 1} d_{SR}(q, e^{\tau f_0}(q)).$$

Since d_{SR} is complete, the trajectory $I \ni z \mapsto e^{zf_0}(q)$ is contained in a pre-compact set and thus γ can be extended on an interval larger than I, proving the completeness of f_0.

Fix then a maximal piece of Reeb orbit Γ (either $\Gamma \simeq \mathbb{S}^1$ or $\Gamma \simeq \mathbb{R}$). Since $\kappa \leq 0$, according to Lemma 6.4, $E : A\Gamma \to M$ is an immersion, thus the triple

$$(A\Gamma, E^*\omega, E^*g) \tag{57}$$

is a contact sub-Riemannian manifold. We show that (57) is a complete sub-Riemannian manifold. First, we claim that $(A\Gamma, E^*g)$ is a complete Riemannian manifold. Let $d_R : A\Gamma \times A\Gamma \to \mathbb{R}$ be the Riemannian distance function determined by E_R. Observe that the zero section $\Gamma_0 \subset A\Gamma$ is an embedded sub-manifold. Moreover $(\Gamma_0, d_R|_{\Gamma_0})$ has transitive isometry group, namely the restriction of the flow of the Reeb field of (57), which is $E_*^{-1} f_0$. Thus $(\Gamma_0, d_R|_{\Gamma_0})$ is a locally compact metric space with transitive isometry group and hence it is complete. We need the following generalization of classical Riemannian Hopf-Rinow theorem.

Theorem 6.6 (Hopf-Rinow from a submanifold)**.** *Let M be a Riemannian manifold, with distance function d_R. Let $S \subset M$ be an embedded sub-manifold (without boundary) and assume that $(S, d_R|_S)$ is a complete metric space. Then the following are equivalent:*

- *(M, d_R) is complete;*
- *The Riemannian tubular neighbourhood map $E_R : AS \to M$ is well-defined.*

The classical Hopf-Rinow corresponds to the case $S = \{pt\}$. The proof of Theorem 6.6 is analogous to the standard one [dC92, Thm. 2.8, Sec. 7.2], replacing the normal neighbourhood of a point with a normal tubular neighbourhood around S.

Since the curves $r \mapsto E(r\lambda)$ for $\lambda \in A^1\Gamma$ are sub-Riemannian geodesics from $\Gamma \subset M$, then radial lines $r \mapsto r\lambda$ on $A\Gamma$ are sub-Riemannian geodesics from $\Gamma_0 \subset A\Gamma$. Then according to Remark 6.2, they are also Riemannian geodesics normal to Γ_0. It follows that the Riemannian tubular neighbourhood map of (57) from Γ_0 is well-defined. Theorem Theorem 6.6 then implies that $(A\Gamma, d_R)$ is a complete Riemannian manifold. Moreover denoting with $d_{SR}^{A\Gamma} : A\Gamma \times A\Gamma \to \mathbb{R}$ the sub-Riemannian distance of (57), we have $d_R \leq d_{SR}^{A\Gamma}$. Therefore $(A\Gamma, d_{SR}^{A\Gamma})$ is a complete metric space as well. Consequently, according to Theorem 2.13, the sub-Riemannian exponential map $\exp^{A\Gamma} : T^*(A\Gamma) \to A\Gamma$ is well-defined.

Being a local isometry, $E : A\Gamma \to M$ sends geodesics to geodesics. Thus for a fixed $\lambda \in A\Gamma$ we have the equality

$$E \circ \exp_\lambda^{A\Gamma} \circ E^* = \exp_{E(\lambda)}.$$

Since M is complete and there are no non-trivial abnormal length-minimizers the exponential map is surjective [ABB20, Prop. 8.38]). We deduce that $E : A\Gamma \to M$ is surjective as well. Now we need the following result from Riemannian geometry [dC92, Lemma 3.3], that holds in the sub-Riemannian case as well.

Lemma 6.7. *Let $f : M \to N$ be a surjective local diffeomorphism of complete sub-Riemannian manifolds satisfying $\|f_* v\| \geq \|v\|$ for every $v \in TM$, then f is a covering map.*

To conclude the proof we observe that $E : A\Gamma \to M$ satisfies the hypothesis Lemma 6.7, thus it is a covering map. Since M is simply connected, E is a diffeomorphism. (In particular, as a consequence, we note that Γ cannot be periodic.) Thus $r_{\mathrm{inj}}(\Gamma) = +\infty$ and according to Theorem 6.3 also $r_{\mathrm{tight}}(\Gamma) = +\infty$. From the explicit expression of $E^*\omega$

$$E^*\omega = \omega_r(\partial_\theta)d\theta + \omega_r(\partial_z)dz = dz + \omega_r(\partial_\theta)d\theta.$$

We deduce that the map

$$\Phi : \mathbb{R}^3 \to A\Gamma \to M, \qquad \Phi(z, x, y) = E\left(\frac{\sqrt{2\omega_r(\partial_\theta)}}{r}x, \frac{\sqrt{2\omega_r(\partial_\theta)}}{r}y, z \right),$$

is a diffeomorphism, satisfying $\Phi^*\omega = \omega_{\mathrm{st}}$. $\qquad\square$

Corollary 6.8. *Let (M, ω, g) be complete K-contact sub-Riemannian manifold. If $\kappa \leq 0$ then any closed orbit of the Reeb field is the generator of an infinite cyclic subgroup of the fundamental group $\pi_1(M)$.*

Proof. Let $\gamma : \mathbb{S}^1 \to M$ be a closed integral curve of the Reeb field. Then, according to the proof of Theorem 6.5, the map $E : A\Gamma \simeq \mathbb{S}^1 \times \mathbb{R}^2 \to M$ is a covering which maps the generator of $\pi_1(A\Gamma)$, which is $[z \mapsto (z, 0, 0)] \in \pi_1(A\Gamma)$, to $[\gamma] \in \pi_1(M)$. $\qquad\square$

6.1. **Examples with prescribed non-positive curvature.** Next we would like to show that there are indeed many examples of sub-Riemannian manifolds with vanishing χ and non positive curvature κ, to which Theorem 6.5 can be applied, see Corollary 6.11.

Theorem 6.9. *Let (B, η) be an orientable Riemannian surface of curvature κ_η. Assume that the area form Ω of g defines an integral cohomology class (which can always be achieved by constant rescaling of the metric). Then there exists a principal circle bundle $\pi : M \to B$, with connection $\omega \in \Omega^1(M)$, $d\omega = \pi^*\Omega$. Moreover the following triple $(M, \omega, g = \pi^*\eta + \omega \otimes \omega)$ defines a sub-Riemannian structure having invariants $\chi = 0, \kappa = \pi^*\kappa_\eta$.*

Proof. We make use of the following result from [Kob56], as stated in [BW58, Thm. 3].

Theorem 6.10. *Let (B, Ω) be a 2-dimensional symplectic manifold and assume that $[\Omega] \in H^2(B, \mathbb{Z})$. Then there is a principal circle bundle $\pi : M \to B$ over B with connection ω satisfying $d\omega = \pi^*\Omega$. In particular ω determines a contact structure and its Reeb field f_0 generates the right translations of the bundle by $\mathbb{S}^1$.*

Let $\pi : (M, \omega) \to B$ be the principal $\mathbb{S}^1$-bundle with contact connection ω, obtained applying Theorem 6.10 to the symplectic manifold (B, Ω). Observe that the quadratic form $g = \pi^*\eta + \omega \otimes \omega$ is positive definite over M, therefore the triple (M, ω, g) defines a sub-Riemannian structure. By construction f_0 generates a one-parameter group of isometries so that $\chi = 0$. The corresponding orbit space is isometric to (B, η), thus $\kappa = \pi^*\kappa_\eta$ by Remark 2.8. $\qquad\square$

By applying Theorem 6.5 to the universal cover of (M, ω, g) equipped with the pull-back sub-contact Riemannian structure, we obtain the following consequence.

Corollary 6.11. *The contact structures in Theorem 6.9, with $\kappa_\eta \leq 0$, are universally tight.*

6.2. **Examples with positive curvature.** In the proof of Theorem 6.5, the assumption $\kappa \leq 0$ can be replaced by asking that $E : A\Gamma \to M$ has no critical points (this fact is implied by $\kappa \leq 0$ through Theorem 6.3). Arguing as in the proof of the latter, it is sufficient that there exists a Reeb orbit $e^{zf_0}(q)$ such that the Riemannian manifold obtained by taking the quotient by the action of the Reeb flow (locally along any sub-Riemannian geodesic $E : A\Gamma \to M$) has no conjugate points along (Riemannian) geodesics starting from the corresponding point in the quotient. The curvature of the quotient, in this case, can also be positive.

We give an example. Let (N, h) be a complete, orientable, non-compact Riemannian surface. There exists $\alpha \in \Lambda^1 N$ such that $d\alpha = \mathrm{vol}_h$ is the volume form of N. Let then $M = N \times \mathbb{R}$, with projection $p(q, z) = q$. Define

$$\omega := dz + p^*\alpha, \qquad g = p^*h + \omega \otimes \omega.$$

One can check that $\omega \wedge d\omega = \mathrm{vol}_g$ so that (M, ω, g) is a contact sub-Riemannian manifold. The Reeb vector field is $f_0 = \partial_z$, which is clearly a generator of isometries for (M, ω, g). Thus, the sub-Riemannian invariants are

$$\chi = 0, \qquad \kappa = p^* \kappa_h,$$

where κ_h is the Riemann curvature of (N, η). Assume now that (N, η) has a *pole*, namely, there exists a point q such that any geodesic emanating from q has no conjugate points along it. Let Γ the complete Reeb orbit passing through from $(q, 0) \in M$. Then the construction in the proof of Theorem 6.3 shows that the map $E : A\Gamma \to M$ is an immersion and thus also the proof of Theorem 6.5 proceeds unchanged. As an example, one can choose as (N, h) the Riemannian surface given by $N = \{(x, y, w) \in \mathbb{R}^3 \mid w = x^2 + y^2\}$ with the induced metric from $\mathbb{R}^3$. The origin is a pole in the above sense, and the curvature κ_h is positive.

Appendix A. Lie derivatives of contact Jacobi curves

We prove here some technical facts needed in the proof of Theorem 5.12.

Lemma A.1. *Let ω_r be as in Definition 5.8, and let $(r\cos\theta, r\sin\theta, z)$ be cylindrical coordinates on $A\Gamma$. Then the following hold:*

$$\omega_0(\partial_\theta) = 0, \qquad\qquad \ddot{\omega}_0(\partial_\theta) = 1, \qquad\qquad \omega_0(\partial_z) = 1,$$
$$\dot{\omega}_0(\partial_\theta) = 0, \qquad\qquad \dddot{\omega}_0(\partial_\theta) = 0, \qquad\qquad \dot{\omega}_0(\partial_z) = 0.$$

Proof. Remember that $\omega_r = \pi \circ e^{r\vec{H}}|_{A^1\Gamma}$. We denote with the same symbol the un-restricted form on T^*M. To prove the statement we compute Lie derivatives of $\pi^*\omega$. Note that

$$\mathscr{L}_{\vec{H}}(\pi^*\omega) = \left(i_{\vec{H}} \circ d + d \circ i_{\vec{H}}\right)(\pi^*\omega) = (i_{\vec{H}} \circ d)(\pi^*\omega),$$

and similarly higher order derivatives w.r.t. r are higher order Lie derivatives:

$$\mathscr{L}_{\vec{H}}^n(\pi^*\omega) = \underbrace{(i_{\vec{H}} \circ d) \circ \cdots \circ (i_{\vec{H}} \circ d)}_{n \text{ times}}(\pi^*\omega).$$

Let f_0, f_1, f_2 be a local orthonormal frame, with dual frame $\nu_0 = \omega, \nu_1, \nu_2$. Let also $h_0, h_1, h_2 : T^*M \to \mathbb{R}$ the linear-on-fiber functions corresponding to the vector fields f_0, f_1, f_2. We adopt the following convention: free latin indices are understood to range from 1 to 2, while greek ones from 0 to 2. Repeated indices are summed over their range. For example, $[f_\alpha, f_\beta] = c_{\alpha\beta}^\gamma f_\gamma$, where $c_{\alpha\beta}^\gamma \in C^\infty(M)$ are the structure coefficients (see Section 2.2).

Recall that since $2H = h_i h_i$ then $\vec{H} = h_i \vec{h}_i$. Furthermore $\vec{H}(h_\mu) = \{H, h_\mu\} = h_\ell c^\alpha_{\ell\mu} h_\alpha$. We start with a few preliminary and straightforward computations:

$$\mathscr{L}_{\vec{H}}\omega = \mathscr{L}_{\vec{H}}\nu_0 = h_\ell c^0_{\ell j}\nu_j, \tag{58}$$

$$\mathscr{L}_{\vec{H}}\nu_j = dh_j + h_i c^j_{i\alpha}\nu_\alpha,$$

$$\mathscr{L}_{\vec{H}}(dh_j) = h_\ell c^\alpha_{j\ell}dh_\alpha + h_\alpha c^\alpha_{j\ell}dh_\ell + h_\ell h_\alpha dc^\alpha_{j\ell},$$

where the pull-back of forms is omitted from the notation, for brevity. More precisely, we adopted the shorthand $\pi^*\nu_\alpha = \nu_\alpha$ and similarly $df = d\pi^* f$ for $f \in C^\infty(M)$, so that the above formulas are understood as identities between forms on T^*M. For the second derivative we have

$$\begin{aligned}
\mathscr{L}^2_{\vec{H}}\omega &= \mathscr{L}_{\vec{H}}\left(h_i c^0_{ij}\nu_j\right) \\
&= \vec{H}(h_i c^0_{ij})\nu_j + h_i c^0_{ij}(\mathscr{L}_{\vec{H}}\nu_j) \\
&= \vec{H}(h_i c^0_{ij})\nu_j + h_i c^0_{ij}\left(dh_j + h_\ell c^j_{\ell\alpha}\nu_\alpha\right) \\
&= \vec{H}(h_i c^0_{ij})\nu_j + h_i c^0_{ij}dh_j + h_i h_\ell c^0_{ij}c^j_{\ell\alpha}\nu_\alpha.
\end{aligned} \tag{59}$$

For the third Lie derivative we take one Lie derivative of (59) and apply Leibniz rule:

$$\begin{aligned}
\mathscr{L}^3_{\vec{H}}\omega &= \vec{H}(h_i c^0_{ij})(\mathscr{L}_{\vec{H}}\nu_j) + \vec{H}(h_i c^0_{ij})dh_j + h_i c^0_{ij}(\mathscr{L}_{\vec{H}}dh_j) + h_i h_\ell c^0_{ij}c^j_{\ell\alpha}(\mathscr{L}_{\vec{H}}\nu_\alpha) + X \\
&= \vec{H}(h_i c^0_{ij})dh_j + \vec{H}(h_i c^0_{ij})dh_j + h_i c^0_{ij}\left(h_\ell c^\alpha_{j\ell}dh_\alpha + h_\alpha c^\alpha_{j\ell}dh_\ell\right) + h_i h_\ell c^0_{ij}c^j_{\ell k}dh_k + X \\
&= 2\vec{H}(h_i c^0_{ij})dh_j + h_i c^0_{ij}\left(h_\ell c^\alpha_{j\ell}dh_\alpha + h_\alpha c^\alpha_{j\ell}dh_\ell + h_\ell c^j_{\ell k}dh_k\right) + X \\
&= 2\vec{H}(h_i c^0_{ij})dh_j - 2H dh_0 + h_i c^0_{ij}\left(h_\ell c^k_{j\ell}dh_k + h_\alpha c^\alpha_{j\ell}dh_\ell + h_\ell c^j_{\ell k}dh_k\right) + X.
\end{aligned} \tag{60}$$

Here, X denotes a linear combination of $\nu_\alpha = \pi^*\nu_\alpha$ and $df = \pi^*df$ for $f \in C^\infty(M)$, since these terms will give no contribution for the relevant computations.

Now consider the vector fields $\partial_z, \partial_\theta$ on $A\Gamma$, in terms of the cylindrical coordinates of Section 3.2. In the notation of this appendix, one can see that

$$\pi_*\partial_\theta = 0, \qquad \pi_*\partial_z = f_0. \tag{61}$$

We also note that, in the notation of this appendix

$$\partial_\theta = h_1\partial_{h_2} - h_2\partial_{h_1} = h_i c^0_{ij}\partial_{h_j}. \tag{62}$$

Recall that $\omega_r = \pi \circ e^{r\vec{H}}|_{A^1\Gamma}$. Using (61) one obtains

$$\omega_0(\partial_\theta) = 0, \qquad \omega_0(\partial_z) = 1.$$

Furthermore, using (58) and (61) we obtain

$$\dot\omega_0(\partial_\theta) = 0, \qquad \dot\omega_0(\partial_z) = 0.$$

Finally, using (59), (61) and (62) one obtains

$$\ddot\omega_0(\partial_\theta) = h_i c^0_{ij}dh_j\left(h_\ell c^0_{\ell k}\partial_{h_k}\right) = h_i h_\ell c^0_{ij}c^0_{\ell j} = 2H,$$

where we used the fact that $c^0_{ij}c^0_{\ell j} = \delta_{i\ell}$. In particular $\ddot\omega_0(\partial_\theta) = 1$ when compute at $\lambda \in A^1\Gamma$.

We now compute the last term we need, $\dddot\omega_0(\partial_\theta)$. We use (60), (61) and (62). We obtain

$$\begin{aligned}
\dddot\omega_0(\partial_\theta) &= 2\vec{H}(h_i c^0_{ij})h_\ell c^0_{\ell j} + h_i c^0_{ij}\left(\overline{h_\ell c^k_{j\ell}h_s c^0_{sk}} + h_\alpha c^\alpha_{jk}h_\ell c^0_{\ell k} + \overline{h_\ell c^j_{\ell k}h_s c^0_{sk}}\right) \\
&= \vec{H}\left(h_i c^0_{ij}h_\ell c^0_{\ell j}\right) + \overline{h_\alpha (h_i c^0_{ij})(h_\ell c^0_{\ell k})c^\alpha_{jk}} = \vec{H}(2H) = 0.
\end{aligned}$$

This concludes the proof of the lemma. $\qquad\square$

Lemma A.2. *The one-forms ω_r and two-forms $\omega_r \wedge \dot{\omega}_r$ on $A^1\Gamma$ (see Definition 5.8) have finite order at all r. More precisely, for any $\lambda \in A^1\Gamma$ and $\bar{r} \geq 0$ the following hold:*

(i) *there exists $m \in \{0, 1, 2, 3\}$, and a non-zero one-form β on $A^1\Gamma$ such that*

$$\omega_r|_\lambda = (r - \bar{r})^m \beta|_\lambda + O\left((r - \bar{r})^{m+1}\right), \tag{63}$$

(ii) *there exist $n \in \{0, 1, 2, 3, 4\}$, and a non-zero two-form α on $A^1\Gamma$ such that*

$$\omega_r \wedge \dot{\omega}_r|_\lambda = (r - \bar{r})^n \alpha|_\lambda + O\left((r - \bar{r})^{n+1}\right),$$

where $O(r - \bar{r})$ denote smooth remainders.

Proof. We first show that $\mathscr{L}_{\vec{H}}^n(\pi^*\omega)$, for $n = 0, 1, 2, 3$, the tautological one-form τ, and dH, are a basis of one-forms on T^*M (out of the zero section), namely for any λ with $H(\lambda) \neq 0$ it holds

$$\text{span}\{\tau, dH, \mathscr{L}_{\vec{H}}^{(n)}(\pi^*\omega) \mid n = 0, 1, 2, 3\} = T_\lambda^*(T^*M). \tag{64}$$

We check (64) by inspection of the computations already done in the proof of Lemma A.1. Set $(Jh)_i = h_\ell c_{\ell i}^0$, and note that Jh and h are independent. We rewrite (58) as

$$\mathscr{L}_{\vec{H}}(\pi^*\omega) = (Jh)_\ell \nu_\ell.$$

Since $\tau = h_i \pi^* \nu_i + h_0 \pi^* \nu_0$, and $\omega = \nu_0$, the above implies

$$\text{span}\{\pi^*\omega, \mathscr{L}_{\vec{H}}(\pi^*\omega), \tau\} = \text{span}\{\pi^*\nu_0, \pi^*\nu_1, \pi^*\nu_2\}.$$

Furthermore we rewrite (59) as

$$\mathscr{L}_{\vec{H}}^2(\pi^*\omega) = (Jh)_j dh_j + \text{span}\{\pi^*\nu_0, \pi^*\nu_1, \pi^*\nu_2\}.$$

Since $dH = h_j dh_j$, the above implies

$$\text{span}\{\pi^*\omega, \mathscr{L}_{\vec{H}}(\pi^*\omega), \mathscr{L}_{\vec{H}}^2(\pi^*\omega), \tau, dH\} = \text{span}\{\pi^*\nu_0, \pi^*\nu_1, \pi^*\nu_2, dh_1, dh_2\}.$$

Finally, we rewrite (60) as

$$\mathscr{L}_{\vec{H}}^3(\pi^*\omega) = 2H dh_0 + \text{span}\{\pi^*\nu_0, \pi^*\nu_1, \pi^*\nu_2, dh_1, dh_2\}.$$

Since $\{\pi^*\nu_\alpha, dh_\alpha\}_{\alpha=0}^3$ is a basis of one-forms on T^*M, (64) follows.

Note that the sub-bundle of one-forms generated by dH, τ is stable under the action of the Hamiltonian flow. More precisely, since $\mathscr{L}_{\vec{H}}(dH) = 0$, $\mathscr{L}_{\vec{H}}(\tau) = dH$, it holds

$$e^{r\vec{H}*} dH = dH, \qquad e^{r\vec{H}*}\tau = \tau + r\, dH. \tag{65}$$

Proof of (i). Recall that $\omega_r = e^{r\vec{H}*} \circ \pi^*\omega|_{A^1\Gamma}$ (see Definition 5.8). Taking the pull-back with $e^{r\vec{H}*}$ of (64), using the stability of $\text{span}\{dH, \tau\}$, taking the restriction on $A^1\Gamma$, and noting that $\tau|_{A^1\Gamma} = dH|_{A^1\Gamma} = 0$. We conclude that

$$\text{span}\{\omega_r, \dot{\omega}_r, \ddot{\omega}_r, \dddot{\omega}_r\}|_\lambda = T_\lambda^*(A^1\Gamma), \qquad \forall \lambda \in A^1\Gamma, \quad \forall r \in \mathbb{R}. \tag{66}$$

In particular, at least one (actually, two) of the generators in (66) is non-zero, and (63) follows.

Proof of (ii). Let $B = \{\lambda \in T^*M \mid H(\lambda) \neq 0\}$ and E be the rank 4 vector bundle over B:

$$E = \ker\{dH, \tau\}|_B.$$

It follows from (65) that $e^{r\vec{H}*}E = E$. It is convenient to introduce the following one-parameter family of one-forms:

$$\chi_r := e^{r\vec{H}*}(\pi^*\omega), \qquad r \in \mathbb{R},$$

to distinguish them from the restriction $\omega_r = \chi_r|_{A^1\Gamma}$. We are interested in the evolution of the following one-parameter family of two-forms

$$\alpha_r := \chi_r \wedge \dot{\chi}_r, \qquad r \in \mathbb{R},$$

which (when restricted to E) is a global section of $\Lambda^2 E^*$. Let σ denote the restriction of the symplectic form of T^*M to E. Using (58) in the proof of Lemma A.2 one sees that $\alpha_0 \wedge \sigma = 0$. Consider the following vector bundle over B:

$$L = \{\eta \in \Lambda^2 E^* \mid \eta \wedge \sigma = 0\}.$$

Since σ is non-degenerate on E, and since E has rank 4, the condition $\eta \wedge \sigma = 0$ is one non-trivial linear equation. Thus L has rank 5. It follows from $e^{r\vec{H}*}E = E$ and $e^{r\vec{H}*}\sigma = \sigma$ that $e^{r\vec{H}*}L = L$. Consequently, since $\alpha_0 \in L$, then $\alpha_r = e^{r\vec{H}*}\alpha_0 \in L$ for all $r \in \mathbb{R}$. It follows that

$$\alpha_r^{(i)} := \frac{d^i}{dr^i}\alpha_r \in \Gamma^\infty(L), \qquad \forall\, r \in \mathbb{R}, \quad i \in \mathbb{N}.$$

We claim that $\alpha_r^{(i)}$, for $i = 0, \ldots, 4$, are linearly independent sections of L. To see this, by elementary computations using the Leibniz property of the Lie derivative one can show that:

$$
\begin{pmatrix} \alpha_r \\ \alpha_r^{(1)} \\ \alpha_r^{(2)} \\ \alpha_r^{(3)} \\ \alpha_r^{(4)} \end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
* & * & * & 0 & 2 & 0 \\
* & * & * & * & * & 2
\end{pmatrix}
\begin{pmatrix}
\chi_r \wedge \dot{\chi}_r \\
\chi_r \wedge \ddot{\chi}_r \\
\chi_r \wedge \dddot{\chi}_r \\
\dot{\chi}_r \wedge \ddot{\chi}_r \\
\dot{\chi}_r \wedge \dddot{\chi}_r \\
\ddot{\chi}_r \wedge \dddot{\chi}_r
\end{pmatrix},
\tag{67}
$$

where $*$ denotes a possibly non-zero entry. By (64), for any $r \in \mathbb{R}$, the one-forms $\{\chi_r^{(i)}\}_{i=0}^3$ are independent and, using also (65), when restricted to E yield a trivialization of E^*. Therefore, for any $r \in \mathbb{R}$, the two forms $\{\chi_r^{(i)} \wedge \chi_r^{(j)}\}_{0 \le i < j \le 3}$ are independent sections of $\Lambda^2 E^*$. The matrix in (67) has rank 5, and the claim follows.

As a consequence, since L has rank 5, for any $r \in \mathbb{R}$, the forms $\{\alpha_r^{(k)}\}_{k=0}^4$ constitute a trivializing basis for L. It follows that, for any $r \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$

$$\alpha_\varepsilon \in \operatorname{span}\left\{\alpha_r^{(k)} \,\middle|\, k = 0, \ldots, 4\right\}. \tag{68}$$

Note that, since $dH|_{A^1\Gamma} = \tau|_{A^1\Gamma} = 0$, $T(A^1\Gamma) \subset E$. Therefore one can take the restriction of (68) to $A^1\Gamma$. Recalling that, by definition, $\alpha_\varepsilon|_{A^1\Gamma} = \omega_\varepsilon \wedge \dot{\omega}_\varepsilon$, we obtain:

$$\omega_\varepsilon \wedge \dot{\omega}_\varepsilon \in \operatorname{span}\left\{\frac{d^k}{dr^k}\omega_r \wedge \dot{\omega}_r \,\middle|\, k = 0, \ldots, 4\right\}, \qquad \forall\, r \in \mathbb{R}.$$

We recall that by the contact condition (28), the left hand side of the above equation is non-zero for sufficiently small $\varepsilon > 0$, then at least one of the generators of the right hand side is non-zero. This concludes the proof. $\qquad\square$

Appendix B. A formula for the Schwarzian derivative of contact Jacobi curves

Let (M, ω, g) be a contact sub-Riemannian manifold and let $\Gamma \subset M$ be an embedded piece of Reeb orbit. We assume here that $\rho = r_{\mathrm{inj}}(\Gamma) > 0$, which is always the case if Γ has compact closure by Theorem 3.3. According to (8) the distance function δ is smooth on $E(A^{<\rho}\Gamma) \setminus \Gamma$, for $\rho = r_{\mathrm{inj}}(\Gamma)$. Therefore, in such region, we define its horizontal gradient, i.e., the unique horizontal vector field $\nabla\delta$ satisfying

$$d\delta(Y) = g(\nabla\delta, Y), \qquad \forall\, Y \in \Delta,$$

which is also called the *horizontal normal* to Γ, and hence denoted by $N = \nabla\delta$.

Definition B.1. *Let $N = \nabla\delta$ be the horizontal normal, and let $J : \Delta \to \Delta$ be the almost-complex structure* (3)*. The ordered triple $\{f_0, N, JN\}$, which is an oriented orthonormal frame defined on $E(A^\rho\Gamma) \setminus \Gamma$, is called* adapted frame *associated to Γ.*

The next proposition shows that the Schwarzian derivative of contact Jacobi curves can be expressed in terms of the adapted frame and the extended metric.

Proposition B.2. *Given $\lambda \in A^1\Gamma$ and $r \in (0, r_{\mathrm{inj}}(\Gamma))$ the Schwarzian derivative of the contact Jacobi curve* (25) *can be expressed in terms of the adapted frame Definition B.1 as*

$$\frac{1}{2}\mathcal{S}(\Omega_\lambda)(r) = g([N, f_0], JN) - \frac{1}{2}Ng([N, JN], JN) - \frac{1}{4}g([N, JN], JN)^2, \qquad (69)$$

where the right hand side is evaluated at $E(r\lambda)$.

To prove Proposition B.2 we write the adapted frame in cylindrical coordinates.

Proposition B.3. *Let (M, ω, g) be a complete contact sub-Riemannian structure. Let $\Gamma \subset M$ be an embedded piece of a Reeb orbit, with $\rho = r_{\mathrm{inj}}(\Gamma) > 0$. Let $E : A^{<\rho}\Gamma \to M$ be the tubular neighbourhood map. Let ω_r be the following smooth 1-parameter family of one-forms on $A^1\Gamma$:*

$$\omega_r := (\pi \circ e^{r\vec{H}})^*\omega|_{A^1\Gamma}, \qquad r \in \mathbb{R}. \qquad (70)$$

The adapted frame $\{f_0, N, JN\}$ is oriented and orthonormal, and in cylindrical coordinates at $\lambda = (z, r\cos\theta, r\sin\theta)$ it holds

$$E_*^{-1}N = \partial_r, \qquad (71)$$

$$E_*^{-1}JN = \frac{1}{a}\left(\omega_r(\partial_z)\partial_\theta - \omega_r(\partial_\theta)\partial_z\right), \qquad (72)$$

$$E_*^{-1}f_0 = \frac{1}{a}\left(\dot\omega_r(\partial_\theta)\partial_z - \dot\omega_r(\partial_z)\partial_\theta + d\omega_r(\partial_\theta, \partial_z)\partial_r\right), \qquad (73)$$

where $a := \omega_r \wedge \dot\omega_r(\partial_z, \partial_\theta)$, the dot denoting the derivative w.r.t. r.

Remark B.4. Since ω_r is a one-parameter family of one-forms on $A^1\Gamma$ one must be careful in evaluation of Eqs. (71) to (73). For clarity, e.g. (72) evaluated at a point $r\lambda \in A\Gamma$ for some $\lambda \in A^1\Gamma$, it corresponds to

$$E_*^{-1}JN|_{r\lambda} = \frac{1}{a}\left(\omega_r(\partial_z|_\lambda)\partial_\theta|_{r\lambda} - \omega_r(\partial_\theta|_\lambda)\partial_z|_{r\lambda}\right), \qquad a = \omega_r \wedge \dot\omega_r(\partial_z|_\lambda, \partial_\theta|_\lambda).$$

Proof. The fact that the frame $\{f_0, N, JN\}$ is smooth, orthonormal and oriented on $E(A^\rho\Gamma) \setminus \Gamma$ follows immediately from the definitions. Thus we only have to compute it in a set of cylindrical coordinates. It follows from the homogeneity property (7) that in cylindrical coordinates

$$E(r\cos\theta, r\sin\theta, z) = \pi \circ e^{r\vec{H}}(\cos\theta\nu_1 + \sin\theta\nu_2)|_{\gamma(z)}.$$

Let $r\lambda \in A\Gamma$ for some $\lambda = (z, \cos\theta, \sin\theta) \in A^1\Gamma$. Let $\lambda_r = e^{r\vec{H}}(\lambda)$. As explained in [ACEAG98, Lemma 2.3], it follows from the Pontryagin maximum principle that

$$\langle\lambda_r, E_*\partial_\theta\rangle = \langle\lambda_r, E_*\partial_z\rangle = 0, \qquad \langle\lambda_r, E_*\partial_r\rangle = 1,$$

and thus $E^*\lambda_r = dr$.

The vector field $E_*\partial_r = \pi_*\vec{H}$ is horizontal and has norm one, hence $E_*\partial_r, JE_*\partial_r$ is an horizontal oriented orthonormal frame. Therefore using this frame in (6) we obtain

$$E_*\partial_r = \pi_*\vec{H} = \langle\lambda_r, E_*\partial_r\rangle E_*\partial_r + \langle\lambda_r, JE_*\partial_r\rangle JE_*\partial_r.$$

It follows that $\langle \lambda_r, JE_*\partial_r \rangle = 0$.

We compute the pull-back of the contact form. Using the definition (70) of ω_r we obtain

$$
\begin{aligned}
(E^*\omega)|_{r\lambda} &= \omega(E_*\partial_r|_{r\lambda})dr + \omega(E_*\partial_\theta|_{r\lambda})d\theta + \omega(E_*\partial_z|_{r\lambda})dz \\
&= \omega\left(\pi_*\vec{H}|_{e^{\vec{H}}(r\lambda)}\right)dr + \omega\left((\pi \circ e^{r\vec{H}})_*\partial_\theta|_\lambda\right)d\theta + \omega\left((\pi \circ e^{r\vec{H}})_*\partial_z|_\lambda\right)dz \\
&= \omega_r(\partial_\theta|_\lambda)d\theta + \omega_r(\partial_z|_\lambda)dz.
\end{aligned}
\tag{74}
$$

Thus, the equalities $E^*\lambda_r = dr$, $\langle \lambda_r, JE_*\partial_r \rangle = 0$ and $\omega(JE_*\partial_r) = 0$ together imply

$$
\tilde{J}\partial_r := E_*^{-1}JE_*\partial_r \propto \omega_r(\partial_z)\partial_\theta - \omega_r(\partial_\theta)\partial_z.
$$

The factor of proportionality is found imposing $E^*d\omega(\partial_r, \tilde{J}\partial_r) = 1$, and since

$$
E^*d\omega|_{r\lambda} = dr \wedge \left(\dot{\omega}_r(\partial_\theta|_\lambda)d\theta + \dot{\omega}_r(\partial_z|_\lambda)dz\right) + d\omega_r(\partial_\theta|_\lambda, \partial_z|_\lambda)d\theta \wedge dz,
\tag{75}
$$

we find $\iota_{\partial_r}E^*d\omega = \dot{\omega}_r(\partial_\theta)d\theta + \dot{\omega}_r(\partial_z)dz$ (we omit evaluations for brevity) and consequently

$$
\tilde{J}\partial_r = \frac{1}{a}\left(\omega_r(\partial_z)\partial_\theta - \omega_r(\partial_\theta)\partial_z\right).
$$

To prove that $N = \nabla\delta$ satisfies $N = E_*\partial_r$, observe that according to (8) we have

$$
\delta(E(r\cos\theta, r\sin\theta, z)) = r.
$$

Thus for any horizontal vector field Y, since $\langle dr, \tilde{J}\partial_r \rangle = 0$, we have

$$
Y(\delta) = \langle dr, \left(g(Y, E_*\partial_r)\partial_r + g(Y, JE_*\partial_r)\tilde{J}\partial_r\right)\rangle = g(Y, E_*\partial_r).
$$

This yields (71) and (72). Finally from the coordinate presentations (74) for $E^*\omega$ and (75) for $E^*d\omega$, one can check that the expression of f_0 appearing in equation (73) satisfies the characterizing equations $\omega(f_0) = 1$, $d\omega(f_0, \cdot) = 0$. $\qquad\square$

Proof of Proposition Proposition B.2. To lighten notation, let us set

$$
A = g([N, JN], JN), \quad \dot{A} = N(A), \quad B = g([N, f_0], JN).
$$

An elementary computation exploiting the expressions (71)-(72)-(73) shows

$$
A = \frac{1}{a}\omega_r \wedge \ddot{\omega}_r(\partial_\theta, \partial_z), \quad B = -\frac{1}{a}\dot{\omega}_r \wedge \ddot{\omega}_r(\partial_\theta, \partial_z),
$$

where $a = \dot{\omega}_r \wedge \omega_r(\partial_\theta, \partial_z)$. We want to prove formula (69), which is

$$
\mathcal{S}(\Omega_\lambda) = S := B - \frac{\dot{A}}{2} - \frac{A^2}{4}.
$$

Let $f : (0, \rho) \to (0, +\infty)$ be a positive function, define $\bar{\omega}_r = f\omega_r$ as well as the corresponding barred quantities

$$
\bar{A} = \frac{1}{\bar{a}}\bar{\omega}_r \wedge \ddot{\bar{\omega}}_r(\partial_\theta, \partial_z), \quad \bar{B} = -\frac{1}{\bar{a}}\dot{\bar{\omega}}_r \wedge \ddot{\bar{\omega}}_r(\partial_\theta, \partial_z),
$$

where $\bar{a} = \dot{\bar{\omega}}_r \wedge \bar{\omega}_r(\partial_\theta, \partial_z)$. We claim that $\bar{S} = S$. Indeed, computing shows the relations

$$
\bar{A} = A - 2\frac{\dot{f}}{f}, \quad \bar{B} = B - \frac{\dot{f}}{f}\bar{A} - \frac{\ddot{f}}{f}.
$$

Therefore we can compute

$$\bar{S} = \bar{B} - \frac{\dot{\bar{A}}}{2} - \frac{\bar{A}^2}{4} = \left(B - \frac{\dot{f}}{f}\bar{A} - \frac{\ddot{f}}{f} \right) - \frac{1}{2}\frac{d}{dr}\left(A - 2\frac{\dot{f}}{f} \right) - \frac{1}{4}\left(A - 2\frac{\dot{f}}{f} \right)^2$$

$$= S - \left(\frac{\dot{f}}{f}\bar{A} + \frac{\ddot{f}}{f} \right) + \frac{d}{dr}\frac{\dot{f}}{f} - \left(\left(\frac{\dot{f}}{f} \right)^2 - A\frac{\dot{f}}{f} \right)$$

$$= S - \frac{\dot{f}}{f}\bar{A} - 2\left(\frac{\dot{f}}{f} \right)^2 + A\frac{\dot{f}}{f} = S.$$

If we choose $f = 1/\omega_r(\partial_\theta)$, then, denoting $v = \omega_r(\partial_z)/\omega_r(\partial_\theta)$, we have

$$\bar{\omega}_r = v\,dz + d\theta, \quad \dot{\bar{\omega}}_r = \dot{v}\,dz, \quad \ddot{\bar{\omega}}_r = \ddot{v}\,dz,$$

consequently

$$\dot{\bar{\omega}}_r \wedge \bar{\omega}_r = -\dot{v}\,d\theta \wedge dz, \quad \bar{\omega}_r \wedge \ddot{\bar{\omega}}_r = \ddot{v}\,d\theta \wedge dz \quad \dot{\bar{\omega}}_r \wedge \ddot{\bar{\omega}}_r = 0.$$

We deduce that $\bar{A} = -\ddot{v}/\dot{v}$ and $\bar{B} = 0$, therefore

$$S = \bar{S} = -\frac{\dot{\bar{A}}}{2} - \frac{\bar{A}^2}{4} = \frac{1}{2}\left(\frac{\dddot{v}}{\dot{v}} - \left(\frac{\ddot{v}}{\dot{v}} \right)^2 \right) - \frac{1}{4}\left(\frac{\ddot{v}}{\dot{v}} \right)^2$$

$$= \frac{1}{2}\left(\frac{\dddot{v}}{\dot{v}} - \frac{3}{2}\left(\frac{\ddot{v}}{\dot{v}} \right)^2 \right) = \frac{1}{2}\mathcal{S}(\Omega_\lambda).$$

Formula (69) is thus proved. □

References

[ABB20]   A. Agrachev, D. Barilari, and U. Boscain. *A comprehensive introduction to sub-Riemannian geometry*, volume 181 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020. From the Hamiltonian viewpoint, With an appendix by Igor Zelenko.

[ABR17]   A. Agrachev, D. Barilari, and L. Rizzi. Sub-Riemannian curvature in contact geometry. *J. Geom. Anal.*, 27(1):366–408, 2017.

[ABR18]   A. Agrachev, D. Barilari, and L. Rizzi. Curvature: a variational approach. *Mem. Amer. Math. Soc.*, 256(1225):v+142, 2018.

[ACEAG98] A. Agrachev, E.-H. Chakir El-A., and J. P. Gauthier. Sub-Riemannian metrics on $\mathbf{R}^3$. In *Geometric control and non-holonomic mechanics (Mexico City, 1996)*, volume 25 of *CMS Conf. Proc.*, pages 29–78. Amer. Math. Soc., Providence, RI, 1998.

[AG97]   A. Agrachev and R. V. Gamkrelidze. Feedback-invariant optimal control theory and differential geometry. I. Regular extremals. *J. Dynam. Control Systems*, 3(3):343–389, 1997.

[AL14]   A. Agrachev and P. W. Y. Lee. Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds. *Math. Ann.*, 360(1-2):209–253, 2014.

[AL15]   A. Agrachev and P. W. Y. Lee. Bishop and Laplacian comparison theorems on three-dimensional contact sub-Riemannian manifolds with symmetry. *J. Geom. Anal.*, 25(1):512–535, 2015.

[AZ02a]   A. Agrachev and I. Zelenko. Geometry of Jacobi curves. I. *J. Dynam. Control Systems*, 8(1):93–140, 2002.

[AZ02b]   A. Agrachev and I. Zelenko. Geometry of Jacobi curves. II. *J. Dynam. Control Systems*, 8(2):167–215, 2002.

[BB23]   E. Bellini and U. Boscain. Surfaces of genus $g \geq 1$ in 3D contact sub-Riemannian manifolds. *ESAIM Control Optim. Calc. Var.*, 29:Paper No. 79, 8, 2023.

[BBC22]   D. Barilari, U. Boscain, and D. Cannarsa. On the induced geometry on surfaces in 3D contact sub-Riemannian manifolds. *ESAIM Control Optim. Calc. Var.*, 28:Paper No. 9, 28, 2022.

[Bel24]   E. Bellini. *Tightness of three-dimensional contact manifolds: a sub-Riemannian approach*. PhD thesis, Università Milano Bicocca, 2024.

[Ben83]   D. Bennequin. Entrelacements et équations de Pfaff. In *Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982)*, volume 107 of *Astérisque*, pages 87–161. Soc. Math. France, Paris, 1983.

[BR16]   D. Barilari and L. Rizzi. Comparison theorems for conjugate points in sub-Riemannian geometry. *ESAIM Control Optim. Calc. Var.*, 22(2):439–472, 2016.

[BR17]   D. Barilari and L. Rizzi. On Jacobi fields and a canonical connection in sub-Riemannian geometry. *Arch. Math. (Brno)*, 53(2):77–92, 2017.

[BR20]   D. Barilari and L. Rizzi. Bakry-Émery curvature and model spaces in sub-Riemannian geometry. *Math. Ann.*, 377(1-2):435–482, 2020.

[BW58]   W. M. Boothby and H. C. Wang. On contact manifolds. *Ann. of Math. (2)*, 68:721–734, 1958.

[CL55]   E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1955.

[dC92]   M. P. a. do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[EKM12]   J. B. Etnyre, R. Komendarczyk, and P. Massot. Tightness in contact metric 3-manifolds. *Invent. Math.*, 188(3):621–657, 2012.

[EKM16]   J. B. Etnyre, R. Komendarczyk, and P. Massot. Quantitative Darboux theorems in contact geometry. *Trans. Amer. Math. Soc.*, 368(11):7845–7881, 2016.

[Eli89]   Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98(3):623–637, 1989.

[FPR20]   V. Franceschi, D. Prandi, and L. Rizzi. On the essential self-adjointness of singular sub-Laplacians. *Potential Anal.*, 53(1):89–112, 2020.

[Gei08]   H. Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[Kob56]   S. Kobayashi. Principal fiber bundles with the 1-dimensional toroidal group. *Tohoku Mathematical Journal*, 8:29–45, 1956.

[LLZ16]   P. W. Y. Lee, C. Li, and I. Zelenko. Ricci curvature type lower bounds for sub-Riemannian structures on Sasakian manifolds. *Discrete Contin. Dyn. Syst.*, 36(1):303–321, 2016.

[LV11]   M. Lesch and B. Vertman. Regular singular Sturm-Liouville operators and their zeta-determinants. *J. Funct. Anal.*, 261(2):408–450, 2011.

[LZ11]   C. Li and I. Zelenko. Jacobi equations and comparison theorems for corank 1 sub-Riemannian structures with symmetries. *J. Geom. Phys.*, 61(4):781–807, 2011.

[Mas14]   P. Massot. Topological methods in 3-dimensional contact geometry. In *Contact and symplectic topology*, volume 26 of *Bolyai Soc. Math. Stud.*, pages 27–83. János Bolyai Math. Soc., Budapest, 2014.

[Nie06]   K. Niederkrüger. The plastikstufe—a generalization of the overtwisted disk to higher dimensions. *Algebr. Geom. Topol.*, 6:2473–2508, 2006.

[NP09]   K. Niederkrüger and F. Pasquotto. Resolution of symplectic cyclic orbifold singularities. *J. Symplectic Geom.*, 7(3):337–355, 2009.

[ZL09]   I. Zelenko and C. Li. Differential geometry of curves in Lagrange Grassmannians with given Young diagram. *Differential Geom. Appl.*, 27(6):723–742, 2009.