Spectrum of the Second Variation

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We consider a smooth control system of the form:

$$\dot{q} = f_u^t(q), \quad q \in M, \ u \in U, \tag{1}$$

with a fixed initial point $q_0 \in M$. Here M and U are smooth manifolds (without border) and the vector $f_u(q) \in T_qM$ smoothly depends on $(q,u) \in M \times U$ and is measurable bounded with respect to $t \in [0,1]$.

We denote by $\mathcal{U}$ the set of all admissible controls; then $\mathcal{U}$ is an open subset of $L^\infty([0,1];U)$. Hence $\mathcal{U}$ is a smooth Banach manifold modelled on the space $L^\infty\left([0,1];\mathbb{R}^{\dim U}\right)$.

Given $t \in [0,1]$ we define the "evaluation map" $F_t : \mathcal{U} \to M$ by the formula $F_t(u(\cdot)) = q(t;u(\cdot))$; then F_t is a smooth map from the Banach manifold $\mathcal{U}$ to M.

Let $\ell : M \times U \to \mathbb{R}$ be a smooth "Lagrangian". We consider functionals $\varphi_t : \mathcal{U} \to \mathbb{R},\ 0 \le t \le 1$, defined by the formula:

$$\varphi_t(u(\cdot)) = \int_0^t \ell(q(\tau; u(\cdot)), u(\tau))\, d\tau. \tag{2}$$

Definition 1. *We say that $u \in \mathcal{U}$ is a normal extremal control if there exists $\lambda_1 \in T^*_{F_1(u)}M$ such that $\lambda_1 D_u F_1 = d_u \varphi_1$; here $\lambda_1 D_u F_1$ is the composition of $D_u F_1 : T_u\mathcal{U} \to T_{F_1(u)}M$ and $\lambda_1 : T_{F_1(u)}M \to \mathbb{R}$. We say that a normal extremal control is strictly normal if it is a regular point of F_1.*

A family of Hamiltonians $h^t_u : T^*M\mathbb{R},\ u \in U$, is defined by the formula:

$$h^t_u(\lambda) = \langle \lambda, f^t_u(q) \rangle - \ell(q, u), \quad g \in M,\ \lambda \in T^*_q M.$$

Let σ be the canonical symplectic form on T^*M and $\pi : T^*M \to M$ be the standard projection, $\pi(T_q^*M) = q$. Recall that $\sigma = ds$ where s is a Louville (or tautological) 1-form, $\langle s_\lambda, \eta \rangle = \langle \lambda, \pi_* \eta \rangle$, $\forall\, \lambda \in T^*M,\ \eta \in T_\lambda(T^*M)$. Given a smooth function $h : T^*M \to \mathbb{R}$, the Hamiltonian vector field $\vec{h}$ on T^*M is defined by the identity: $dh = \sigma(\cdot, \vec{h})$.

Proposition 1. *Let $\tilde{u} \in \mathcal{U}$ and $\tilde{q}(t) = q(t; \tilde{u}),\ 0 \le t \le 1$; then $\tilde{u}$ is a normal extremal control if and only if there exists $\tilde{\lambda}_t \in T^*_{\tilde{q}(t)}M$ such that*

$$\dot{\tilde{\lambda}}_t = \vec{h}^t_{\tilde{u}(t)}, \quad \left. \frac{\partial h^t_u(\tilde{\lambda}_t)}{\partial u} \right|_{u=\tilde{u}(t)} = 0, \quad 0 \le t \le 1. \tag{4}$$

The time-varying Hamiltonian system $\dot{\lambda} = \vec{h}^t_{\tilde{u}(t)}(\lambda)$ defines a flow

$$\tilde{\Phi}^t : T^*M \to T^*M, \quad \tilde{\Phi}^t : \lambda(0) \mapsto \lambda(t),$$

where $\dot{\lambda}(\tau) = \vec{h}^t_{\tilde{u}(\tau)}(\lambda(\tau))$, $0 \le \tau \le t$. Obviously, $\tilde{\Phi}^t(\tilde{\lambda}_0) = \tilde{\lambda}_t$. Moreover, $\pi_* \vec{h}^t_{\tilde{u}(t)}(\lambda) = f^t_{\tilde{u}(t)}(q)$, $\forall\, q \in M$, $\lambda \in T^*_q M$.

Let $\tilde{P}^t : M \to M$ be the flow generated by the time-varying system $\dot{q} = f^t_{\tilde{u}(t)}(q)$, i. e. $\tilde{P}^t : q(0) \mapsto q(t)$, where $\dot{q}(\tau) = f^\tau_{\tilde{u}(\tau)}(q(\tau))$, $0 \le \tau \le t$. It follows that $\tilde{\Phi}^t$ are fiberwise transformations and $\tilde{\Phi}^t(T^*_q M) = T^*_{\tilde{P}^t(q)} M$.

Now we consider Hamiltonian functions

$$g_u^t = (h_u^t - h_{\tilde{u}(t)}^t) \circ \tilde{\Phi}^t, \quad u \in U,\ t \in [0,1].$$

A time-varying Hamiltonian vector field $\vec{g}_u^t$ generates the flow $(\tilde{\Phi}^t)^{-1} \circ \Phi_u^t$, where Φ_u^t is the flow generated by the field $\vec{h}_u^t$. We have:

$$g_{\tilde{u}(t)}^t \equiv 0, \quad \frac{\partial g_{\tilde{u}(t)}^t(\tilde{\lambda}_0)}{\partial u} = 0, \quad \frac{\partial^2 g_{\tilde{u}(t)}^t(\tilde{\lambda}_0)}{\partial^2 u} = \frac{\partial^2 h_{\tilde{u}(t)}^t(\tilde{\lambda}_t)}{\partial^2 u}.$$

We introduce a simplified notation $H_t \doteq \frac{\partial^2 h_{\tilde{u}(t)}^t(\tilde{\lambda}_t)}{\partial^2 u}$; recall that $H_t : T_{\tilde{u}(t)}U \to T_{\tilde{u}(t)}^*U$ is a self-adjoint linear map.

More notations. Consider the map $u \mapsto \vec{g}_u^t(\tilde{\lambda}_0)$ from U to $T_{\tilde{\lambda}_0}(T^*M)$ We denote by Z_t the differential of this map at $\tilde{u}(t)$, $Z_t \doteq \frac{\partial \vec{g}^t_{\tilde{u}(t)}(\tilde{\lambda}_0)}{\partial u}$; then Z_t is a linear map from $T_{\tilde{u}(t)}U$ to $T_{\tilde{\lambda}_0}(T^*M)$. We also set $X_t = \pi_* Z_t$; then X_t is a linear map from $T_{\tilde{u}(t)}U$ to $T_{\tilde{q}(t)}M$. Finally, we denote by $J : T_{\tilde{\lambda}_0}(T^*M) \to T^*_{\tilde{\lambda}_0}(T^*M)$ the anti-symmetric linear map defined by the identity $\sigma_{\tilde{\lambda}_0}(\cdot,\cdot) = \langle J\cdot,\cdot\rangle$.

Note that $T_{u(\cdot)}\mathcal{U}$ is the space of measurable bounded mappings $t \mapsto v(t) \in T_{u(t)}U,\ 0 \le t \le 1$. We have:

$$(D_{\tilde{u}}F_t)\, v = \tilde{P}^t_* \int_0^t X_\tau v(\tau)\, d\tau.$$

Now we introduce a "Gramm matrix", a self-adjoint linear map $\Gamma_t : T^*_{q_0}M \to T_{q_0}M$ defined by the formula:

$$\Gamma_t \doteq \int_0^t X_\tau X^*_\tau\, d\tau.$$

We see that $\tilde{u}$ is a regular point of F_t (i. e. $D_{\tilde{u}}F_t$ is surjective) if and only if Γ_t is invertible.

The Hessian of $\varphi_1|_{F_1^{-1}(\tilde{q}(1))}$ at $\tilde{u}$ is a quadratic form D^2 : ker $D_{\tilde{u}}F_1 \to \mathbb{R}$. This is the "second variation" of the optimal control problem at $\tilde{u}$, the main object of this paper. It has the following expression:

$$D^2_{\tilde{u}}\varphi_1(v) = -\int_0^1 \langle H_t v(t), v(t)\rangle \, dt - \int_0^1 \left\langle J \int_0^t Z_\tau v(\tau)\, d\tau, Z_t v(t) \right\rangle dt, \tag{5}$$

where $v \in T_{\tilde{u}}\mathcal{U}$ and $\int_0^1 X_t v(t)\, dt = 0$.

Definition 2. *The extremal $\tilde{\lambda}_t$, $0 \le t \le 1$, is regular if $\tilde{u}(\cdot)$ is smooth and H_t is invertible for any $t \in [0,1]$.*

Linear map $D_{\tilde{u}}F_1 : T_{\tilde{u}}\mathcal{U} \to T_{\tilde{q}(1)}M$ and quadratic form $D^2_{\tilde{u}}\varphi_1$ are continuous in the topology L^2. Let $\mathcal{V}$ be the closer of $\ker D_{\tilde{u}}F_1$ in the topology L^2. Then $\mathcal{V}$ is a Hilbert space equipped with a Hilbert structure

$$< v_1|v_2 > \doteq \int_0^1 \langle -H_t v_1(t), v_2(t)\rangle \, dt.$$

Formula (5) implies that

$$D^2_{\tilde{u}}\varphi_1(v) = < (I+K)v|v >, \quad v \in \mathcal{V}, \tag{6}$$

where K is a compact symmetric operator on $\mathcal{V}$. In particular, the spectrum of K is real, the only limiting point of the spectrum is 0, and any nonzero eigenvalue has a finite multiplicity.

Example 1. Let $M = U = \mathbb{R}^2$, $q = (q^1, q^2)$, $u = (u^1, u^2)$, $f_u(q) = u$, $\ell(q,u) = \frac{1}{2}|u|^2 + r(q^1u^2 - q^2u^1)$, $q_0 = 0$, $\tilde{u}(t) = 0$. Then

$$\mathcal{V} = \{v \in L^2([0,1];\mathbb{R}^2) : \int_0^1 v(t)\,dt = 0\}.$$

It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ as follows: $(v^1, v^2) = v^1 + iv^2$. A simple calculation gives the following expression for the operator K:

$$Kv(t) = \int_0^t 2riv(\tau)\,d\tau - \int_0^1 \int_0^t 2riv(\tau)\,d\tau dt.$$

The eigenfunctions of this operator have a form $t \mapsto ce^{2\pi nit}$, $0 \leq t \leq 1$, $c \in \mathbb{C}$, $n = \pm 1, \pm 2, \ldots$, where the eigenfunction $ce^{2\pi nit}$ corresponds to the eigenvalue $\frac{r}{\pi n}$.

We denote by $\bar{\zeta}_t$ is the sum of positive roots (counted according to multiplicity) of the equation

$$\det\left(\left\{\frac{\partial h^t_{\tilde{u}(t)}}{\partial u}, \frac{\partial h^t_{\tilde{u}(t)}}{\partial u}\right\}(\tilde{\lambda}_t) + si\frac{\partial^2 h^t_{\tilde{u}(t)}}{\partial u^2}(\tilde{\lambda}_t)\right) = 0$$

with unknown s.

Let $\mathsf{Sp}(K) \subset \mathbb{R}$ be the spectrum of the operator K, $\mathsf{Sp}(K)\setminus\{0\} = \mathsf{Sp}_+(K) \cup \mathsf{Sp}_-(K)$, where $\mathsf{Sp}_\pm(K) \subset \mathbb{R}_\pm$. If $\mathsf{Sp}_\pm(K)$ is an infinite set, then we introduce a natural ordering of $\mathsf{Sp}_\pm(K)$ that is a monotone decreasing sequence α_n, $n \in \mathbb{Z}_\pm$, with the following properties:

$$\bigcup_{n\in\mathbb{Z}_\pm} \{\alpha_n\} = \mathsf{Sp}_\pm(K), \quad \#\{n \in \mathbb{Z}_\pm : \alpha_n = \alpha\} = m_\alpha. \tag{7}$$

Theorem 1. *If $\bar{\zeta}_t \equiv 0$, then $\alpha_n = O(|n|^{-2})$ as $n \to \pm\infty$. If $\bar{\zeta}_t$ is not identical zero and H_t and Z_t are piecewise real analytic with respect to t then $\mathsf{Sp}_+(K)$ and $\mathsf{Sp}_-(K)$ are both infinite and*

$$\alpha_n = \frac{1}{\pi n}\int_0^1 \bar{\zeta}_t\, dt + O\left(|n|^{-5/3}\right) \quad \text{as} \quad n \to \pm\infty. \tag{8}$$

A cancellation of slow convergent to zero terms of the opposite sign in the expansion (8) gives the following:

Corollary 1. *The depending on $\varepsilon > 0$ families of real numbers*

$$\sum_{\substack{\alpha \in \mathsf{Sp}(K) \\ |\alpha| \geq \varepsilon}} m_\alpha \alpha, \qquad \prod_{\substack{\alpha \in \mathsf{Sp}(K) \\ |\alpha| \geq \varepsilon}} (1+\alpha)^{m_\alpha}$$

have finite limits as $\varepsilon \to 0$.

We use natural notations for these limits:

$$\mathrm{tr}K = \lim_{\varepsilon\to 0} \sum_{\substack{\alpha\in\mathrm{Sp}(K)\\ |\alpha|\geq\varepsilon}} m_\alpha \alpha, \quad \det(I+K) = \lim_{\varepsilon\to 0} \prod_{\substack{\alpha\in\mathrm{Sp}(K)\\ |\alpha|\geq\varepsilon}} (1+\alpha)^{m_\alpha}.$$

Assume that $h^\tau(\lambda) = \max\limits_{u\in U} h^\tau_u(\lambda)$ is smooth with respect to $\lambda \in T^*M$. We define the *exponential map* $\mathcal{E}^t_q : T^*_q M \to M$ by the formula $\mathcal{E}^t_q(\lambda_0) = \pi(\lambda_t)$, where $\dot\lambda_\tau = \vec{h}^\tau(\lambda_\tau)$, $0 \leq \tau \leq t$, and set: $Q_t = (\tilde{P}^t_*)^{-1} D_{\tilde\lambda_0} \mathcal{E}^t_{q_0}$.

Theorem 2. *Under conditions of Theorem 1, the following identities are valid:* $\det(I+K) = \det(Q_1 \Gamma_1^{-1})$,

$$\mathrm{tr}K = \mathrm{tr}\left(\int_0^1 \int_0^t X_t H_t^{-1} Z_t^* J Z_\tau H_\tau^{-1} X_\tau^* \, d\tau dt \Gamma_1^{-1} \right).$$

Let us consider a very simple example, a harmonic oscillator.

Example 2. $M = U = \mathbb{R}$, $f_u(q) = u$, $\ell(q,u) = \frac{1}{2}(u^2 - rq^2)$, $q_0 = 0$, $\tilde{u}(t) = 0$. Operator K has a form:

$$Kv(t) = r\int_0^t (t-\tau)v(\tau)\,d\tau - r\int\limits_0^1\int\limits_0^t (t-\tau)v(\tau)\,d\tau dt.$$

The eigenfunctions of this operator have a form $t \mapsto c\cos(\pi n t)$, $c \in \mathbb{R}$, $n = 1, 2, \ldots$, where the eigenfunction $c\cos(\pi n t)$ corresponds to the eigenvalue $-\frac{r}{(\pi n)^2}$. Moreover, $Q_1 = \frac{\sin\sqrt{r}}{\sqrt{r}}$ if $r > 0$ and $Q_1 = \frac{\mathrm{sh}\sqrt{|r|}}{\sqrt{|r|}}$ if $r < 0$. The determinant formula from Theorem 2 coincides with the Euler identity: $\prod\limits_{n=1}^{\infty}\left(1 - \frac{r}{(\pi n)^2}\right) = \frac{\sin\sqrt{r}}{\sqrt{r}}$ or its hyperbolic version. The trace formula gives another famous Euler observation: $\sum\limits_{n=1}^{\infty}\frac{r}{(\pi n)^2} = \frac{r}{6}$.

Example 3. $M = U = \mathbb{R}^m$, $f_u(q) = Aq + u$, $\ell(q,u) = \frac{1}{2}(|u|^2 - \langle q, Rq\rangle)$, $q_0 = 0$, $\tilde{u}(t) = 0$, where A and R are symmetric $m \times m$-matrices. The determinant and trace identities take the form:

$$\prod_{n=1}^{\infty} \det\left(I - R(A^2 + (\pi n)^2 I)^{-1}\right) = \frac{2\det\left(\sin\sqrt{R - A^2}\right)}{\det\left(\sqrt{R - A^2}\int_{-1}^{1} e^{tA}\,dt\right)},$$

$$\sum_{n=1}^{\infty} \operatorname{tr}(R(A^2 + (\pi n)^2 I)^{-1}) =$$

$$\operatorname{tr}\Big(\iiint\limits_{0\le\tau_1\le\tau_2\le t\le 1} e^{(\tau_2 - 2t)A} R e^{(\tau_2 - 2\tau_1)}\, d\tau_2 d\tau_1 dt \big(\int_0^1 e^{-2tA}\,dt\big)^{-1}\Big).$$

The right-hand side of the determinant formula has an obvious meaning also in the case of a degenerate $R - A^2$. If $m = 1$, $A = a$, $R = a^2 + b^2$, we get:

$$\prod_{n=1}^{\infty}\left(1 - \frac{a^2 + b^2}{a^2 + (\pi n)^2}\right) = \frac{a \sin b}{b \operatorname{sh} a}$$

an "interpolation" between the classical Euler identity and its hyperbolic version. The trace identity is essentially simplified if the matrices R and A commute. In the commutative case we obtain:

$$\sum_{n=1}^{\infty} \operatorname{tr}(R(A^2 + (\pi n)^2 I)^{-1}) = \frac{1}{2}\operatorname{tr}\left(R(A \operatorname{cth} A - I)A^{-2}\right).$$

Definition 3. *We say that a compact quadratic form* $b(u) = \langle Bu, u\rangle$, $u \in \mathcal{V}$, *has the spectrum of capacity* $\varsigma > 0$ *with the remainder of order* $\nu > 1$ *if* $\mathsf{Sp}_+(b)$ *and* $\mathsf{Sp}_-(b)$ *are both infinite and*

$$\beta_n = \frac{\varsigma}{n} + O(n^{-\nu}) \quad \text{as} \quad n \to \pm\infty. \tag{11}$$

We say that b *has the spectrum of zero capacity with the remainder of order* ν *if either* $\mathsf{Sp}_\pm(b)$ *is finite or* $\beta_n = O(n^{-\nu})$ *as* $n \to \pm\infty$.

Proposition 2. (i) *If b has the spectrum of capacity $\varsigma \geq 0$, then sb has the spectrum of capacity sc with the remainder of the same order as b, for any $s \in \mathbb{R}$.*

(ii) *If b_1, b_2 have the spectra of capacities ς_1, ς_2 with the remainders of equal orders, then $b_1 \oplus b_2$ has the spectrum of capacity $\varsigma_1 + \varsigma_2$ and the remainder of the same order as b_1, b_2.*

(iii) *Let $\mathcal{V}_0$ be a Hilbert subspace of the Hilbert space $\mathcal{V}$ and* $\dim(\mathcal{V}/\mathcal{V}_0) < \infty$. *Assume that one of two forms b or $b|_{\mathcal{V}_0}$ has the spectrum of capacity $\varsigma \geq 0$ with a remainder of order $\nu \leq 2$. Then the second form has the spectrum of the the same capacity ς with a remainder of the same order ν.*

(iv) *Let the forms b and $\widehat{b}$ be defined on the same Hilbert space $\mathcal{V}$, where b has the spectrum of capacity ς and $\widehat{b}$ has the spectrum of zero capacity, both with the reminder term of order ν. Then the form $b+\widehat{b}$ has the spectrum of capacity ς with the reminder term of order $\frac{2\nu+1}{\nu+1}$.*

Proof. Statement (i) is obvious. To prove (ii) we re-write asymptotic relation (11) in a more convenient form. An equivalent relation for positive n reads:

$$\#\left\{k \in \mathbb{Z} : 0 < \frac{1}{\beta_k} < n\right\} = \varsigma n + O(n^{2-\nu}), \quad \text{as} \quad n \to \infty$$

and similarly for negative n. Statement (ii) follows immediately.

Statement (iii) follows from the Rayleigh–Courant minimax principle for the eigenvalues and the relation: $\left|\frac{\varsigma}{n} - \frac{\varsigma}{n+j}\right| = O(\frac{1}{n^2})$ as $|n| \to \infty$ for any fixed j.

To prove (iv) we use the Weyl inequality for the eigenvalues of the sum of two forms. Weyl inequality is a straightforward corollary of the minimax principle, it claims that the positive eigenvalue number $i+j-1$ in the natural ordering of the sum of two forms does not exceed the sum of the eigenvalue number i of the first summand and the eigenvalue number j of the second summand. Of course, we may equally works with naturally ordered negative eigenvalues simply changing the signs of the forms.

In our case, to have both sides estimates we first present $b+\widehat{b}$ as the sum of b and $\widehat{b}$ and then present b as the sum of $b+\widehat{b}$ and $-\widehat{b}$. In the first case we apply the Weyl inequality with $i = n-[n^\delta]$, $j=[n^\delta]$ for some $\delta \in (0,1)$, and in the second case we take $i=n$, $j=[n^\delta]$. The best result is obtained for $\delta = \frac{1}{\nu+1}$. $\square$

We have to prove that the spectrum of operator K (see (6), (5)) has capacity $\frac{1}{\pi}\int_0^1 \bar{\zeta}_t\,dt$ with the remainder of order $\frac{5}{3}$.

Let $0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = 1$ be a subdivision of the segment $[0,1]$. The subspace

$$\{v \in \mathcal{V} : \int_{t_i}^{t_{i+1}} X_t v(t)\,dt = 0,\ i = 0,1,\dots,l\} = \bigoplus_{i=0}^{l} \mathcal{V}_i,\ \mathcal{V}_i \subset L^2([0,1];\mathbb{R}^m)$$

has a finite codimension in $\mathcal{V}$. The quadratic form

$$< Kv|v >= \int_0^1 \left\langle JZ_t v(t), \int_0^t Z_\tau v(\tau)\,d\tau \right\rangle dt \tag{12}$$

restricted to this subspace turns into the direct sum of the forms

$$< K_i v|v >= \int_{t_i}^{t_{i+1}} \left\langle JZ_t v(t), \int_{t_i}^t Z_\tau v(\tau)\,d\tau \right\rangle dt, \qquad v_i \in \mathcal{V}_i,\ i = 0,1\dots l.$$

Under the analyticity condition we may assume that

$$Z_t^* J Z_t = \bigoplus_{j=1}^{k} \begin{pmatrix} 0 & -\zeta_j(t) \\ \zeta_j(t) & 0 \end{pmatrix},$$

where $0 \le 2k \le m$ and $\zeta_j(t)$ are not identical zero. Indeed, according to the Rayleigh theorem, there exists an analytically depending on t orthonormal basis in which our anti-symmetric matrix takes a desired form.

The functions $\zeta_j(t)$, $j = 1, \dots, k$, are analytic and may have only isolated zeros. Hence we may take a subdivision of $[0,1]$ in such a way that $\zeta_j(t)$, $j = 1, \dots, k$, do not change sign on the segments $[t_i, t_{i+1}]$. Actually, to simplify notations a little bit, we may simply assume that $\zeta_j(t) \ge 0$, $0 \le t \le 1$, $j = 1, \dots, k$. In this case $\bar{\zeta}(t) = \sum_{j=1}^{k} \zeta_j(t)$.

Let us study our quadratic form on the space $\{v \in L^2([0,1]; \mathbb{R}^m) : \int_0^1 v(t)\,dt = 0\}$. Recall that we are allowed by Proposition 3 to work on any subspace of $L^2([0,1]; \mathbb{R}^m)$ of a finite codimension. We set $w(t) = \int_0^t v(\tau)\,d\tau$; a double integration by parts gives:

$$\int_0^1 \left\langle JZ_t v(t), \int_0^t Z_\tau v(\tau)\,d\tau \right\rangle dt = \int_0^1 \langle JZ_t v(t), Z_t w(t)\rangle\, dt +$$

$$\int_0^1 \left\langle JZ_t w(t), \dot{Z}_t w(t) \right\rangle dt + \int_0^1 \left\langle J\dot{Z}_t w(t), \int_0^t \dot{Z}_\tau w(\tau)\,d\tau \right\rangle dt.$$

Moreover, we have:

$$\left| \int_0^1 \left\langle JZ_t w(t), \dot{Z}_t w(t) \right\rangle dt + \int_0^1 \left\langle J\dot{Z}_t w(t), \int_0^t \dot{Z}_\tau w(\tau)\,d\tau \right\rangle dt \right| \le c \int_0^1 |w(t)|^2\, dt$$

Let $\lambda_n^{\pm},\ n \in \mathbb{Z} \setminus \{0\}$, be naturally ordered non zero eigenvalues of the quadratic form

$$\int_0^1 \langle JZ_t v(t), Z_t w(t)\rangle \, dt \pm c \int_0^1 |w(t)|^2 \, dt.$$

The minimax principle implies that $\lambda_n^- \leq \alpha_n \leq \lambda_n^+,\ n \in \mathbb{Z} \setminus \{0\}$. Moreover, the form

$$\int_0^1 \langle JZ_t v(t), Z_t w(t)\rangle \, dt \pm c \int_0^1 |w(t)|^2 \, dt$$

splits in the direct sum of the forms

$$\int_0^1 \zeta_j(t)\langle Jv_j(t), w_j(t)\rangle \, dt \pm c \int_0^1 |w_j(t)|^2 \, dt, \quad j = 1, \ldots, k, \tag{13}$$

where $v_j(t) \in \mathbb{R}^2,\ w_j(t) = \int_0^t v_j(t),\ w_j(1) = 0$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

It remains to estimate the spectrum of the forms (13). To do that, we'll study the spectrum of the forms

$$\int_0^1 \zeta_j(t)\langle Jv_j(t), w_j(t)\rangle\, dt, \qquad c\int_0^1 |w_j(t)|^2\, dt, \tag{14}$$

and then use the Weyl inequality for the eigenvalues of the sum of two forms. Weyl inequality is a straightforward corollary of the minimax principle: it claims that the eigenvalue number $i+j-1$ in the natural ordering of the sum of two forms does not exceed the sum of the eigenvalue number i of the first summand and the eigenvalue number j of the second summand.