A novel dark matter mechanism to produce X-ray lines with unique morphology and spectrum
To my unforgettable grandparents.
Acknowledgements

Thanks to my Comet and Lighthouse.
Thanks to Tacita Intesa and listen to FARO.
Thanks to my friends of a lifetime and to my adventure’s companions.
Thanks to my family that never stops loving me.
Abstract

The aim of this thesis is to propose a novel class of dark matter candidates. We introduce a dark sector comprised of two states: one lighter stable particle that accounts for the observed abundance of dark matter and one heavier unstable partner with a small relative mass splitting. Inelastic processes with the interstellar plasma (e.g. electrons or protons) happening today can up-scatter the stable dark matter state into the heavier partner, which suddenly decays back to the lighter state by emitting a quasi-monochromatic photon. In order to have an appreciable rate, the splitting between the two states should be of the order of the energy of the free electrons in the plasma, so the emitted photon should have an energy near the keV, that is in the X-ray range of the electromagnetic spectrum. Photon lines produced through this mechanism have many peculiar features: the lines can only be produced from regions where the plasma is present and rates can be significant only in environments hot enough to be able to excite the lighter state. Interestingly, these are precisely the features of the tentative 3.5 keV line from galaxy clusters detected in 2014. Another peculiar feature is that the width of the line, caused by the Doppler effect, should contain information about not only the velocities of the dark matter particles but also of the ones in the plasma. Thus a measurement of the width would provide a way to discriminate spectral lines of such model from others with different origins. Experimental tests scheduled in the near future will probe this scenario.
# Contents

## Introduction

### 1 Evidence of DM

1.1 From Lord Kelvin to Zwicky ................................. 1
1.2 Rotation curves ............................................. 2
1.3 X-rays from the intracluster medium ......................... 3
1.4 Gravitational lensing ........................................ 4
1.5 The Bullet Cluster ............................................ 4
1.6 CMB, BAO, SNe and overall fit ............................. 5

### 2 Sterile neutrino: the keV range

2.1 The Standard Model of massless neutrinos .................. 9
2.2 Neutrino oscillation ......................................... 10
2.3 Massive neutrinos beyond the SM ............................ 10
2.4 Sterile neutrino as DM ....................................... 11
   2.4.1 Upper bounds to the mass .............................. 12
   2.4.2 Lower bounds to the mass .............................. 12
   2.4.3 Production mechanism ................................. 13
   2.4.4 X-rays from sterile neutrinos ......................... 14
2.5 The 3.5 keV line ............................................ 15
   2.5.1 The first detections: sterile neutrino? ................ 16
   2.5.2 The non-detections: just bananas! ................. 17
   2.5.3 Another way out: a novel DM mechanism ............ 18

### 3 Fluxes of X-rays

3.1 The novel X-ray production mechanism ..................... 20
   3.1.1 Kinematics for the scattering ......................... 21
   3.1.2 Kinematics for the decay ............................. 21
3.2 Differential fluxes ......................................... 22
   3.2.1 Differential flux from decays ......................... 23
   3.2.2 Differential flux from the novel mechanism ........ 25
3.3 Energy spectra ............................................. 27
   3.3.1 Spectrum from decays ................................ 27
   3.3.2 Spectrum from the novel mechanism .................. 28
3.4 Experimental effects ...................................... 31
3.5 Morphology ................................................ 31
   3.5.1 Differential J-factor for a sphere .................. 31

### 4 Plots: energy spectra and morphologies

4.1 Neutrino-like decay ....................................... 35
4.2 Potassium-like decay ...................................... 37
4.3 The novel DM mechanism .................................... 39
   4.3.1 Inelastic up-scattering with electrons ............ 39
   4.3.2 Inelastic up-scattering with protons ............. 45
Introduction

One of the most interesting open problems in modern physics is that ordinary matter accounts for barely the 15% of the total mass of the Universe. The rest of the matter has an unknown origin and composition. It is not luminous, otherwise we would have seen it. For this reason, it has been dubbed Dark Matter (DM).

The evidence for the existence of such a component in the Universe comes from different branches of astronomy, astrophysics and cosmology, each providing clues at different scales. It is difficult to detect its presence in the Universe since it interacts very weakly with the other particles, if not for nothing, unless through gravity. Indeed these pieces of evidence are based mostly on the gravitational effects the DM exercises on the other components in the Universe.

It would not be the first time that something unknown gets discovered thanks to its gravitational influence over the other celestial bodies. In 1844 Friederich Bessel was able to predict the presence of a companion for each of the stars Sirius and Procyon, which had to exist in order to explain their observed proper motion [10]. It was perhaps the first time that the presence of celestial unseen objects was predicted just by considering their gravitational influences. Soon after in 1846, Urban Le Verrier and John Couch Adams, in order to explain a deviation in the motion of Uranus, proposed independently the existence of another planet, which was then observed and called Neptune [1]. The DM problem would be analogous if it were not that the observations seem to paint a more complicated picture, still unexplainable by the models that today describe best the cosmos and the subatomic world.

 Nowadays most of the scientific community is in favour of a particle nature for DM, a fact that is strongly supported by observations. Therefore the research has moved to the microscopic realm of particle physics. Here not even the Standard Model (SM) is able to provide a good candidate to solve the DM mystery. It is true though that also the SM has its issues and typically what physicists try to do is to kill two birds with one stone, giving answers both to the DM problem and to one of the SM's open questions. This is the case for axions and for sterile neutrinos in the keV range.

To admit a particle nature for DM usually means to introduce a very weak interaction with the other particles, thanks to which one can think of detecting DM whether directly, by searching for its reactions with the nuclei here on Earth, indirectly, by looking for products of its reactions happened somewhere else in the outer space, or by producing it at colliders.

This thesis is about a novel class of DM candidates that are able to produce X-ray signals from blasting environments like the hot plasma of a galaxy cluster. We also propose methods to discriminate the origin of X-ray excitation lines from mechanisms such as our novel one, DM decays or elemental transitions. Chapter 1 is about the evidence of DM at present. In chapter 2 we summarize the neutrino mass problem and discuss how to solve it with sterile neutrinos as possible candidates of DM. We briefly present also the history of the (un)detecteds of the tentative 3.5 keV line in 2014, its connection with sterile neutrinos and other possible explanations. In chapter 3 we describe in detail the novel DM mechanism to produce X-ray lines as the 3.5 keV one. We then develop a formalism to compute the differential flux of incoming photons, their spectrum and the morphology of the line in the case of our novel DM mechanism, the neutrino and the potassium-like decays. In chapter 4 we show the plots arising from the formulas obtained in the chapter before and compare the features of the lines for the various cases. In chapter 5 we study an Effective Field Theory (EFT) predicting the novel DM mechanism.
Chapter 1

Evidence of DM

In this chapter, we present a list of what are considered to be the most relevant pieces of evidence that today concur on proving the existence of DM and outlining the features that a good particle candidate should have. At the galactic scale, we observe too dispersive galaxies, which could not be gravitationally bounded unless an enormous amount of DM is there to hold them together. At the intergalactic scale, hot plasma is in hydrostatic equilibrium, which is not possible without an invisible massive component providing the correct gravitational force to balance the pressure. Evidence at this scale also arises from the collisions of clusters of galaxies, where ordinary matter behaves differently from the gravitationally dominant component, again DM. At the cosmological scale, the Cosmic Microwave Background (CMB) which is a picture of the Universe at the time of last scattering, gives the most precise information about the parameters of the cosmological model, including the abundance of DM.

In section 1.1 we discuss the first historical signs of DM’s existence; section 1.2 is about the rotation curves; sections from 1.3 to 1.5 concern the ICM X-ray emission, the gravitational lensing and the Bullet Cluster; finally, in the last section we describe what we can learn from CMB, BAO and SNe observations about the different species composing the Universe.

1.1 From Lord Kelvin to Zwicky

Nowadays, the astronomer Fritz Zwicky is considered by many to be the first one who found the first evidence of DM in our Universe. It is true that he has provided essentially the first convincing proof about its existence, but one cannot forget the early works of other important scientists of the first half of the 20th century, who helped to increase the interest on the issue over the years. Actually, the first one we know who attempted to estimate the amount of unobserved matter was Lord Kelvin, although he believed to be made essentially of faint stars, and he did it by applying the theory of gases to the stellar system of our galaxy [46]. It was then Henri Poincaré to explicitly use the term “matière obscure” in 1906 to describe the unseen matter for the first time [60]. Another worth mentioning work is the one by Jon Oort [55], who followed and improved the work of his teacher, Jacobus Kapteyn, and in 1932, using a thermodynamical approach to describe the motion of the stars in the galactic plane, he concluded that there had to be a non-luminous component in our galaxy. However, more recent works proved that there is actually no significant amount of DM in the galactic disk [35], showing that Oort’s work was actually incorrect. Detailed reviews about the history of DM and its evidence can be found in [8, 9, 29, 64].

It is just in 1933 that the Swiss astronomer Fritz Zwicky stated to have found excesses of DM in the galaxies of the Coma Cluster. His innovating work consisted of the use of the virial theorem for the estimate of the average mass of the galaxies in the cluster [76]. Here we try to reproduce its result, following [61]. The virial theorem tells us that the average over time of the kinetic energy is equal to minus half of the total averaged potential energy, i.e.

$$
\overline{E_K} = -\frac{1}{2} \overline{U}.
$$

(1.1)
By assuming a spherical uniform distribution for the cluster, the potential energy can be written as

\[ U = -\frac{3GM^2}{5R}, \]

where \( M \) and \( R \) are respectively the total mass and radius of the cluster. By averaging over the velocities of the galaxies, the kinetic energy becomes

\[ \overline{E_K} = \frac{1}{2}M\langle v^2 \rangle, \]

where \( \langle v^2 \rangle^{1/2} \) is the average velocity dispersion of the galaxies in the cluster.

So, by using (1.1) and dividing by the number \( N \) of galaxies, we are able to obtain an equation for \( m \), the average mass of a galaxy in the cluster:

\[ m \equiv \frac{M}{N} = \frac{5R\langle v^2 \rangle}{3NG}. \quad (1.2) \]

Zwicky used data of the velocity dispersions of the Coma Cluster to estimate \( \langle v^2 \rangle^{1/2} \approx 707 \text{ km/s} \), he considered a radius of \( R \approx 2 \cdot 10^9 \text{ light-years} \) and a number of galaxies \( N = 1000 \) and obtained \( m \approx 4.5 \cdot 10^{10} \, M_\odot \). He compared it with the Hubble’s estimate of the average luminous mass for a cluster of galaxies \( (M_L \approx 8.5 \cdot 10^7 \, M_\odot) \) and obtained a mass-to-light ratio of about 500.\footnote{To be precise, Zwicky in his estimates used a Hubble’s constant value of 558 km s\(^{-1}\) Mpc\(^{-1}\). Now from CMB analyses \( H_0 \approx 67.4 \) [3], so the mass-to-light ratio found by Zwicky should be divided by a factor \( \approx 8 \).} From this result, Zwicky concluded that the galaxies of the Coma Cluster must have contained DM with a mass of almost five hundreds times greater than the luminous one, in order to keep the system bounded with such high velocities. This is considered by most physicists and astronomers to be the first true evidence of DM in our Universe.

### 1.2 Rotation curves

The rotation curves are essentially plots of the circular velocity of stars or gases orbiting around a galaxy versus their distance from the centre. Even though they were being studied already in the 30s thanks to spectroscopy (e.g. Babcock in 1939 plotted a rotation curve of Andromeda Galaxy [7]), it is with the advent of the radio telescopes that they became objects of study for a great number of astronomers, leading them to convince the scientific community about the existence of DM halos around spiral galaxies. The method thanks to which they could estimate the circular velocity with radiometry was to extract it from the Doppler shift affecting the 21cm hydrogen line (HI). In this way, one could also plot the density of neutral hydrogen in the galaxy as a function of the radius.

If we start from the Newton’s law for a gravitating body in uniform circular motion, it is simple to obtain the speed as a function of the radius:

\[ v(r) = \sqrt{\frac{GM(r)}{r}}, \quad (1.3) \]

where \( M(r) \) is the total mass enclosed in a sphere of radius \( r \).

Typically, the baryonic matter of a spiral galaxy is concentrated below a certain radius. If the mass of a galaxy was mainly given by the luminous matter, assuming that this was concentrated in a sphere of radius \( R \) around the centre, one would expect to see \( v \) decreasing as \( r^{-1/2} \) for \( r > R \) (Keplerian fall), but rotation curves of a plethora of galaxies show instead how velocity remains practically constant even for large values of the radius.

Figure 1.1a clearly illustrates how, from radiometry, one would expect the mass of the galaxy to be confined within a certain radius (left), but, on the contrary, rotation velocities do not decrease with the increasing distance (right), remaining constant instead. From other analyses, it can be seen that essentially every spiral galaxy exhibits no Keplerian fall of the velocity.
Figure 1.1: a) Hydrogen density profile (left) and rotation curves (right) of five different galaxies, as resulted in the work of Rogstad and Shostak in 1972 [63]. The bars under the name of the galaxies stand for the effective spatial resolution, while $R_{80}$ is the radius containing the 80% of hydrogen. b) Data on the rotation curve of the Andromeda Galaxy, from different works. The pink points are from Babcock (1939) [7], the black ones are from Rubin and Ford (1970) [65], the red from Roberts and Whitehurst (1975) [62] and lastly the green from Carignan et al. (2006) [14]. The blue line is the Freeman’s theoretical curve [28]. Here $1^\circ = 60$ arcmin $\simeq 13.6$ kpc. The pictures are taken from [8].

The inner region of the rotation curves can be described, in first approximation, by a linear dependence on the radius coming from a homogeneous density. The plateau at large radii can be interpreted as an indication of the presence of a DM halo surrounding every spiral galaxy, characterized by a density $\rho \propto r^{-2}$, which gives $M \propto r$ and therefore a constant velocity. This is considered to be one of the strongest evidence of the existence of DM in our Universe at the galactic scale.

In figure 1.1b are shown different data coming from the most notable analyzes on the rotation curve of the Andromeda galaxy (M31), from 1939 to 2006. The blue line represents the Freeman’s theoretical curve obtained by assuming an exponential disk distribution for the baryonic matter [28] and no DM. The deviation from the observed values at large distances is here evident.

Anyway, the rotation curves are limited within the region where neutral hydrogen is present. This implies that it is fair to assume the existence of a DM halo, but we cannot say how it expands beyond this region. Such limitation is cut out with gravitational lensing, as described in section 1.4.

1.3 X-rays from the intracluster medium

A cluster of galaxies is typically immersed in a cloud of gas, the intracluster medium (ICM), so hot to be detectable by X-rays emission. From the spectrum of such gas, one can extrapolate information about the abundance of the matter in the cluster and, by combining it with the one of the luminous matter, obtain the total baryonic mass $M_b$. A relation between the temperature $T$ and the total gravitational mass $M_g$ that keeps the gas bounded to the cluster can be found by using the hydrostatic equilibrium equation, namely

$$\frac{1}{\rho} \frac{dp}{dr} = -\frac{GM_g(r)}{r^2}, \quad (1.4)$$
where $\rho$ is the gas density, $p$ is the pressure, $G$ is the Newton’s gravitational constant and $M_g(r)$ is the total mass contained within the radius $r$. From the equation of state of ideal gases, we get the relation

$$p = \frac{k_BT}{\mu m_p \rho}$$

where $\mu \approx 0.6$ is the average molecular weight, $m_p$ the proton’s mass and obviously $k_B$ is the Boltzmann constant. The temperature is almost constant in regions outside the core, while the density profile roughly follows a power-law with index $(-\alpha)$ between -2 and -1.5. By massaging (1.4), one can get to a relation between the temperature $T$, the radius $r$ and the total mass $M_g$ [9]:

$$T(r) = \left(\frac{\mu m_p G}{\alpha}\right) \frac{M_g(r)}{r} \approx (1.3 \div 1.8) \text{ keV} \left(\frac{M_g(r)}{10^{14} M_\odot}\right) \left(\frac{1 \text{ Mpc}}{r}\right).$$

(1.5)

Here the evidence of non-baryonic matter comes from the fact that the measured temperature is greater than the one expected by substituting $M_g(r)$ at large radii $r \approx 1 \text{ Mpc}$ with $M_b \approx 10^{14} M_\odot$ (typically $T_{exp} \approx 10 \text{ keV}$). In other words, the gas is so hot that the baryonic matter alone is not enough to keep it bounded to the cluster, hence there must be also a dark component to ensure hydrostatic equilibrium.

### 1.4 Gravitational lensing

As predicted by General Relativity, spacetime is curved by gravity in such a way that also light is affected by it. Light travels along null geodesics which are not straight lines in a curved spacetime. This implies that light emitted by a source, which is situated beyond an object with a great gravitational potential (lens), bends and can form interesting images when it reaches the observer. Figure 1.2a is a sketch of the phenomenon, which takes the name of Gravitational Lensing.

Technically, based on how intense the potential of the lens is, there are two different types of lensing: weak and strong. It is thanks to the strong lensing that one can see the fascinating images like the one pictured in figure 1.2b, while the weak is more difficult to detect. Anyway, in both cases, gravitational lensing can be used to support the existence of DM between the source and the observer: because of the universality of gravitation, lensing occurs independently of the composition of the lens, making it a powerful method to detect DM. Figure 1.3 illustrates, on the left, a cluster of galaxies lensing a blue galaxy into multiple stretched images; on the right, a computer reconstruction of the lens in which a smooth background of DM component is evident, not accounted for by the mass of the luminous matter identified by the peaks.

Another important feature of gravitational lensing is that it can be used to study DM halos in galaxies well beyond the limitations of the rotation curves. Indeed it has been found that DM can be present out to 200 kpc from the centre of a galaxy [29].

### 1.5 The Bullet Cluster

Another fascinating evidence comes from the synergy between different observations. The figure 1.4 is a combined image showing the collision of two clusters: the Bullet Cluster [17]. The optical part comes from Magellan and Hubble Telescopes; the pink region represents the hot gas of baryonic matter as detected by the Chandra X-ray Observatory; the blue one is what can be extracted from the analysis of gravitational lensing and shows the area where most of the mass of the clusters is concentrated.

This image shows not only that the matter in the clusters is mainly dark, but also reveals how DM behaves differently from the baryonic one (a proof that among other things put MOND theory [52] and related at risk). Indeed the hot gas has been slowed down during the collision, remaining at the centre, while the DM moved ahead after the impact, revealing how it does not interact (or at least, if it does, very weakly) with itself and with the gas, if not gravitationally.
Figure 1.2: a) A simple illustration of how gravitational lensing can occur. Image taken from [34]. b) A “happy face” caught in the Universe by Hubble telescope: the two eyes are the galaxies SDSSCGB 8842.3 and SDSSCGB 8842.4, while the smile lines are the arcs of the Einstein’s ring caused by the strong gravitational lensing of a galaxy situated behind [53].

Figure 1.3: a) Galaxy cluster 0024+1654 on the centre lensing a blue galaxy into multiple stretched images, as seen by Hubble telescope [18]. b) A computer reconstruction of the mass distribution of the cluster. The peaks identify the luminous matter. A smooth background of DM can be noticed. Pictures taken from [29].

1.6 CMB, BAO, SNe and overall fit

The most accurate cosmological evidence in support of the existence of DM comes from the analyses of the CMB anisotropies. The CMB has essentially a black body spectrum with an average temperature of about $T \approx 2.73$ K. Anyway, there are departures from this value: the fit with a black-body spectrum at various directions in the sky gives different temperature with a deviation of order $10^{-5}$, namely

$$\Theta(\vec{x}, \hat{p}, \eta) \equiv \frac{\Delta T}{T} \approx 10^{-5},$$

where $\vec{x}$ is the position of the observer, $\hat{p}$ is the direction of the momentum of the incoming photon and $\eta$ is the conformal time. Such deviations come from primordial density perturbations originated during inflation that froze out after they exit the horizon and began to oscillate again just before photon decoupling, at redshift $z \approx 1100$, so that they could be implanted in the CMB spectrum at the time of last scattering. The temperature anisotropy is usually expanded
Figure 1.4: An image of the so called Bullet Cluster from the combination of optical and X-ray observations (pink), respectively made by Magellan and Hubble telescopes and Chandra X-ray Observatory, and a gravitational lensing analysis (blue) [75].

in terms of the spherical harmonics $Y_{\ell m}^m$:

$$\Theta(\vec{x}, \hat{p}, \eta) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\vec{x}, \eta) Y_{\ell m}(\hat{p}).$$

From such decomposition one can evaluate the angular power spectrum:

$$\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{mm'} C_\ell.$$

One can study $C_\ell$ as a function of $\ell$, which is related to the angular size of the observed patch of the sky and obtain the first plot in figure 1.5. From the peaks shown in the graphic one can extract information about the parameters of the cosmological model. See [38] for a detailed review about the CMB temperature anisotropies.

The ratios between the heights of the first and the second peaks and of the second and the third give respectively values for $\Omega_b h^2$ and $\Omega_m h^2$, where $\Omega_b$ and $\Omega_m$ are in order the present baryonic and total matter abundances, while $h$ is the Hubble constant divided by 100. The value for $\Omega_{DM} h^2$ regarding the DM abundance can be found by simply subtracting the baryons from the total matter. The most recent measurements combine information from temperature and polarization anisotropies, plus other effects. The numbers taken from the latest Planck results (2018) [3] are

$$\Omega_b h^2 = 0.0224 \pm 0.0001, \quad \Omega_{DM} h^2 = 0.120 \pm 0.001.$$

The position of the first peak at around $1^\circ$ proves the flatness of the Universe, hence $\Omega_0 \approx 1$, and gives a value for the amount of dark energy in a $\Lambda$CDM Universe:

$$\Omega_{DE} = \Omega_\Lambda = 0.685 \pm 0.007.$$ 

Planck experiment also gave a value for the Hubble constant today, that is $H_0 = 67.4 \pm 0.5$.

The results are surprisingly interesting. Firstly, the baryonic matter is not dominant at all in the Universe, although it is the component that prevails in our lives on Earth ($\Omega_b \approx 0.05$). Secondly, there is something else other than DM ($\Omega_{DM} \approx 0.25$) that reigns in the Universe with an abundance $\Omega_{DE}^0 = \Omega_0 - \Omega_{DM}^0 - \Omega_b^0 \approx 0.7$. Figure 1.5 shows the variations on the angular power spectrum with the cosmological parameters.
Figure 1.5: On the top the plot of $\ell(\ell + 1)T^2C_\ell/2\pi$ describing the temperature fluctuations of the CMB as a function of the multipole $\ell$ or the angular scale. The dots are the data from Planck observations while the line is the best fit [24]. The images below show respectively the effects on the spectrum with the changing of the baryons amount, the total matter abundance, the dark energy density and its equation of state and finally the curvature. Pictures taken from [37].
Figure 1.6: Overall fit combining data from CMB, BAO and SNe in the $(\Omega_\Lambda, \Omega_m)$ plane. The brightness of the contours defines respectively 99.7%, 95.4% and 68.3% of confidence level. The straight line represents a flat Universe with $\Omega_\Lambda + \Omega_m = 1$. Image taken from [48] and then readapted. The CMB data are from WMAP, hence before Planck’s measurements.

It is worth mentioning that there are other constraints on the baryon density also from the primordial nucleosynthesis, which are [25]

$$0.021 \lesssim \Omega_bh^2 \lesssim 0.025 \quad (95\% \text{ CL}),$$

in perfect agreement with CMB observations.

The same density perturbations, from which the CMB’s anisotropies arise, are also responsible for the Baryonic Acoustic Oscillations (BAO). Before decoupling, photons were tightly bounded to the baryons thanks to the Thomson scattering with the electrons and the Coulomb interactions of the latter with the nuclei. The matter not influenced by the pressure and interacting only gravitationally, alias DM, concentrated around the centre of those perturbations, while outside the pressure given by the photons led to the formation of sound waves propagating in the baryon-photon fluid. These oscillations propagated until the decoupling, after which the photons started to travel on their own, leaving a disk of baryons around the density perturbation at a distance given by the sound horizon at that time, i.e. the maximum distance a sound wave could travel from its formation to the time of the last scattering. Gravity from these configurations formed the large-scale structures we see today in the Universe.

As a consequence of BAO, one would expect to find most of the large structures in the Universe separated by an average distance given by the sound horizon at decoupling. This is why BAO is used as a cosmological standard ruler.

Large-scale structure surveys can be analyzed and constraints on the cosmological parameters can be extracted from the BAO power spectrum [57].

Another evidence of DM comes from the fact that galaxies couldn’t have been formed in a purely baryonic fluid after the decoupling. The density perturbations in the baryonic fluid are in fact exactly the ones that can be measured with the CMB and are too small to enter in a non-linear regime. Thus galaxies could only have been formed thanks to the presence of a gravitating DM free from the pressure of the radiation [64].

Other constraints on the cosmological parameters come from observations of the Supernovae Ia (SNe) and from measurements of the deceleration parameter, which depends on $\Omega_0$. As a final test, one can combine data from CMB, BAO and SNe in an overall fit and check the cosmological parameters. In figure 1.6 the fit in the $(\Omega_\Lambda, \Omega_m)$ plane is shown. The intersection between the different data lies around the line describing a flat Universe. The overall fit gives a value for the matter density parameter $\Omega_m$ in a flat Universe [48]. The knowledge of $\Omega_b$ permits to find the present DM abundance $\Omega_{DM} = \Omega_m - \Omega_b$, which corresponds almost to the 85% of the total matter in the Universe.
Chapter 2

Sterile neutrino: the keV range

The Standard Model of Particle Physics (SM) does not provide any good candidate for DM. Neutrinos would seem to have all the features needed, but actually, they were relativistic at the time of equivalence between matter and radiation, so they cannot explain large scale structure formation.

On the other hand, the SM has its issues too that at first glance are not even related to the DM mystery. The research, therefore, moves beyond the SM and a good method to search for a viable DM candidate would be to solve at the same time also one of its problems.

An example of such an approach would be to give mass to active neutrinos by introducing a DM candidate. In this scenario, an interesting viable DM candidate would be the sterile neutrino in the keV range, which besides giving small masses to active neutrinos, if combined with the existence of two other sterile neutrinos in the GeV range as in the $\nu_{MSM}$, would also provide a natural explanation for the baryogenesis. Sterile neutrinos with a mass of a few keV can also be observed thanks to their decay into an X-ray and an active neutrino, thus providing a method to search for their presence in the Universe.

In this chapter we discuss why in the SM neutrinos are massless (sec. 2.1); the phenomenon of neutrino oscillation (sec. 2.2) which instead tells us that in the real world neutrinos do have mass; how to give mass to neutrinos beyond the SM (sec. 2.3), in particular the See-Saw mechanism; sterile neutrino in the keV range as DM and constraints on its mass, its production mechanism and X-rays production via its decay (sec. 2.4).

Rumours about the discovery of sterile neutrinos from astrophysical X-rays observations have spilt out in the past recent years after the detection of a 3.5 keV line of doubtful origin. We present a brief summary of the most important observations of such line and its features and then discuss the possibility of an exotic explanation (sec. 2.5).

2.1 The Standard Model of massless neutrinos

In the SM neutrinos are massless. The reason is that it is impossible to think of a mass term for the neutrinos without losing some of the features of the SM, i.e. without losing its renormalizability, breaking a symmetry or adding new fields to its particle content. A brief review of the SM’s EW sector is presented in Appendix A.

Right-handed neutrinos are not included in the particle content of the SM simply because they would not interact with any other particle in the model. This is why they are called sterile neutrinos. There is also an important accidental symmetry\(^1\) of the SM described by the group $G_{FLAVOUR} = U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau$, that is the conservation of the baryonic number and each leptonic flavours. Obviously, the total leptonic number $L$ and $B - L$ are conserved too.

If neutrinos were massive ($m_\nu \neq 0$), then chirality would not be a good quantum number, simply because $P_L$ would not commute with the Dirac Hamiltonian $H_D = \gamma^0 (i\gamma^j \partial_j - m_\nu)$.

---

\(^1\)An accidental symmetry is a symmetry that arises from the particle content of the theory and from the requirements of renormalizability.
Hence there should be a right-handed component of the field. This could be the right-handed neutrino, which would give rise to Dirac mass terms \( m_\nu \nu_R \nu_L + h.c. \), but there is no such field in the SM. Another possibility is the right-handed anti-neutrino, but the latter case would give rise to Majorana mass terms as \( m_\nu \nu_L \nu_L + h.c. \), which violate lepton number. There are other valid reasons why there should be no terms like the latter in the SM. For example, a way to generate a Majorana mass term is from terms like \( \tilde{\Phi} \tilde{\Phi} L \nu_L \nu_L \), but they are not renormalizable and plus they cannot arise from quantum corrections either, because they violate U(1)_{B-L}, which is not an anomalous symmetry [33].

### 2.2 Neutrino oscillation

The phenomenon of neutrino oscillation, which can only occur if the neutrinos are massive, is an example of something that needs an explanation from beyond the SM. Such phenomenon is analogous to what happens for the quark mixing: to get the correct Dirac mass terms from the ones generated by (A.8), one should perform a rotation of the fields in order to move to the mass basis. One can do the same also for leptons and obtain analogously the so-called Pontecorvo - Maki - Nakagawa - Sakata matrix, given by:

\[
U_{\text{PMNS}} = L_L L_L^\dagger.
\] (2.1)

If the neutrinos are massless though, as in the SM, there is a freedom in the choice of \( L_\nu \) because there are no neutrino mass matrices to be diagonalized, so one can always put \( U_{\text{PMNS}} \) equal to the identity. Hence, in the SM there should be no neutrino oscillation.

If neutrinos were massive instead, \( L_\nu \) must be chosen in order to diagonalize the mass matrix and so \( U_{\text{PMNS}} \neq 1 \) in general, giving the phenomenon of neutrino oscillation.

Recent experimental tests have proved that the phenomenon actually occurs, showing that neutrinos do have masses, even if very small. Historically, the first evidence of neutrino oscillations was observed in the disappearance of solar electron neutrinos [4] and atmospheric muon neutrinos [36]. Other pieces of evidence come now from more recent experiments (see page 8 of [2] for a detailed list).

Solar neutrino oscillations are governed by \( \Delta m_{21}^2 = m_2^2 - m_1^2 \), as these two are electron neutrino rich, while atmospheric oscillations by \( \Delta m_{32}^2 \) and \( \Delta m_{31}^2 \). The mass splitting of the neutrinos are approximately [31]

\[
\Delta m_{21}^2 \simeq +7.5 \times 10^{-5} \text{ eV}^2, \quad \Delta m_{32}^2 \simeq \pm 2.5 \times 10^{-3} \text{ eV}^2.
\]

The plus sign in \( \Delta m_{21}^2 \) dictates the order of solar neutrinos, which is instead not known for the atmospheric ones. Thus, at present, we do not know if the correct order is \( m_1 < m_2 < m_3 \) (Normal Order) or \( m_3 < m_1 < m_2 \) (Inverted Order). The sum of the masses of the neutrinos satisfies [2, 3]

\[
0.05 \text{ eV} \lesssim \sum_{i=1}^{3} m_i \lesssim 0.12 \text{ eV}.
\]

The upper bound comes from the latest Planck results whereas the lower from the fact that in the case of a normal order and with \( m_1 = 0 \) we would have \( \sum_{i=1}^{3} m_i = \sqrt{\Delta m_{32}^2 + \Delta m_{12}^2} \simeq \sqrt{\Delta m_{32}^2} \).

The goal now is to find a mechanism able to give neutrinos such small masses going beyond the Standard Model of particle physics.

### 2.3 Massive neutrinos beyond the SM

There are different ways to generate the mass terms for the neutrinos by extending the SM. One is to consider three singlet neutral fermions, e.g. the right-handed neutrinos \( N_i^R \), and add another term in the Yukawa interaction of the form:

\[
\mathcal{L}_N^Y = - \sum_{i,j=1,2,3} L_i^C (f_\nu)_{ij} N_j^R \tilde{\phi} + h.c.
\]
Unfortunately, this method gives rise to a technical issue, meaning that if one accepts that this is the correct mechanism, then one must also give a reasonable explanation of why $f_\nu \sim 10^{-11} \ll f_u \sim 1$.

Another mechanism that naturally generates small neutrinos’ masses is the See-Saw mechanism. The term in the Lagrangian which generates the masses after the electroweak Spontaneous Symmetry Breaking (ew-SSB) is still the Yukawa one, but this time a new term is added of the form $\sum_{i=1,2,3} \sum_{j=1,2,3} \nu_{\nu i}^j (f_\nu)_{ij} N_R^j + h.c.$ If one considers three families for active and $n$ for sterile neutrinos, 3 of which are responsible for the Dirac masses of the active ones, then the resulting terms after ew-SSB can be grouped as follows:

$$L_\nu = -\sum_{i=1,2,3} \sum_{j=1,2,3} \frac{\nu_{\nu i}^j (f_\nu)_{ij}}{\sqrt{2}} + \sum_{i=1,\ldots,n} \sum_{j=1,\ldots,n} (N_R^i (m_N)_{ij} N_R^j + h.c.).$$  \hspace{1cm} (2.2)

For the sake of simplicity, let us reduce to one leptonic family, but the mechanism can be generalized to more than one generations [2]. Introducing the fields

$$\begin{align*}
\chi &= \nu_L + \nu_L^c, \\
\omega &= N_R + N_R^c,
\end{align*}$$

and assuming everything real, (2.2) for one generation can be rewritten as

$$L_\nu = -\left( \chi \begin{array}{c} \omega \end{array} \right) \begin{pmatrix} 0 & \frac{v_f}{2\sqrt{2}} m_N \end{pmatrix} \left( \begin{array}{c} \chi \\ \omega \end{array} \right).$$  \hspace{1cm} (2.3)

After a diagonalization of the matrix appearing in (2.3), one finds the mass eigenstates and those, in the limit for $m_N \gg \frac{v_f}{\sqrt{2}}$, become

$$\begin{align*}
\eta_1 &= \chi \cos(\theta) + \omega \sin(\theta) \quad \text{with mass} \quad m_1 \simeq \frac{v^2 f_\nu^2}{8m_N}, \\
\eta_2 &= \omega \cos(\theta) - \chi \sin(\theta) \quad \text{with mass} \quad m_2 \simeq m_N.
\end{align*}$$

The more than one family case is done by taking the matrix in (2.3) to be a $(n+3) \times (n+3)$ matrix, where $n$ is the number of sterile neutrinos added to the model, while 3 is the number of active neutrinos, as in the SM.

The mass eigenstates are a combination of active and sterile neutrinos, whose mixing is suppressed by angles of the order

$$\theta \simeq \frac{v_f^2}{2\sqrt{2} m_N}. \hspace{1cm} (2.4)$$

Typically one chooses $m_N$ to be above the electroweak scale, in the so-called See-Saw limit. In this case, the mixing angle is practically zero, meaning that the sterile neutrinos do not mix with the active ones. The mass eigenstates become now two separate Majorana fermions: one with a very small mass, corresponding to the left-handed neutrinos, and one with a huge mass, corresponding to the right-handed neutrinos. It is the See-Saw mechanism, which not only predicts that the left-handed neutrinos should have very small masses, but that they should also be Majorana fermions. The latter result can be tested for example by searching for the double beta decay without neutrinos ($0\nu\beta\beta$ decay) [66].

There is no experimental constraint on the value of $m_N$ though, so it can be chosen essentially at whatever scale [2]. Therefore, one could think of the same mechanism to generate neutrinos’ masses but with sterile neutrinos in the interesting scale of the keV. Indeed, in this range, the sterile neutrinos can be viable DM candidates, as we discuss in the next section.

2.4 Sterile neutrino as DM

If what discussed in the previous section is valid, a good range for the sterile neutrino’s mass as DM candidate would be between the keV and the MeV, as we discuss in the subsections to come.
We review also the main sterile neutrino’s production mechanism and its decay channel into a photon and an active neutrino, which provides a way to detect its presence in the Universe.

2.4.1 Upper bounds to the mass

In order to be a reasonable DM candidate, a particle must be stable, i.e. its lifetime must be roughly greater than the age of the Universe. This imposes a constraint on the mass of the sterile neutrino if one wants it to be a good DM particle. In principle, as discussed in the previous section, sterile neutrinos can mix with active neutrinos. Hence they can interact electroweakly and, in particular, decay into three active neutrinos.

The decay of a sterile neutrino into three active neutrinos, namely \( N \rightarrow 3\nu \), can be represented by the following Feynman diagram:

![Feynman diagram](image)

It is the mixing (here represented by a crossed circle) that makes the process possible: a sterile neutrino has a probability of order \( \theta \) to be an active neutrino. It can be guessed, from a dimensional analysis, what the rate of the decay is:

\[
\Gamma_{N \rightarrow 3\nu} \sim G_F^2 \theta^2 m_N^5. \tag{2.5}
\]

The inverse of the rate must be compared to the age of the Universe \( \tau_u \sim 10^{16} s \sim 10^{40} \text{GeV}^{-1} \). If we insert the numbers and impose that \( \tau = 1/\Gamma \gg \tau_u \) we obtain the following relation:

\[
\theta^{-2} \left( \frac{m_N}{1 \text{ keV}} \right)^{-5} \gg 1. \tag{2.6}
\]

With (2.4) the latter inequality gives an upper bound for the mass of the stable sterile neutrino:

\[
m_N \lesssim 0.1 \text{ MeV} \left( \frac{1 \text{ eV}}{m_1} \right)^{2/3}. \tag{2.7}
\]

The previous condition can also be considered as a bound for the mixing angle [21]:

\[
\theta^2 \lesssim 10^{-7} \left( \frac{50 \text{ keV}}{m_N} \right)^5. \tag{2.8}
\]

Further bound on the mixing angle, much stronger than the latter, comes from the one loop-mediated radiative decay, which we briefly discuss later, that leads to a monochromatic X-ray line signal. If sterile neutrinos are produced via mixing (whether resonantly, or not) with ordinary neutrinos, their mass should be below about 50 keV [12, 50], both to yield the correct DM abundance and not to produce a too strong decay line. Therefore, in the minimal setting, the mass of DM sterile neutrinos should be below 50 keV.

2.4.2 Lower bounds to the mass

Lower bounds to the mass of the sterile neutrino as DM candidate come from inequalities for the phase-space distribution. The first limit that one can arrive to is essentially due to the Pauli exclusion principle. If one considers a generic fermionic DM candidate, with the aim to explain
the total amount of DM in the Universe, then one should consider also that fermions cannot be as close as one wants. This idea transforms into an inequality in terms of the phase-space distribution.

If we consider a galactic system, we may approximately write the velocity distribution of the DM particles as the Maxwell-Boltzmann’s one, normalized to the number of particles:

\[ f_{MB}(\vec{x}, \vec{v}) = (2\pi)^{-3/2}m^{-4}\sigma^{-3}(\vec{x})\rho(\vec{x})e^{-\frac{\vec{v}^2}{2\sigma^2(\vec{x})}}, \]

where \( \sigma(\vec{x}) \) is the velocity dispersion, which in general depends on the position. The Pauli exclusion principle imposes that the maximum of the Maxwellian distribution, which occurs for \( \vec{v} = 0 \), should not exceed the maximum of the Fermi-Dirac phase-space density, namely

\[ f_{F}^{\text{max}} = \frac{g}{(2\pi\hbar)^3}, \]

where \( g = 2 \) for a Majorana fermion. The inequality is

\[ (2\pi)^{-3/2}m_{N}^{-4}\sigma^{-3}(\vec{x})\rho(\vec{x}) \leq \frac{g}{(2\pi\hbar)^3}, \]

which, using typical values, gives the following bound for the mass:

\[ m_N \gtrsim 0.1 \text{ keV}. \]

This is usually referred to as the Tremaine-Gunn bound [69]. It is important to stress out that this limit is valid only for a fermion particle apt to describe the total amount of DM in the galaxies. Also, it does not depend on the production mechanism.

However, if one specifies the production mechanism, the lower limit can be enhanced. The production mechanism gives the correct phase-space distribution that remains constant along geodesics for a collisionless fluid, e.g. a frozen component, while the coarsely grained density decreases instead. By imposing the maximum of the coarsely grained phase-space density to be less than the maximum of the initial distribution, one finds a more general bound to the mass, independent of the spin-statistics. The lower bound to the sterile neutrino’s mass can be strengthened up to \( \sim 1 \text{ keV} \) [32].

Other limits can be put to the mass of the sterile neutrino from the observations of the so-called Lyman-\( \alpha \) forest. The latter is essentially a series of spectral lines of absorption in the spectrum of a far object, such as a quasar (QSO), that arises from the fact that the emitted photons, on their way to Earth, encounter regions of intragalactic medium (IGM) where a great amount of neutral hydrogen is present. A photon is emitted with a certain frequency from the source and then gets redshifted by the expansion during its propagation. If the photon, redshifted to a wavelength of approximately 121.6 nm (the Lyman-\( \alpha \) line), hits a neutral hydrogen atom (H) in its fundamental state, then it gets absorbed, leading H to the first excited state. The IGM extends for very large regions, so many emission frequencies get absorbed, being redshifted to the correct wavelength at different distances. This means that in the observed spectrum there is a series of frequencies missing, each characterized by a negative peak in the observed spectrum and all are concentrated in a certain range corresponding to the IGM extension. The observed flux is related to the matter density in the IGM, which in turn is related to the DM distribution. Statistical quantities can be extracted from observations of the Lyman-\( \alpha \) forest, such as the matter power spectrum. From the data of such observations, one can extract constraints on the model parameters of large-scale structure formation. Recent works tell that in a AWDM Universe, a warm DM candidate, e.g. the sterile neutrino, accounting for the total DM amount should have a mass greater than roughly 3.3 keV [72, 71].

2.4.3 Production mechanism

Sterile neutrinos can be produced out of equilibrium in the early Universe thanks to their mixing with active neutrinos [20]. Other production mechanisms are possible in principle, but the one
via mixing remains unavoidable if the mixing angle is different from zero. There is one minimal request that can be done, namely that there are no sterile neutrinos at the beginning. Such an assumption is valid if no interaction beyond the SM is added for the neutrinos and so only active flavours interact electroweakly.

The mechanism can be simplified as follows. In the thermal bath of the Universe, there is an abundance of active neutrinos as interaction eigenstates because those are the ones interacting electroweakly. Since the interaction eigenstates are not the Hamiltonian eigenstates, the active neutrinos can change their flavour over time and mix with sterile neutrinos too. In other words, starting from a certain flavoured active neutrino there is a non-zero probability to find a differently flavoured neutrino or a sterile one after some time. In the vacuum, the Hamiltonian eigenstates are the mass ones and the mixing angle between active and sterile neutrinos is given by (2.4). In presence of matter the mixing angle changes ($\theta_m$) and it depends on the temperature of the bath. For an active neutrino, there is a probability of a scattering process to not occur given by $\sim \theta_m^2$. Each scattering has a rate of $\sim G_F^2 T^5$, hence the production rate of the sterile neutrinos would approximately be something like

$$\Gamma_N \sim G_F^2 T^5 \theta_m^2(T).$$

The dependence of the mixing angle in matter upon the temperature reads [67]:

$$\theta_m(T) \simeq \frac{\theta}{1 + 2.4 \left( \frac{T}{200 \text{ MeV}} \right)^6 \left( \frac{1 \text{ keV}}{M} \right)^2}. \quad (2.9)$$

Consequently, there is a peak in the production mechanism, namely a maximum for $\Gamma / H$, which is roughly the density of sterile neutrinos, for

$$T \sim 170 \text{ MeV} \left( \frac{M}{1 \text{ keV}} \right)^{1/4}.$$

There is a numerical result for the production of sterile neutrinos via mixing that is [49]:

$$\Omega_N h^2 \sim 0.1 \left( \frac{\theta^2}{3 \times 10^{-9}} \right) \left( \frac{M}{3 \text{ keV}} \right)^{1.8}.$$

Actually, there could be a resonant process that would enhance such production mechanism. This would happen in the case of a lepton asymmetry thanks to the MSW effect, giving the so-called Shi-Fuller resonant production mechanism [68]. Such an effect would be appreciable for a lepton asymmetry of order $\sim 10^{-6}$. There is a scenario that naturally predicts this resonant mechanism and it is the $\nu$MSM model, that is the Minimal SM with massive neutrinos $\nu$, first introduced by Laine and Shaposhnikov [6]. The latter assumes the existence of three sterile neutrinos to be added to the SM. One is at the keV scale solving the DM issue. The other two are in the electroweak scale with slightly different masses accounting for the lepton asymmetry and the observed neutrino oscillation. The lepton asymmetry can then be converted into a baryon asymmetry. In this way, the $\nu$MSM model would be able to solve at the same time three problems with minimal assumptions.

### 2.4.4 X-rays from sterile neutrinos

The primary decay channel of the sterile neutrino is the decay into three active neutrinos, $N \rightarrow 3\nu$. There is another possible decay though, described by a one loop process, that is $N \rightarrow \gamma \nu$, where $\gamma$ is a photon. Two relevant Feynman diagrams of the process are the following:
The rate of the process is given by \cite{61}

\[
\Gamma_{N \rightarrow \gamma \nu_\alpha} = \frac{9\alpha}{1024\pi^4} \theta^2 G_F^2 M^5 \simeq \frac{(2\theta)^2}{1.8 \times 10^{21}\text{ s}} \left( \frac{M}{1\text{ keV}} \right)^5.
\] (2.10)

It is a very slow decay mode with a rate that is almost 130 times smaller than \(\Gamma_{N \rightarrow 3\nu}\). Since it is a two bodies process and in addition \(m_N \gg m_\nu\), then simple kinematics gives \(E_\gamma = m_N/2\). If \(m_N\) is in the keV range, the emitted photon should have an energy in the same range, namely, it is an X-ray. In principle there are several ways to detect such photon, figuring as a line in the energy spectrum.

First of all, there could be an indication of this line in the cosmic X-ray background (CXB). Throughout the history of the Universe, neutrinos could decay at different redshift producing in the spectrum a broad line with a tail. This feature can in principle be detected.

One could look also for this signal from galaxy clusters, where a big amount of DM is thought to be present in the form of halos. The flux from such objects does not differ much from the CXB’s one. In the case of clusters, the width of the line in the spectrum depends on the velocity dispersion of the halo. This feature makes the signal from those object different from signals in the CXB. There is an unavoidable background noise though and that is given by the intracluster medium which emits lines in the same range of the spectrum. To avoid such issue one can consider to observe near dwarf spherical galaxies (dSphs) instead, where such noise is not present. Although there should be less amount of DM in dSph, they are nearer, so the flux is approximately the same as from clusters. The centre of the Milky Way could also be a good source for X-rays produced via sterile neutrinos.

Unfortunately, no discovery of X-ray lines from sterile neutrinos DM has been made yet. At least, non-detections have imposed constraints on the mixing angle \(\theta\) and on the mass \(m_N\). The limits that such observations gave to the parameters \(\theta\) and \(m_N\) are shown in figure 2.1 together with Tremaine-Gunn and Lyman-\(\alpha\) constraints. Anyway, there were some recent observations about a tentative line with an energy of 3.5 keV. Although now it is thought not to be from sterile neutrinos, it is still interesting from the point of view of DM physics. This is the topic of the next section.

2.5 The 3.5 keV line

In 2014, some rumours surfaced about the discovery of the sterile neutrino, after the observation of a strange 3.5 keV line in the X-ray spectrum of many clusters of galaxies, believed to be associated with its one-loop decay. Early that year, two independent groups, Bulbul et al. \cite{13} and Boyarsky et al. \cite{11}, published two articles with which they showed evidence of such
line and proposed the neutrino’s exotic explanation, although elemental transitions were not discarded, rather considered unlikely. Short after, other studies came out such as the work of Jeltema and Profumo [42] and others [51, 5, 43] showing that such line was present in the Milky Way centre, but that could be fully explained by elemental transitions of the He-like K XVIII, and that no evidence could be found in dSphs nor in groups of galaxies, ruling out the sterile neutrino interpretation.

In the next sections, we briefly present the history of such (un)detections, focusing on what were the difficulties that the groups had to deal with while trying to explain the origin of the particular 3.5 keV line. Although nowadays it is known that the keV sterile neutrino cannot fully explain these observations, which in contrast are compatible with atomic transitions of the K XVIII [43], exotic physics is not completely ruled out and there are novel models which can give alternative explanations that are interesting from the point of view of DM indirect searches [16, 19].

2.5.1 The first detections: sterile neutrino?

Bulbul et al. (2014) [13] analyzed the X-ray spectrum data of 73 clusters of galaxies, observed by the XMM-Newton satellite with two different cameras (MOS and PN). The clusters were located at different redshifts ranging from 0.01 to 0.35, so they had to re-calibrate all the data at the same redshift (rest frame, $z = 0$), in order to obtain superimposable different spectra. They accurately subtracted a modelled continuum background from each individual spectrum and only then they could correctly stack all the spectra together in a full sample spectrum.

Bulbul et al. concentrated the analysis of the full sample spectrum in the energy range 2-10 keV. They accounted for the background by fitting the residual continuum emission with a power-law model and added 28 strong known elemental transition lines, each modelled with a Gaussian. They consequently obtained a clean residual spectrum in which they could search for unpredicted lines. Indeed they found one at $E \approx 3.55 - 3.57$ keV.

Before gambling on an exotic origin of such line, they did try to explain it with the near elemental transitions of K XVIII (3.47 keV and 3.57 keV) and Ar XVII (3.62 keV). Considering the relative abundances of the elements in the plasma as proportional to their abundances in the solar photosphere and using a multi-temperature model for the plasma with not so low temperatures, they were able to obtain an upper bound for the flux of each elemental emission lines mentioned above. The excesses of the emission lines were then included in the background model with their own maximum flux, but still, they found evidence for an unpredicted line, this time located at $E = 3.57 \pm 0.02$ keV in the MOS spectrum and $E = 3.51 \pm 0.03$ in the PN one.

Although the lines detected with the two different cameras were not in good agreement (2.8 $\sigma$), they considered reasonable to think of an exotic origin of such line. The most intriguing explanation would have been the sterile neutrino’s decay. In this case, it is straightforward to compute the parameters of the model ($\theta$ and $m_N$) by measuring the flux, which is proportional to the decay rate (2.10).

Anyway, they noted an anomaly in the spectrum of the Perseus cluster. By analyzing its individual MOS spectrum, they did find a line at the same energy 3.57 keV (PN detected no line in Perseus), but with a flux so high that the related parameters where not only in disagreement with the ones found from the full sample, but even outside the permitted limits from other observations. With no desire to abandon the sterile neutrino’s interpretation, they further investigated the excess. They found that such anomaly could be explained by an overabundance of Ar in the plasma of Perseus, in particular, an abundance 30 times greater than expected could fully explain the line, but this seemed very unlikely and unphysical to Bulbul et al. who instead associated the flux’s excess to a not so high overabundance of Ar in the cool core (a smaller temperature could explain the overabundance). Taking this into account, they fitted again the full sample, excluding the Perseus core data from the MOS spectrum, and found a fainter and less energetic line at $E = 3.55 \pm 0.03$ keV, this time in better agreement with the PN value. The sterile neutrino’s decay interpretation was still possible. At the end Bulbul et al. found
the following values for the two sterile neutrinos parameters:

\[ m_N \approx 7.1 \text{ keV}, \quad \sin^2(2\theta) \approx 7 \times 10^{-11}, \]

in good agreement with the limits known at the time.

Almost at the end of their work, they pointed out that the new fainter line was however consistent with the transition line of K XVIII at 3.51 keV, but in order to fully explain the flux with potassium, the abundance of this element would have to be greater by a factor \( 10 \sim 20 \) than expected, so an elemental origin seemed to them still very unlikely.

Meanwhile, the group Boyarsky et al. (2014) [11] independently performed a different (and not so accurate) analysis of the XMM-Newton data of Perseus cluster and Andromeda galaxy (M31). They found a line at consistent energy and they too tried to explain it with the sterile neutrino's decay, finding values of the parameters in agreement with the ones found by Bulbul et al. The important feature of their work was that they detected the line from the spectrum of a single galaxy, namely M31.

The two groups believed to have discovered the sterile neutrino, but something was wrong in their analyses, as revealed by other studies during the same year.

2.5.2 The non-detections: just bananas!

Later that year, other groups published their work on the subject and most of them revealed incongruencies with the results of Bulbul et al. and Boyarsky et al. Two important works were the ones made by Malyshev et al. (2014) [51] and Anderson et al. (2014) [5]. The first analyzed the stacked spectrum of dwarf spheroidal galaxies, while the second single galaxies and group of galaxies. Each found no evidence for a 3.5 keV line.

The reason why they decided to analyze the spectra of galaxies or group of them instead of clusters is very simple. In such systems the signal is cleaner than in galaxy clusters because the plasma is cooler, giving almost no X-ray background at all. Hence, any DM decay signal should in principle be clearly visible in those X-ray spectra. Finding no evidence for the 3.5 keV lines enabled the two groups not only to rule out any DM decay explanation, but also to give upper bounds to the sterile neutrino's parameters. Such upper limits made the results of Bulbul et al. and Boyarsky et al. completely inconsistent.

Anyhow, the most important work which contributed to rule out the neutrino's possibility was the one made by Jeltema and Profumo (2014) [42]. Their work resulted in many important aspects.

Firstly, they detected a 3.5 keV line in the Milky Way centre by analyzing the spectrum observed by the XMM-Newton satellite. The line was compatible with a neutrino's interpretation, but it could be fully explained also by the two near atomic transitions of K XVIII.

Secondly, they analyzed again the M31 spectrum, previously studied by Boyarsky et al., but with a difference: the energy range on which they had to model the background was restricted to 3 - 4 keV (not 2 - 8 as in [11]). They again found no evidence for any 3.5 keV line, contrarily to what Boyarsky et al. reported in their work. On a reply to two comments made on their work [41], Jeltema and Profumo explained the reasons why the restricted energy range they chose turned out to be more accurate: wider energy window could give rise to spurious lines, as a consequence of the incapability to correctly model the background.

They also found that no exotic line was needed in the stacked spectrum of Bulbul et al. when lower temperatures were considered in the intracluster plasma model, allowing overabundances of elements like K, meaning that the detected line could be exclusively explained by known elemental transitions.

Finally, they also observed the emission spectrum of the Tycho's supernova, finding an evidence for a 3.5 keV line. Since no significant amount of DM was expected to be found near Tycho and since instrumental features seemed very unlikely, they argued that elemental transition of K XVIII could explain this observation too and that a DM decay was not needed to interpret the peculiar line.
These three independent studies seemed to indicate essentially that the 3.5 keV line originated from potassium or, using a synecdoche, as Jeltema and Profumo ironically did in [40], from bananas.

Later, other works contributed to validating the bananas hypothesis. Philips et al (2015) [59] pointed out that the photospheric K abundances are smaller by one order of magnitude than the coronal abundances. These are the more correct ones to be used in the analyses of the plasma emission lines.

Urban et al. (2015) [70] studied Suzaku satellite observations of X-ray bright galaxies, finding that the 3.5 keV line could be simply associated with a potassium emission.

Carlson, Jeltema and Profumo (2015) [15] found an independent proof against a DM decay signal. They studied the morphology of the 3.5 keV line from Perseus cluster and from the Milky Way, finding a correlation with the plasma distribution, excluding with no doubt any DM decay signal which instead should be correlated only with DM halos.

Another recent work made by Jeltema and Profumo [43] showed no evidence for a 3.5 keV line in the dwarf spheroidal galaxy Draco, improving the constraints on the DM lifetime. In particular, they found that $\tau \gtrsim 2.7 \times 10^{28}$ at 95% CL for a DM particle with a mass of 7 keV which radiatively decays into a two bodies final state with one photon.

At this point, the most simple solution would be provided by the near transitions of the K XVIII, while the DM decay is ruled out by the observations. However, it is not strictly necessary to abandon the path of exotic physics. Indeed there are other interesting mechanisms that can explain the origin of the 3.5 keV, like the one that is the topic of this thesis.

### 2.5.3 Another way out: a novel DM mechanism

Summing up the results mentioned in the previous two sections, a mechanism with the aim to predict the 3.5 keV line must satisfy the following features:

1. The line is expected to be present in the spectra of galaxy clusters and of the Milky Way centre, where the temperature of the plasma can be so high to permit elemental transitions at the X-ray range, as in the Milky Way centre or in galaxy clusters.
2. The line is not expected to be present in the spectra of systems where the plasma has low temperature and density, as in dSphs and group of galaxies.
3. The predicted morphology of the line must be strictly correlated to the plasma distribution.

Such properties are in agreement with the assumption of a K XVIII origin of the line if one considers that:

- multi-temperature model for the plasma must not be biased towards large temperatures (by allowing lower temperature, compatible with the ratios of elements like Ca XIX and Ca XX, the abundance of K is expected to be one order of magnitude greater);
- the correct abundances of reference to set the appropriate elemental ratios are the coronal one, and not the photospheric, which are one order of magnitude smaller.

A DM decay interpretation cannot instead be an explanation of the line, because properties 2 and 3 would not be satisfied.

Although an elemental transition of K XVIII seems to be the most conservative explanation, other exotic options are still available, as long as properties 1-3 are fulfilled. An interesting Effective Field Theory (EFT) of DM that does so is the one described by F. D’Eramo et al. in [19]. Such theory introduces a dark sector comprised of two states, one lighter stable particle that accounts for the observed abundance of DM and one heavier unstable partner with a small relative mass splitting. Inelastic processes with the plasma (e.g. electrons or protons) happening today can up-scatter the stable DM state into the heavier partner, which suddenly decays back to the lighter state by emitting a quasi-monochromatic photon. In order to have an appreciable
rate, the splitting between the two states should be of order the energy of the free charged particles in the plasma, so the emitted photon should have an energy near the keV, that is in the X-ray range of the electromagnetic spectrum. Photon lines produced via this novel DM mechanism have just the right features required to described the 3.5 keV line: the lines can only be produced from regions where the plasma is present (3) and rates can be significant only in environment hot enough to be able to efficiently excite the heavier state (1 and 2).

Another peculiar feature of this DM mechanism is that the width of the line, caused by the Doppler effect, should contain information about not only the velocities of the DM particles but also of the ones in the plasma. Thus a measurement of the width would provide a way to discriminate spectral lines of such mechanism from others with different origins. How the width changes with the process that generates the line is the main topic of the next two chapters. In chapter 5 we analyze in detail the EFT described by F. D’Eramo et al. in [19] and study the width of the line in that particular case.
Chapter 3

Fluxes of X-rays

As seen in the previous chapter, exotic explanations regarding the 3.5 keV line are not completely ruled out by observations. The aim of the thesis is indeed to study a new DM mechanism for the production of X-ray lines with peculiar morphology and spectrum, such that the observed line could in principle be discriminated from others of different origin. We introduce the novel DM mechanism and discuss the kinematics of the reactions involved in section 3.1.

We postpone to chapter 5 the study of a particular EFT predicting such mechanism, while in the remaining parts of this chapter we construct a formalism in order to compute the flux of expected photons (sec. 3.2), then the energy spectrum (sec. 3.3) and the morphology (sec. 3.5) of the produced line, from a generic neutrino or potassium-like decay (subsecs. 3.2.1 and 3.3.1) and for the novel DM mechanism (subsecs. 3.2.2 and 3.3.2). For the latter, we consider in this chapter the differential cross-section for the inelastic up-scattering times the relative velocity to be isotropic and energy independent. We discuss the changes to the spectrum when this assumption is not valid at the end of chapter 5. We give also a formula to include the effects of experimental uncertainties in the spectrum (sec. 3.4).

The reader interested more in the quantitative results and direct comparisons between the different processes originating the line can move to chapter 4, where plots and figures obtained from the formulas of the following sections are presented, using typical values for the quantities involved.

3.1 The novel X-ray production mechanism

The new mechanism is based on the existence of two different states of a DM particle, one lighter $\chi_1$ with mass $m_{\chi_1}$ and one heavier $\chi_2$ with mass $m_{\chi_2}$. The lighter state is also the stable one whereas the heavier state $\chi_2$ is very unstable and tends to decay quickly, i.e. in a time that is much less than the Hubble time, into $\chi_1$ and a photon $\gamma$. So the cosmological abundance of such DM candidate is given essentially by the abundance of $\chi_1$. The heavier state is accessible by the lighter one thanks to the interaction with charged particles, like electrons or protons, present in the plasma.

So basically a model describing such mechanism must predict at least the following two interesting processes:

- the inelastic up-scattering with electrons $\chi_1 e^- \rightarrow \chi_2 e^-$ and with protons $\chi_1 p^+ \rightarrow \chi_2 p^+$;
- the decay $\chi_2 \rightarrow \chi_1 \gamma$.

The two processes are schematized by the following diagrams. We fix here the incoming and outgoing momenta of the particles once and for all. We will use a star “∗” when referring to the centre of mass rest frame (CM).
The decay $\chi_2 \rightarrow \chi_1 \gamma$ is exactly analogous to the decay of an excited nucleus, like the potassium transition K XVIII, meaning that the kinematics is the same. We name this kind of process potassium-like or DM-like decay. We will denote instead the simple decay with two massless final products, one of which is a photon, like the one of the sterile neutrino, as neutrino-like decay. The kinematics for the latter process can be simply obtained by setting to zero the mass of the final product of the potassium-like decay. We discuss now the kinematics of the two processes separately.

### 3.1.1 Kinematics for the scattering

Let us first consider the scattering. We define the components of the four momenta in a generic frame and in the CM as:

$$ p_1 = (E_1, \vec{p}_1), \quad k_1 = (E_e, \vec{k}_1), \quad p_2 = (E_2, \vec{p}_2), \quad k_2 = (E'_e, \vec{k}_2); $$

$$ p_1^s = (E_1^s, \vec{p}_1^s), \quad k_1^s = (E_e^s, -\vec{p}_1^s), \quad p_2^s = (E_2^s, \vec{p}_2^s), \quad k_2^s = (E'_e^s, -\vec{p}_2^s). $$

The on-shell conditions are $p_{1,2}^2 = m_{\chi_{1,2}}^2$, and $k_{1,2}^2 = m_{e,p}^2$. For simplicity, we consider just the electron case for the moment. We also introduce the Lorentz invariant Mandelstam variable $s$ and the 4-momentum $q$ of the CM:

$$ s = (p_1 + k_1)^2, \quad q = (q_0, \vec{q}) \equiv \gamma_q \sqrt{s}(1, \vec{\beta}_q), $$

where $\vec{\beta}_q$ is the 3-velocity of the centre of mass in a generic frame and $\gamma_q = (1 - |\vec{\beta}_q|^2)^{-1/2}$ is the Lorentz boost factor connecting the generic frame with the CM.

Using the definition of $s$ and the 4-momentum conservation we can write

$$ s = (p_1 + k_1)^2 = (p_1^s + k_1^s)^2 = m_{\chi_1}^2 + m_e^2 + 2E_1^s E_e^s + 2|p_2^s|^2 = m_{\chi_2}^2 + m_e^2 + 2E_1^s (\sqrt{s} - E_2^s) + 2E_2^s E_1^s - 2m_{\chi_2}^2 = m_e^2 - m_{\chi_2}^2 = 2\sqrt{s}E_1^s, $$

which leads to

$$ E_1^s(s) = \frac{s - m_e^2 + m_{\chi_2}^2}{2\sqrt{s}} \quad (3.1) $$

We can now perform a Lorentz boost to go back to a generic frame, obtaining the energy of $\chi_2$ as a function of $s$, the Lorentz factor $\gamma_q$ and the angle $\theta^*$ between $p_2^s$ and $\vec{q}$:

$$ E_2(s, \gamma_q, \theta^*) = \gamma_q E_1^s(s) + \sqrt{\gamma_q^2 - 1} \sqrt{E_2^s(2(s) - m_{\chi_2}^2) \cos \theta^*}. \quad (3.2) $$

### 3.1.2 Kinematics for the decay

The components of the 4-momenta in a generic frame and in the CM for the decay are:

$$ p_2 = (E_2, \vec{p}_2) \equiv m_2 \gamma_2 (1, \vec{\beta}_2), \quad p_1 = (E_1, \vec{p}_1), \quad k = (E_\gamma, \vec{k}); $$

$$ p_2^* = (m_2, 0), \quad p_1^* = (E_1^*, \vec{p}_1^*), \quad k^* = (E_\gamma^* = |\vec{p}_1^*|, -\vec{p}_1^*). $$

Depending on the kind of process we need to consider after, namely DM, potassium or neutrino-like decay, the on shell-conditions are $p_2^2 = m_2^2 = m_{\chi_2}^2, \quad m_\gamma^2, \quad 0$ and $p_2^2 = m_2^2 = m_{\chi_2}^2, \quad m_\gamma^2, \quad m_\bar{\chi}_1^2$. We consider here just the DM-like decay for brevity. For the photons we have that $k^2 = 0$. 

21
From the conservation of the 4-momentum we get
\[ p^*_2 = m^2_{\chi_2} = (p^*_1 + k^* + E^*_\gamma)^2 = m^2_{\chi_1} + k^* + E^*_\gamma \sqrt{m^2_{\chi_1} + E^*_\gamma^2}, \]
\[ m^2_{\chi_2} - m^2_{\chi_1} - 2E^*_\gamma^2 = 2E^*_\gamma \sqrt{m^2_{\chi_1} + E^*_\gamma^2}, \]
which leads to
\[ E^*_\gamma = \frac{(\delta m_{\chi_2} + 2m_{\chi_1})}{2m_{\chi_2}} \delta m_{\chi}, \tag{3.3} \]
where \( \delta m_{\chi} = m_{\chi_2} - m_{\chi_1} \) is the mass splitting. If \( \delta m_{\chi} \ll m_{\chi_1} \sim m_{\chi_2} \) then \( E^*_\gamma \approx \delta m_{\chi} \). With the latter assumption and \( \delta m_{\chi} \) near the k eV, the emitting photon has an energy of order 1 keV, i.e. is in the X-ray range. Typically the condition is valid for a nucleus decay and will be valid in the EFT discussed in chapter 5 [19]. The mass of the lighter state \( m_{\chi_1} \) and the mass splitting \( \delta m_{\chi} \) are the two mass parameters of the model. In the case of a neutrino-like decay, (3.3) simply becomes to \( E^*_\gamma = m_N/2 \).

Back in a generic rest frame we can write
\[ E^*_\gamma(\gamma_2, \theta^*) = E^*_\gamma(\gamma_2 + \sqrt{\gamma_2^2 - 1 \cos \theta^*}), \tag{3.4} \]
where here \( \theta^* \) is the angle between \( \vec{p}^*_1 \) and \( \vec{\beta}_2 \).

The combination of the two processes, namely the inelastic up-scattering and the decay of \( \chi_2 \), gives then a simple mechanism to produce X-ray lines from the interstellar plasma. The DM galactic halos are essentially made of \( \chi_1 \). Thanks to the presence of electrons and protons in the interstellar plasma, the DM particles \( \chi_1 \) in the halos can up-scatter to \( \chi_2 \), a process which is enhanced if the temperature of the plasma is of the same order as the mass splitting (see subsec. 5.4.1). Then the so formed heavier states decay back quickly to \( \chi_1 \) emitting X-ray photons. This is essentially our novel mechanism.

With \( \delta m_{\chi} \approx 3.5 \text{ keV} \) the mechanism could in principle explain the 3.5 keV line, satisfying all the three requirements we wrote in section 2.5. Indeed the up-scattering can happen efficiently only in the regions where the plasma is concentrated (condition 3.) and where it has enough temperature (conditions 1. and 2.).

### 3.2 Differential fluxes

Let us consider an observer standing far away from a source of X-rays. The flux of incoming photons is the number of photons received by the observer per unit of time (the duration of the observation) and area:
\[ \Phi_\gamma = \frac{dN_\gamma}{dt \cdot d\Sigma}. \tag{3.5} \]
Most of the works about the 3.5 keV line discussed in the previous chapter are based just on the flux of photons coming from different sources. If one wants to discriminate in a more convincing way the origin of such lines though, one needs to investigate further and extrapolate features like the energy width of the line and its correlation to the particles distribution around the source, i.e. the morphology. Of course, a greater precision of the instruments is needed. The JAXA’s X-ray Astronomy Recovery Mission (XARM) will probably reach the requested precision in the near future [45, 30, 39].

To study the spectrum and the morphology of the line we need the differential flux per unit of energy \( E_\gamma \) and radial position \( r \) (or angular if one prefers):
\[ \text{spectrum described by} \quad \frac{d\Phi_\gamma}{dE_\gamma}, \tag{3.6} \]
\[ \text{morphology described by} \quad \frac{d\Phi_\gamma}{dr}. \]

The aim of this section is to find formulas for the spectrum and the morphology in three different cases: the neutrino-like decay, the potassium-like decay and our novel mechanism. We then compare the results for the three different processes in the next chapter.
3.2.1 Differential flux from decays

The neutrino and the potassium-like decays can be treated together: the first can be obtained by the second one simply by setting to zero the mass of the daughter particle.

Let us consider the infinitesimal probability of a single decay to occur given a fixed initial energy in an infinitesimal interval of time \( dt \), which is the time of observation. This is given by the infinitesimal decay rate \( \Gamma \) if the lifetime, i.e. \( 1/\Gamma \), is much greater than the time of observation. The infinitesimal rate is given by:

\[
\frac{dP}{dt} \bigg|_{E_2 \text{ fixed}} = d\Gamma = (2\pi)^4 \frac{|\mathcal{M}|^2}{2E_2} \delta^{(4)} (p_1 + k - p_2) \left[ \frac{d^3p_1}{(2\pi)^3(2E_1)} \right] \left[ \frac{d^3k}{(2\pi)^3(2E_\gamma)} \right],
\]

where \(|\mathcal{M}|^2\) is the averaged Feynman amplitude squared of the process and the delta is for the conservation of the 4-momentum. The last two terms in parentheses are the Lorentz invariant phase-space factors. Going in the CM and after some simplification we arrive to:

\[
d\Gamma = \frac{1}{(2\pi)^2} \frac{|\mathcal{M}|^2}{2m_2} \delta^{(4)} (p_1^* + k^* - p_2^*) \left( \frac{d^3p_1^*}{2E_1^*} \right) \left( \frac{d^3k^*}{2E_\gamma^*} \right).
\]

We integrate over \( d^3p_1^* \) to cancel the 3-delta of the conservation of the 3-momentum. Using spherical coordinates we get:

\[
d\Gamma = \frac{|\mathcal{M}|^2}{32\pi^2 m_2 E_1^* E_\gamma^*} \delta \left( E_1^* + E_\gamma^* - E_2^* \right) p_1^* 2 dp_1^* d\Omega_1^*.
\]

To eliminate the delta of the conservation of energy, we need to integrate over \( dp_1^* \). The Dirac delta function is such that

\[
\delta \left( E_1^* + E_\gamma^* - E_2^* \right) = \delta \left( p_1^* - p_1^* \right) \left| \frac{\partial (E_1^*(p_1^*) + E_\gamma^*(p_1^*))}{\partial p_1^*} \right|^{-1} \bigg|_{p_1^*},
\]

where \( p_1^* \) is such that \( E_1^*(p_1^*) + E_\gamma^*(p_1^*) - E_2^* = 0 \). Given that

\[
E_1^*(p_1^*) + E_\gamma^*(p_1^*) = \sqrt{m_1^2 + p_1^*}^2 + p_1^*,
\]

we get:

\[
\left| \frac{\partial (E_1^*(p_1^*) + E_\gamma^*(p_1^*))}{\partial p_1^*} \right|^{-1} \bigg|_{p_1^*} = \frac{p_1^* + E_1^*}{E_1^*}.
\]

Therefore the integration of (3.7) gives

\[
\frac{d\Gamma}{d\Omega_1^*} = \frac{|\mathcal{M}|^2}{32\pi^2 m_2^2 E_\gamma^*},
\]

which is the infinitesimal probability per unit of time and solid angle to find a photon after a decay in the CM. We would like to have the rate per unit of energy in a generic frame. In order to do so first we rewrite (3.8) as

\[
\frac{d\Gamma}{d\Omega_1^*} = \int_0^{+\infty} dE_\gamma \frac{d\Gamma}{d\Omega_1^*} \delta (E_\gamma - E_\gamma(\gamma_2, \theta^*)) ,
\]

where \( E_\gamma(\gamma_2, \theta^*) \) is given by (3.4). Hence we get that

\[
\frac{d\Gamma}{dE_\gamma} = \int_0^{4\pi} d\Omega_1^* \frac{d\Gamma}{d\Omega_1^*} \delta (E_\gamma - E_\gamma(\gamma_2, \theta^*)) .
\]
Now we need to average over the normalized distribution of the Lorentz boost factor \( f(\gamma) \) of the initial particle, namely
\[
\left\langle \frac{d\Gamma}{d\xi_\gamma} \right\rangle = \int_1^{+\infty} \frac{d\gamma_2}{d\xi_\gamma} f(\gamma_2) \int_{4\pi} d\Omega_1 \frac{d\Gamma}{d\Omega_1} \delta (E_\gamma - E_\gamma(\gamma_2, \theta^*)).
\]
To average over the boost factors means to average over all the possible energies the initial decaying particle can have. Indeed the energy of the initial particle in a generic frame is \( E_2 = m_2 \gamma_2 \).

The rate per unit of solid angle in (3.8) cannot depend on the angles, i.e. the decay is isotropic in the CM. It must also be independent of the energy of the initial particle in order to preserve the Lorentz invariance. Hence we can take it out of the integral and arrive to the following:
\[
\left\langle \frac{d\Gamma}{dE_\gamma} \right\rangle = \frac{\Gamma}{2} \int_1^{+\infty} \frac{d\gamma_2}{d\xi_\gamma} f(\gamma_2) \int_{-1}^{+1} d\mu^* \delta (E_\gamma - E_\gamma(\gamma_2, \mu^*)) \, ,
\]
where \( \mu^* = \cos \theta^* = \frac{E_\gamma(\gamma_2, \mu^*)}{|E_\gamma(\gamma_2, \mu^*)|}, \) \( E_\gamma(\gamma_2, \mu^*) = E_\gamma(\gamma_2, \theta^*) \cos \theta^* = \mu^* \) and \( \Gamma = |\mathcal{M}|^2 E_\gamma^2 / 8 \pi m_2^2 \). The delta can be removed with the integration over \( d\mu^* \). We call \( \mathbf{p}^* \) the cosine of the emitting angle in the CM such that \( E_\gamma(\gamma_2, \mathbf{p}^*) = E_\gamma \). The derivative of the function inside the delta can be taken by differentiating (3.4) with respect to \( \mu^* \):
\[
\frac{\partial E_\gamma(\gamma_2, \mu^*)}{\partial \mu^*} = E_\gamma \sqrt{\gamma_2^2 - 1}.
\]
Using again the property of the delta when applied to a function of the variable of integration and making explicit the extremes of integration using the combination of two Heaviside step functions, we can rewrite (3.9) as
\[
\left\langle \frac{d\Gamma}{dE_\gamma} \right\rangle = \frac{\Gamma}{2} \int_1^{+\infty} \frac{d\gamma_2}{d\xi_\gamma} f(\gamma_2) \int_{-\infty}^{+\infty} d\mu^* \frac{\delta (\mu^* - \mathbf{p}^*)}{E_\gamma^* \sqrt{\gamma_2^2 - 1}} \Theta(\mu^* + 1) \Theta(1 - \mu^*) \, ,
\]
which, after integration, becomes:
\[
\left\langle \frac{d\Gamma}{dE_\gamma} \right\rangle = \frac{\Gamma}{2} \int_1^{+\infty} \frac{d\gamma_2}{d\xi_\gamma} \frac{f(\gamma_2)}{E_\gamma^* \sqrt{\gamma_2^2 - 1}} \Theta(\mathbf{p}^* + 1) \Theta(1 - \mathbf{p}^*) \, .
\]
(3.10)
The condition given by the two Heaviside step functions simply force the energy of the emitting photon to satisfy
\[
E_\gamma^* (\gamma_2 - \sqrt{\gamma_2^2 - 1}) \leq E_\gamma \leq E_\gamma^* (\gamma_2 + \sqrt{\gamma_2^2 - 1}) ,
\]
which can be translated after some tricks in a condition for \( \gamma_2 \). Indeed the inequality (3.11) is equivalent to
\[
\left[ \frac{E_\gamma}{E_\gamma^*} - (\gamma_2 - \sqrt{\gamma_2^2 - 1}) \right] \left[ \frac{E_\gamma}{E_\gamma^*} - (\gamma_2 + \sqrt{\gamma_2^2 - 1}) \right] \leq 0,
\]
Then
\[
\left[ \left( \frac{E_\gamma}{E_\gamma^*} - \gamma_2 \right)^2 - (\gamma_2^2 - 1) \right] \leq 0,
\]
which leads to
\[
\gamma_2 \geq \frac{1}{2} \left( \frac{E_\gamma}{E_\gamma^*} + \frac{E_\gamma^*}{E_\gamma} \right) .
\]
(3.12)
It is straightforward to see that the second term of the inequality is consistently greater than or equal to 1.

By substituting the two Heaviside step functions with the condition just found, (3.10) gets to its final form:
\[
\frac{d\mathcal{P}}{dt \cdot dE_\gamma} = \left\langle \frac{d\Gamma}{dE_\gamma} \right\rangle = \frac{\Gamma}{2} \int_{\frac{1}{2}}^{+\infty} \frac{d\gamma_2}{d\xi_\gamma} \frac{f(\gamma_2)}{E_\gamma^* \sqrt{\gamma_2^2 - 1}} \, d\gamma_2 .
\]
(3.13)
Finally, if we multiply (3.13) for \( N_2 = \int_V n_2 \, dV \) and divide by \( 4\pi D^2 \), where \( n_2 \) is the density of the decaying particle and \( D \) is the distance between the observer and the centre of the source, we obtain the flux per unit of energy:

\[
\frac{d\Phi}{dE_\gamma} = \Gamma \times \left( \frac{1}{4\pi D^2} \int_V n_2 \, dV \right) \times \left( 2 \int_0^{+\infty} \frac{f(\gamma_2)}{E_\gamma \sqrt{\gamma_2^2 - 1}} \, \frac{d\gamma_2}{dE_\gamma} \right). \tag{3.14}
\]

Here we have separated the three different contributions to the differential flux. The first one comes from the underlying subatomic model and it is the decay rate, the second one comes from the astrophysics and it is called \( J \)-factor, while the third comes from the statistics and we call it \( d\mathcal{P}/dE_\gamma \). All the information about the energy spectrum, e.g. the line width, lies in the third term, while the morphology is described by the differential \( J \)-factor, \( dJ/dr \).

### 3.2.2 Differential flux from the novel mechanism

We want now to perform the analogous calculation for the scattering process, which is the first reaction that takes place in our new mechanism. The rate of the scattering with fixed initial momenta is given by the product of the density of the targets \( n_e \) (for simplicity we consider the scattering with electrons), the cross-section \( \sigma \) of the process and the relative velocity \( v_{rel} \) of the two incoming particles. The differential rate for the scattering process is then

\[
d\Gamma = n_e \cdot d\sigma_{rel} = n_e (2\pi)^4 \frac{|\mathcal{M}|^2}{4E_1 E_e} \delta(4) (p_2 + k_2 - p_1 - k_1) \left[ \frac{d^3 p_2}{(2\pi)^3(2E_2)} \right] \left[ \frac{d^3 k_2}{(2\pi)^3(2E_2)} \right].
\]

With the same steps as for the decay, we can arrive to:

\[
d\sigma_{rel} = \frac{|\mathcal{M}|^2}{64\pi^2 E_1^2 E_e^2 E_2^2} \delta(E_2^s - E_1^s + E_1^e) \; p_2^* \; dp_2^* d\Omega_2^*.
\]

Now we have

\[
E_1^e (p_2^*) + E_e^* (p_2^*) = \sqrt{m_1^2 + p_2^*} \; \sqrt{E_1^e (p_2^*)},
\]

therefore

\[
\frac{\partial}{\partial p_2^*} (E_1^e (p_2^*) + E_e^* (p_2^*)) = p_2^* E_1^e (p_2^*) E_e^* (p_2^*).
\]

By applying the property of the delta and integrating over \( dp_2^* \) we get the differential cross-section times the relative velocity in the CM:

\[
\frac{d\sigma_{rel}}{d\Omega_2^*} = \frac{|\mathcal{M}|^2}{64\pi^2 E_1^2 E_e^2 (E_1^e + E_1^e)} \sqrt{E_2^s + m_1^2}.
\tag{3.15}
\]

We would like to have the cross-section per unit of energy of the scattered particle with energy \( E_2 \) in a generic frame, so we write:

\[
\frac{d\sigma_{rel}}{d\Omega_1^*} = \int_0^{+\infty} \frac{dE_2}{d\Omega_1^*} \frac{d\sigma_{rel}}{d\Omega_1^*} \delta(E_2 - E_2(s, \gamma_2, \theta^*)) \delta(E_2 - E_2(s, \gamma_2, \theta^*)),
\]

where \( E_2(s, \gamma_2, \theta^*) \) is taken from (3.2). It follows that

\[
\frac{d\sigma_{rel}}{dE_2} = \int_{4\pi} d\Omega_1^* \frac{d\sigma_{rel}}{d\Omega_1^*} \delta(E_2 - E_2(s, \gamma_2, \theta^*)).\]

Now we average over the distribution \( f_Q(s, \gamma_2) \) of \( s \) and \( \gamma_2 \), i.e. we average over all the possible energies and 3-momenta of the initial particles, and get:

\[
\left\langle \frac{d\sigma_{rel}}{dE_2} \right\rangle = \int_{(m_2 + m_e)^2}^{+\infty} ds \int_1^{+\infty} d\gamma_2 f_Q(s, \gamma_2) \int_{4\pi} d\Omega_1^* \frac{d\sigma_{rel}}{d\Omega_1^*} \delta(E_2 - E_2(s, \gamma_2, \theta^*)). \tag{3.16}
\]
One can check that \( 1 \) of energy, we arrive to
\[
\frac{\langle d\sigma_{\text{rel}} \rangle}{dE_2} = \frac{\sigma_0}{2} \int_{(m_\chi^2 + m_e)^2}^{+\infty} ds \frac{d\gamma_q}{\sqrt{\gamma_q^2 - 1}} \int_1^{+\infty} \int_{-1}^{+1} d\mu^* f_Q(s, \gamma_q) \delta (E_2 - \bar{E}_2(s, \gamma_q, \mu^*)) ,
\]
(3.17)
where \( \mu^* \) is the usual cosine of the angle of emission, while \( \bar{E}_2(s, \gamma_q, \mu^*) = E_2(s, \gamma_q, \theta^*)|_{\cos \theta^* = \mu^*} \).

After some manipulation we can get to a condition for \( \gamma_q \), which, in terms of the energy, becomes
\[
\gamma_q E_2^2(s) - \sqrt{\gamma_q^2 - 1} \sqrt{E_2^2(s) - m_\chi^2} \leq E_2 \leq \gamma_q E_2^2(s) + \sqrt{\gamma_q^2 - 1} \sqrt{E_2^2(s) - m_\chi^2}.
\]
After some manipulation we can get to a condition for \( \gamma_q \), that is
\[
\gamma_q^- \leq \gamma_q \leq \gamma_q^+ ,
\]
where
\[
\gamma_q^\pm = \frac{E_2 \cdot E_2^2(s)}{m_\chi^2} \pm \sqrt{\left( \frac{E_2^2(s)}{m_\chi^2} - 1 \right) \left( \frac{E_2^2}{m_\chi^2} - 1 \right)}.
\]
One can check that \( 1 \leq \gamma_q^- \leq \gamma_q^+ \).

Therefore, by integrating (3.17) over \( d\mu^* \) and multiplying for \( n_e \) to recover the rate per unit of energy, we arrive to
\[
\frac{dP}{dt \cdot dE_2} = \frac{\langle d\Gamma \rangle}{dE_2} = \frac{\sigma_0}{2} \cdot n_e \cdot \int_{(m_\chi^2 + m_e)^2}^{+\infty} ds \frac{d\gamma_q}{\sqrt{\gamma_q^2 - 1}} \frac{f_Q(s, \gamma_q)}{\sqrt{E_2^2(s) - m_\chi^2}}.
\]
The flux of \( \chi_2 \) per unit of its energy can be obtained from the previous formula by averaging over the density distribution \( n_1 \) of \( \chi_1 \) and dividing by \( 4\pi D^2 \):
\[
\frac{d\Phi_2}{dE_2} = \sigma_0 \left( \frac{1}{4\pi D^2} \int_V n_1 \cdot dV \right) \times \left( \frac{1}{2} \int_{(m_\chi^2 + m_e)^2}^{+\infty} \int_{\gamma_q^-}^{+\infty} \frac{d\gamma_q}{\sqrt{\gamma_q^2 - 1}} \frac{f_Q(s, \gamma_q)}{\sqrt{E_2^2(s) - m_\chi^2}} d\gamma_q \right) \cdot \frac{dP}{dE_2},
\]
(3.18)
If the decay rate of the particle \( \chi_2 \) into \( \chi_1 \) and a photon is much less than the time of observation, then we can say that all the \( \chi_2 \) particles are decayed, emitting photons in all the directions. This means that the flux of photons is equal to the flux of \( \chi_2 \). Therefore we can write
\[
\frac{d\Phi_\gamma}{dE_2} = \frac{d\Phi_2}{dE_2} = \int_{-1}^{+1} \frac{d\mu^*}{2} \frac{d\Phi_2}{dE_2}.
\]
from which we obtain the flux of photons per unit of energy:

\[
\frac{d\Phi_\gamma}{dE_\gamma} = \sigma_0 \times J \times \frac{m_{i_2}}{2} \int_{1}^{+\infty} d\gamma_2 \int_{-1}^{+1} d\mu^* \left( E_\gamma - E_\gamma(\gamma_2, \mu^*) \right) \frac{dP}{dE_2} \bigg|_{E_2 = m_{i_2} \gamma_2},
\]

where \( E_\gamma(\gamma_2, \mu^*) = E_\gamma(\gamma_2, \theta^* = \cos^{-1} \mu^*) \).

Finally, the delta can be removed with the integration over \( d\mu^* \) giving the same factor and conditions on \( \gamma_2 \) as in the decay case discussed in the previous section:

\[
\frac{d\Phi_\gamma}{dE_\gamma} = \sigma_0 \times \left( \frac{1}{4\pi D^2} \int_{V} n_1 \cdot n_e \, dV \right) \times \left[ \frac{m_{i_2}}{2} \int_{1}^{+\infty} \frac{1}{\left( \frac{E_\gamma}{m_{i_2}} \right)^{\frac{3}{2}} + \frac{e^{\frac{1}{E_\gamma}}}{m_{i_2}}} \right] \frac{dP}{dE_2} \bigg|_{E_2 = m_{i_2} \gamma_2} \frac{d\gamma_2}{\sqrt{\gamma_2^2 - 1}}.
\]

(3.19)

### 3.3 Energy spectra

All the information about the spectrum of the incoming photons lies in the term \( dP/dE_\gamma \). In order to study it, we need to specify the distributions of the momenta of the initial particles. We assume from now on that the initial particles have a Maxwell-Jüttner statistics for the momenta, which is nothing but the relativistic extension of the Maxwell-Boltzmann [44]. We do it because in this way the integrations we need to perform are easier in terms of the gamma factors. The distribution for the \( i \)-species reads:

\[
f_i(\vec{p}) = \frac{dP}{d^3p} = B_i e^{-\frac{\sqrt{\vec{p}^2 + m_i^4}}{T_i}},
\]

(3.20)

where \( m_i \) and \( T_i \) are respectively the mass and the temperature of the species. We can always get the non-relativistic limit, which is the interesting case in the scenarios we are dealing with, letting \( m_i/T_i \to +\infty \) and recover the Maxwell-Boltzmann distribution.

The constant \( B_i \) can be found by imposing the normalization condition:

\[
B_i = \left( \int_{\mathbb{R}^3} d^3p_i e^{-\frac{\sqrt{\vec{p}_i^2 + m_i^4}}{T_i}} \right)^{-1} = \left( 4\pi \int_{0}^{+\infty} dp \, p^2 \, e^{-\frac{\sqrt{p^2 + m_i^4}}{T_i}} \right)^{-1}.
\]

(3.21)

### 3.3.1 Spectrum from decays

The decay case is quite straightforward. All we need to do is to find the distribution of the boost factor \( f(\gamma_2) \) for the initial decaying particle from (3.20) and then inserting it in (3.14). The normalization condition for the momentum distribution reads:

\[
\int_{\mathbb{R}^3} d^3p_2 \, f_2(p_2) = 1
\]

According to (3.20), the distribution only depends on the modulus of the momentum. Hence we shall use spherical coordinates:

\[
4\pi \int_{0}^{+\infty} dp_2 \, p_2^2 \, f_2(p_2).
\]

Then we use the fact that \( p_2 dp_2 = E_2 dE_2 \) and that \( E_2 = m_2 \gamma_2 \) and we arrive to:

\[
4\pi m_2^2 B_2 \int_{1}^{+\infty} d\gamma_2 \gamma_2 \sqrt{\gamma_2^2 - 1} \cdot e^{-\frac{m_2 \gamma_2}{m_2 \gamma_2}} = 1.
\]
We have just found the boost factor distribution for the initial decaying particle:

\[ f(\gamma_2) = 4\pi m_2^3 B_2 \gamma_2 \sqrt{\gamma_2^2 - 1} \cdot e^{-\frac{m_2^2 \gamma_2^2}{\gamma_2^2}}. \] (3.22)

Now we can plug this expression in \( d\mathcal{P} / dE_\gamma \) of equation (3.14), getting the following:

\[ \frac{d\mathcal{P}}{dE_\gamma} = \frac{2\pi m_2^3 B_2}{E_\gamma} \int_{\frac{1}{2}(E_\gamma + E_\gamma^*)}^{+\infty} \gamma_2 \cdot e^{-\frac{m_2^2 \gamma_2^2}{\gamma_2^2}} \, d\gamma_2. \]

For the plots, it is better to define the variable \( x := E_\gamma / E_\gamma^* \) and study instead

\[ \frac{d\mathcal{P}}{dx} = 2\pi m_2^3 B_2 \int_{\frac{1}{2}(x+\frac{1}{x})}^{+\infty} \gamma_2 \cdot e^{-\frac{m_2^2 \gamma_2^2}{\gamma_2^2}} \, d\gamma_2. \]

The last integral is analytical and can be easily computed. First we perform a change of variable defined by \( y = m_2 \gamma_2 / T_2 \). The integral then becomes:

\[ \frac{d\mathcal{P}}{dx} = 2\pi m_2 T_2^2 B_2 \int_{\frac{m_2}{2T_2}(x+\frac{1}{x})}^{+\infty} y \cdot e^{-y} \, dy. \]

After one integration by parts we get that

\[ \frac{d\mathcal{P}}{dx} = 2\pi m_2 T_2^2 B_2 \left[ \frac{m_2}{2T_2} \left( x + \frac{1}{x} \right) + 1 \right] e^{-\frac{m_2^2}{4T_2^2} \left( x + \frac{1}{x} \right)}. \] (3.23)

We can get the non-relativistic limit by expanding in terms of the variable \( z = m_2 / T_2 \) around \(+\infty:\)

\[ \frac{d\mathcal{P}}{dx} \xrightarrow{z \to +\infty} \frac{1}{4\pi \sqrt{z}} \left( x + \frac{1}{x} \right) e^{z \left[ 1 - \frac{1}{2} \left( x + \frac{1}{x} \right) \right]}. \] (3.24)

### 3.3.2 Spectrum from the novel mechanism

In the case of our mechanism first we need to find the distribution \( f_Q(s, \gamma_Q) \) of the CM boost factor in terms of \( \gamma_Q \) and the Mandelstam variable \( s \). Let us begin with the infinitesimal probability to find the two initial particles at the same place with momenta \( \vec{p}_1 \in [\vec{p}_1, \vec{p}_1 + d\vec{p}_1] \) and \( \vec{k}_1 \in [\vec{k}_1, \vec{k}_1 + d\vec{k}_1] \):

\[ \frac{d\mathcal{P}}{d^3 p_1 d^3 k_1} = \frac{d\mathcal{P}}{d^3 p_1} \cdot \frac{d\mathcal{P}}{d^3 k_1} = f_1(\vec{p}_1) \cdot f_1(\vec{k}_1). \]

The normalization condition gives:

\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_1(\vec{p}_1) \cdot f_1(\vec{k}_1) \, d^3 p_1 \, d^3 k_1 = 1. \]

We want now to use different variables, namely \( \vec{q} \) and \( s \). Therefore we write:

\[ \int_{m_1 + m_e}^{+\infty} \int_{\mathbb{R}^3} \left\{ \left[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_1(\vec{p}_1) \cdot f_1(\vec{k}_1) \, d^3 p_1 \, d^3 k_1 \right] \times \delta(3) \left( \vec{q} - \vec{p}_1 - \vec{k}_1 \right) \delta \left( s - (E_1 + E_e)^2 + \left( \vec{p}_1 + \vec{k}_1 \right)^2 \right) \right\} \, d^3 q \, ds = 1. \]

Hence the infinitesimal probability to find a CM with momentum \( \vec{q} \) and Mandelstam variable \( s \) is

\[ \frac{d\mathcal{P}}{ds d^3 q} = \int_{\mathbb{R}^3} d^3 p_1 \int_{\mathbb{R}^3} d^3 k_1 \ f_1(\vec{p}_1) \cdot f_1(\vec{k}_1) \ \delta(3) \left( \vec{q} - \vec{p}_1 - \vec{k}_1 \right) \delta \left( s - (E_1 + E_e)^2 + \left( \vec{p}_1 + \vec{k}_1 \right)^2 \right). \]
The three dimensional Dirac delta can be removed thanks to the integration over $d^3k_1$, which leads to:
\[
\frac{dP}{ds\,d^3q} = \int_{\mathbb{R}^3} d^3p_1 \, f_1(\vec{p}_1) \cdot f_e(\vec{q} - \vec{p}_1) \, \delta \left( s - (E_1 + E_e)^2 + |\vec{q}|^2 \right).
\] (3.25)

The on-shell conditions on the energies are
\[
E_1 = E_1(|\vec{p}_1|) = \sqrt{|\vec{p}_1|^2 + m_{\chi_1}^2}
\]
and
\[
E_e = E_e(|\vec{p}_1|, |\vec{q}|) = \sqrt{|\vec{q}|^2 + |\vec{p}_1|^2 - 2|\vec{q}||\vec{p}_1|\mu + m_e^2},
\]
where we have defined $\mu$ as $\frac{\vec{q} \cdot \vec{p}_1}{|\vec{q}||\vec{p}_1|}$.

If we move to spherical coordinates, we can remove the last delta by integrating over $d\mu$.

This as usual leaves a condition on $p_1 \equiv |\vec{p}_1|$, which can be expressed with two Heaviside step functions, and the factor coming from the delta, that is
\[
\left\{ \frac{\partial}{\partial \mu} \left[ E_1(p_1) + E_e(p_1, \mu) \right]^2 \right\}^{-1} = \frac{\sqrt{s + q^2} - \sqrt{m_{\chi_1}^2 + p_1^2}}{2p_1q\sqrt{s + q^2}} = \frac{\sqrt{s \gamma_q - m_{\chi_1} \gamma_1}}{2m_{\chi_1} s \gamma_q \gamma_1^2 - 1},
\]
where $\vec{p}$ is such that
\[
E_1(|\vec{p}_1|) + E_e(|\vec{p}_1|, |\vec{q}|, \mu) = \sqrt{s + |\vec{q}|^2},
\]
which is the condition imposed by the delta.

The product of the two initial distribution functions evaluated in $\vec{p}$ is simply
\[
f_1(\vec{p}_1) \cdot f_e(\vec{q} - \vec{p}_1) \big|_{\vec{p}} = B_1B_e \cdot e^{-\frac{E_1(|\vec{p}_1|)}{m_{\chi_1}}} \cdot e^{-\frac{E_e(|\vec{p}_1|, |\vec{q}|)}{\mu}} = B_1B_e \cdot e^{-\frac{\sqrt{s + |\vec{q}|^2}}{\sqrt{s}} \cdot e^{-\sqrt{m_{\chi_1}^2 (\frac{T_e - T_{\chi_1}}{T_{\chi_1}})}}}.
\]

Therefore the integral in (3.25) becomes:
\[
\frac{dP}{ds\,d^3q} = 2\pi B_1 B_e \int_0^{+\infty} dp_1 \, p_1^2 \, e^{-\frac{\sqrt{s + q^2}}{\sqrt{s}} \cdot e^{-\sqrt{m_{\chi_1}^2 (\frac{T_e - T_{\chi_1}}{T_{\chi_1}})} \times \theta (\vec{p} + 1) \theta (1 - \vec{p})}.
\] (3.26)

If we express everything in terms of $\gamma_1$ and $\gamma_q$ we get that
\[
f_q(s, \gamma_q) = \frac{dP}{ds\,d\gamma_q} = 4\pi^2 m_{\chi_1}^3 B_1 B_e \sqrt{s} \, e^{-\frac{\sqrt{s}}{\sqrt{s}} \times \int_1^{+\infty} d\gamma_1 \, \gamma_1 \left( \frac{\sqrt{s}}{m_{\chi_1}} \gamma_q - \gamma_1 \right) \, e^{-m_{\chi_1} \gamma_1 (\frac{T_e - T_{\chi_1}}{T_{\chi_1}})} \theta (\vec{p} + 1) \theta (1 - \vec{p})},
\] (3.27)
where we have used the relation $d^3q = 4\pi s^{3/2} \gamma_q \sqrt{\gamma_q^2 - 1} \, d\gamma_q$.

Let us now analyze the condition imposed by the two Heaviside step functions. First we need to express $\vec{p}$ in terms of $\gamma_1$, $\gamma_q$ and $s$. By solving the condition imposed by the Dirac delta we get:
\[
\vec{p} = \frac{\gamma_q \gamma_1 - \varepsilon(s)/m_{\chi_1}}{\sqrt{\gamma_q^2 - 1}},
\]
where $\varepsilon(s)$ is the same function as $E_2^2(s)$ in (3.1) with the substitution of $m_2$ with $m_{\chi_1}$. The inequality given by the Heaviside functions reads:
\[
\vec{p}^2 \leq 1,
\]
which leads to the following condition on $\gamma_1$:

$$\gamma_1^- \leq \gamma_1 \leq \gamma_1^+,$$

where

$$\gamma_1^\pm = \gamma_1 \frac{e(s)}{m_{X_1}} \pm \sqrt{\left(\frac{q^2}{m_{X_1}^2} - 1\right) \left(\frac{\varepsilon^2(s)}{m_{X_1}^2} - 1\right)}.$$

One can verify that $1 \leq \gamma_1^- \leq \gamma_1^+$.

Summing up, the distribution for the Mandelstam variable $s$ and the boost factor $\gamma_q$ is

$$f_Q(s, \gamma_q) = \frac{dP}{ds \, dq} = 4\pi^2 m_{X_1}^3 B_{X_1} B_e \sqrt{s} e^{-\frac{\sqrt{s}}{T_e}} \times$$

$$\frac{1}{(s - m_e^2 + m_{X_2}^2)^2 - 4m_{X_2}^2 \sqrt{s} \sqrt{\gamma_q^2 - 1}}$$

$$\int_{\gamma_1^-}^{\gamma_1^+} d\gamma_1 \gamma_1 \left(\frac{\sqrt{s}}{m_{X_1} \gamma_q - \gamma_1}\right) e^{-m_{X_1} \gamma_1 \left(\frac{T_e - T_{X_1}}{T_e \gamma_q}\right)}.$$  (3.28)

For the purpose of illustration, in the next chapter we present the plot for the latter distribution in terms of $s$ and the momentum of the centre of mass $q$, namely

$$f_Q(s, q) = \frac{dP}{ds \, dq} = 4\pi^2 m_{X_1}^3 B_{X_1} B_e \frac{q}{\sqrt{s^2 + q^2}} e^{-\frac{\sqrt{s \gamma_q}}{T_e}} \times$$

$$\frac{1}{(s - m_e^2 + m_{X_2}^2)^2 - 4m_{X_2}^2 \sqrt{s} \sqrt{\gamma_q^2 - 1}}$$

$$\int_{\gamma_1^-}^{\gamma_1^+} d\gamma_1 \gamma_1 \left(\frac{\sqrt{s}}{m_{X_1} \gamma_q - \gamma_1}\right) e^{-m_{X_1} \gamma_1 \left(\frac{T_e - T_{X_1}}{T_e \gamma_q}\right)}.$$  (3.29)

The latter integral is analytical and can be solved with an integration by parts, but we do not compute it since it is meaningless to carry with us the very ugly expression we would obtain by solving it.

The insertion of (3.28) in $dP/dE_2$ of (3.18) gives the following expression for the differential flux of the scattered particle:

$$\frac{dP}{dE_2} = 4\pi^2 m_{X_1}^3 B_{X_1} B_e \int_{(s + m_e^2)^2}^{+\infty} ds \int_{\gamma_1^-}^{\gamma_1^+} d\gamma_q \frac{s e^{-\frac{\sqrt{s}}{T_e}}}{\sqrt{(s - m_e^2 + m_{X_2}^2)^2 - 4m_{X_2}^2 \sqrt{s} \sqrt{\gamma_q^2 - 1}}} \times$$

$$\int_{\gamma_1^-}^{\gamma_1^+} d\gamma_1 \gamma_1 \left(\frac{\sqrt{s}}{m_{X_1} \gamma_q - \gamma_1}\right) e^{-m_{X_1} \gamma_1 \left(\frac{T_e - T_{X_1}}{T_e \gamma_q}\right)}.$$  (3.30)

For the plots, it is convenient to write it in terms of the momentum instead of the energy:

$$\frac{dP}{dp^2} = 4\pi^2 m_{X_1}^3 B_{X_1} B_e \int_{(m_{X_2} + m_e)^2}^{+\infty} dp^2 \int_{\gamma_1^-}^{\gamma_1^+} d\gamma_q \frac{s e^{-\frac{\sqrt{s}}{T_e}}}{\sqrt{(s - m_e^2 + m_{X_2}^2)^2 - 4m_{X_2}^2 \sqrt{s} \sqrt{\gamma_q^2 - 1}}} \times$$

$$\int_{\gamma_1^-}^{\gamma_1^+} d\gamma_1 \gamma_1 \left(\frac{\sqrt{s}}{m_{X_1} \gamma_q - \gamma_1}\right) e^{-m_{X_1} \gamma_1 \left(\frac{T_e - T_{X_1}}{T_e \gamma_q}\right)}.$$  (3.31)

If we plug the result in (3.19) we get the spectrum of the photon emitted via the novel mechanism:

$$\frac{dP}{dE_{\gamma}} = 2\pi^2 m_{X_1}^3 m_{X_2} B_{X_1} B_e \int_{(m_{X_2} + m_e)^2}^{+\infty} dp^2 \int_{\gamma_1^-}^{\gamma_1^+} d\gamma_q \left\{ \frac{1}{E_\gamma \sqrt{\gamma_2^2 - 1}} \times$$

$$\left[ \frac{s e^{-\frac{\sqrt{s}}{T_e}}}{\sqrt{(s - m_e^2 + m_{X_2}^2)^2 - 4m_{X_2}^2 \sqrt{s} \sqrt{\gamma_2^2 - 1}}} \right] \times$$

$$\int_{\gamma_1^-}^{\gamma_1^+} d\gamma_1 \gamma_1 \left(\frac{\sqrt{s}}{m_{X_1} \gamma_q - \gamma_1}\right) e^{-m_{X_1} \gamma_1 \left(\frac{T_e - T_{X_1}}{T_e \gamma_q}\right)} \right\}.$$  (3.32)
and also
\[
\frac{dP}{dx} = 2\pi^2 m_{\chi_1}^3 m_{\chi_2} B_{\chi_1} B_{\chi_2} \int_{\frac{1}{2}(1+\frac{1}{2})}^{+\infty} d\gamma_2 \left\{ \frac{1}{\sqrt{\gamma_2^2 - 1}} \times \right. \\
\int_{(m_{\chi_2} + m_e)^2}^{+\infty} ds \int_{\gamma_q}^{\gamma_q^+} d\gamma_1 \left[ \frac{s e^{-\frac{\sqrt{\gamma_q}}{\tau_e}}}{\sqrt{(s - m_e^2 + m_{\chi_2}^2)^2 - 4m_{\chi_2}^2 \sqrt{s\gamma_q}} - 1} \times \\
\left. \int_{\gamma_q^+}^{\gamma_1} d\gamma_1 \left( \frac{\sqrt{s}}{m_{\chi_1}} \gamma_q - \gamma_1 \right) e^{-m_{\chi_1} \gamma_1 \left( \frac{\tau_e - \tau_{\chi_1}}{4\pi c_{\chi_1}} \right)} \right] \left. \right\} \\
\right\}
\tag{3.33}
\]

### 3.4 Experimental effects

Until now we have described the theoretical features of the spectrum only, but experimental effects due to the error of the measurements are inevitable and will contribute to the total width of the observed line.

Let us assume a Gaussian distribution in energy with variance \(\sigma_E\) for the data of a hypothetical experiment apt to observe a line as the one we are interested in. Each point of the theoretical spectrum \(dP/dx\) will then be a Gaussian centred in the point and with variance \(\sigma_x = \sigma_E/E_\gamma^*\). Therefore, the experimental spectrum is the infinite sum of all the Gaussians centred in the different points of the theoretical spectrum, namely a convolution between the theoretical spectrum and the Gaussian distribution of the data:

\[
\frac{dP}{dx}^{\exp} = \frac{1}{\sqrt{2\pi}\sigma_x^2} \int dy \frac{dP}{dy} e^{-\frac{(y-y_0)^2}{2\sigma_x^2}}.
\tag{3.34}
\]

### 3.5 Morphology

All the information about the morphology of the line lies in the \(J\)-factor appearing in the equations (3.14) and (3.19). The \(J\)-factor as the general form

\[
J = \frac{1}{4\pi D^2} \int_V N(\vec{x}) \, dV,
\tag{3.35}
\]

where \(N(\vec{x})\) is a function of the point and is different from process to process. Namely in the case of the decay \(N(\vec{x}) = n_2(\vec{x})\), where \(n_2(\vec{x})\) is the number density of the decaying particles (e.g. potassium or the sterile neutrino), while in the case of our mechanism \(N(\vec{x}) = n_{\chi_1}(\vec{x}) \cdot n_{e,p}(\vec{x})\), where \(n_{\chi_1}(\vec{x})\) is the number density of the DM lighter state \(\chi_1\) whereas \(n_{e,p}(\vec{x})\) the one of the electrons or protons in the plasma.

Using the coordinates of an observer placed far away from the source, we can write the infinitesimal volume element as follows:

\[
dV = d\Sigma \, d\ell = D^2 d\Omega \, d\ell,
\]

where \(\ell\) is the line of sight (\(\ell\, o\, s\)). Hence the \(J\)-factor defined in equation (3.35) becomes:

\[
J = \frac{1}{4\pi} \int_{\Delta\Omega} d\Omega \int_{\ell\, o\, s.} d\ell \, N(\theta, \phi, \ell).
\tag{3.36}
\]

#### 3.5.1 Differential \(J\)-factor for a sphere

Let us compute the \(J\)-factor in the case of a sphere. Picture 3.1a illustrates the geometry of the problem. First of all, thanks to the azimuthal symmetry, we can write:

\[
J = \frac{1}{2} \int_0^\alpha d\theta \sin \theta \int_{\ell\, o\, s.} d\ell \, N(\theta, \ell),
\]

31
where $\alpha$ is the angle of aperture of the telescope looking at the source. Typically, the densities are functions of only $R$, the distance from the centre of the source. This means that:

$$\mathcal{N}(\theta, \ell) \equiv \mathcal{N}(R(\theta, \ell)) .$$

The extrema of integration over the line of sight can be made explicit with $\ell_-$ and $\ell_+$, that are respectively the beginning and the end of the source along the line of sight. If we assume the source to be very far away from the observer, meaning that $D \gg R_g$, where $R_g$ is the radius of the source, then we can say that $\ell_-$ and $\ell_+$ lie on the line parallel to the satellite-source line. If we choose the origin of the line of sight in the middle between $\ell_-$ and $\ell_+$ we simply get that $\ell_- = -\ell_+$. The approximation is represented in figure 3.1b.

Under the approximation $D \gg R_g$, we can confuse the angle $\theta$ and its sine with the ratio $r/D$, where $r$ is the radial position (see figure).

We can now write the distance from the centre of the galaxy in terms of $\ell$ and $r$ with the Pythagoras’s theorem:

$$R(\theta, \ell) = \sqrt{\ell^2 + r^2} .$$

We can also write $\ell_{\pm}$ in terms of $R_g$ and $r$, namely:

$$\ell_{\pm} = \pm \sqrt{R_g^2 - r^2} .$$

We are not interested in the integration over $r$ since we want to study the morphology of the line, described by the differential $J$-factor, that is

$$\frac{dJ}{dr} = \frac{r}{D^2} \int_0^{\sqrt{R_g^2 - r^2}} d\ell \mathcal{N}(\sqrt{\ell^2 + r^2}) . \quad (3.37)$$
Figure 3.1: a) The geometry of the problem. The satellite (XMM-Newton in the picture) is staring at a certain direction, along with the line of sight (ℓ. o. s.), with an aperture dΩ, the blue cone. The sphere represents the spherical source of X-rays (Perseus cluster); D is the distance of it from the satellite; r is the projection of the line of sight along the direction perpendicular to the satellite-source line and we call it radial position. The background [54], the satellite and the Earth [23] are taken from ESA’s website and adapted here for the purpose of our illustration. The size scale is exaggerated to permit a better understanding. b) The fact that $D \gg R_g$ enables us to consider $\ell^- = -\ell^+$ as shown in the figure. This simplifies the integral along the line of sight, which otherwise would have been dependent upon $r$. It follows from Pythagoras’s theorem that $\ell_\pm = \pm \sqrt{R_g^2 - r^2}$. 

33
Chapter 4

Plots: energy spectra and morphologies

In the previous chapter, we have developed a formalism in order to study the energy spectrum of photons emitted via different mechanisms. In particular, we have examined the neutrino-like decay \( N \to \nu \gamma \), the potassium-like decay \( K^* \to K \gamma \) describing the transition \( K_{\text{III}} \), and a novel DM mechanism, which consists of an inelastic up-scattering between DM and electrons or protons in the plasma \( \chi_1 e^-, p^+ \to \chi_2 e^-, p^+ \), followed by a DM-like decay \( \chi_2 \to \chi_1 \gamma \). For each of these processes, we have found formulas for the differential flux of photons emitted from a very far away source (like a cluster of galaxies): (3.14) for the decay and (3.19) for the novel mechanism. We have separated the three different contributions coming from the particular subatomic model (rate \( \Gamma \) or cross-section \( \sigma \)), the astrophysics (\( J \)-factor), connected to the morphology of the line, and the statistics \( (dP/dE_\gamma) \), where all the information about the energy spectrum lies. Both the formulas (3.14) and (3.19) are general and can be used once the momentum distributions of the initial particles are known.

We have then focused the study on the case of relativistic Maxwell-Jüttner distributions, which in the non-relativistic limit give the correct Maxwell-Boltzmann distributions for the particles in the plasma and the DM in the galactic halos. We have thus obtained two analytical formulas for the energy spectrum of photons: (3.23) for the simple decay case and the quite complicated expression (3.33), to be solved numerically, for the novel mechanism. Actually, for our mechanism, we have simplified the problem by considering an isotropic and independent of the energy differential inelastic up-scattering cross-section times relative velocity. This permitted us to construct a model-independent spectrum. However, in chapter 5 we study in detail a particular EFT where the latter assumption does not yield and hence we discuss whether it leads to important changes in the spectrum or not.

Effects due to experimental uncertainties have been taken into account with formula (3.34), describing the true experimental energy spectrum.

We have also found a general formula for the differential \( J \)-factor in the case of a spherical symmetry, see (3.37). This allows us to study the morphology of the line from sources like the Perseus cluster.

The width of the line is due to the Doppler shift caused by the motion of the particles in the initial state. Hence, we expect it to be different for each of the mechanism we have considered so far since they do not share the same initial states. In addition, the morphologies should be different too because they depend on the spatial distribution of the initial particles. The aim of this chapter is to compare lines of different origins and to see the differences in the widths, thus proposing a method to determine which mechanism is at the origin of the line based on the measurement of the FWHM (Full Width Half Maximum) of the energy spectrum. We show also that the morphologies are different, as expected, hence providing another method to discriminate the origin of the line. We discuss the main features of the energy spectrum for the neutrino-like decay in section 4.1, for the potassium-like decay in section 4.2 and for the novel mechanism in section 4.3. We present also the experimental effects on the line and its width in section 4.4. Then, in section 4.5, we compare the morphologies of the lines for the three different cases considering the Perseus cluster as the source.
Figure 4.1: The Maxwell-Boltzmann momentum distribution for a sterile neutrino of mass \( m_N = 7 \text{ keV} \) and dispersion velocity \( \langle v^2 \rangle^{1/2} = 10^{-3} \).

For the purpose of illustration, we set the temperature of the plasma at 1 keV and the dispersion velocity of the DM at \( \langle v_{DM}^2 \rangle^{1/2} = 10^{-3} \), which in our formalism is connected to the temperature thanks to the equipartition theorem, namely \( T_{DM} = \frac{1}{3} m_{DM} \langle v_{DM}^2 \rangle \) (where DM becomes \( \chi \) or \( N \) for the DM or neutrino-like decay cases respectively). To permit a better comparison between the three different mechanisms, we have expressed the energy spectra in terms of the variable \( x = E_\gamma / E^*_{\gamma} \).

4.1 Neutrino-like decay

Let us begin with the distribution for the sterile neutrino as DM. We suppose that the velocity distribution for the non-relativistic DM in the galactic halo can be described by the Maxwell-Boltzmann [19]. Hence for the sterile neutrino we have:

\[
    f_N(v) = \frac{dP_N}{d^3v} = \frac{1}{(2\pi)^{3/2}} \frac{p_N^{-3/2}}{\langle v_{DM}^2 \rangle^{-3/2}} e^{-\frac{3p^2}{2\langle v_{DM}^2 \rangle}}.
\]

We can write it in terms of the momentum \( p \equiv |\vec{p}| = m_N |\vec{v}| \):

\[
    f_N(p) = \frac{dP_N}{dp} = \left( \frac{2}{27\pi} \right)^{1/2} \frac{m_N^{-3}}{\langle v_{DM}^2 \rangle^{-3/2}} \frac{p^{2} e^{-\frac{3p^2}{2m_N^2 \langle v_{DM}^2 \rangle}}}{2\sqrt{2 \log 2} \cdot \sigma_{E_\gamma} \cdot 2 \sqrt{2 \log 2} \cdot [\langle x^2 \rangle - \langle x \rangle^2]^{1/2}}.
\]

In figure 4.1 we show the momentum distribution for a sterile neutrino of mass \( m_N = 7 \text{ keV} \). One can obtain practically the same plot using (3.20) simply by setting \( T_{DM} = \frac{1}{3} m_{DM} \langle v_{DM}^2 \rangle \) and fixing the values. Indeed we have that \( T_{DM} / m_{DM} = \frac{1}{3} \langle v_{DM}^2 \rangle \sim 10^{-7} \ll 1 \).

Next in figure 4.2 we show the plot for the photon’s spectrum emitted via neutrino-like decay as a function of the variable \( x = E_\gamma / E^*_{\gamma} \). The plot comes from (3.23) or directly from its non-relativistic expansion (3.3.1). Notice that there is a peak at \( x = 1 \) meaning that most of the photons are produced with the energy they have in the rest frame.

The width arises because in the rest frame of the medium the neutrinos are not still, hence the photons gain a Doppler shift depending on whether the neutrinos are moving forward or backwards.

We define the FWHM as for a Gaussian, namely:

\[
    \text{FWHM} \approx 2 \sqrt{2 \log 2} \cdot \sigma_{E_\gamma} = E^*_{\gamma} \cdot 2 \sqrt{2 \log 2} \cdot [\langle x^2 \rangle - \langle x \rangle^2]^{1/2},
\]
where here \( \langle \cdot \rangle \) denotes the average over the photon’s spectrum \( dP/dx \). In figure 4.3 we show the width of the spectrum as a function of the mass at different values of the velocity dispersion for the neutrino-like decay. Remember that each mass value \( m_N \) corresponds to a photon’s energy in the neutrino’s rest frame of \( E^*_\gamma \approx m_N/2 \).

**Figure 4.2:** The energy spectrum of the photon emitted via the sterile neutrino decay. Here we have used the variable \( x = E_\gamma / E^*_\gamma \) to permit a better comparison between the other mechanisms. Notice the peak at \( x = 1 \) meaning that the photon’s energy is peaked at its value in the neutrino’s rest frame. The width of the line arises from the Doppler shift the photons gain in the medium rest frame.

**Figure 4.3:** The width of the photon’s energy spectrum as a function of the mass \( m_N \) at different values of the velocity dispersion \( \langle v_{DM}^2 \rangle^{1/2} \). The vertical dashed line corresponds to a mass of \( m_N = 7 \) keV, thus to a photon’s energy in the neutrino’s rest frame of \( E^*_\gamma \approx 3.5 \) keV. The horizontal dashed lines indicate the respective values of the FWHM for such mass at different velocities.
Figure 4.4: The Maxwell-Boltzmann momentum distribution for the excited potassium with mass $m_{K^*} \approx 40$ GeV. The temperature is set to the value $T_{K^*} = 1$ keV.

4.2 Potassium-like decay

We want here to generalize our analysis not only to the K XVIII transition but to any elemental line. We thus consider an initial excited state ($A^*$) with mass $m_{A^*}$ and a final state ($A$) with mass $m_A$. We define the mass splitting between the two as $\delta m_A$. For a K XVIII transition producing the observed line, we have $m_K \approx 40$ GeV and $\delta m_K \approx 3.5$ keV.

The particles in the plasma have a non-relativistic Maxwell-Boltzmann distribution with a temperature of about $T \sim 1$ keV. The Maxwell-Boltzmann distribution in terms of the momentum for the initial excited state of a nucleus $A^*$ reads:

$$f_{K^*}(p) = \frac{dP_{A^*}}{dp} = \left( \frac{2}{\pi} \right)^{1/2} [(m_A + \delta m_A) T_{A^*}]^{-3/2} p^2 e^{-\frac{p^2}{2(m_A + \delta m_A) T_{A^*}}},$$ \hspace{1cm} (4.4)

and it is shown in figure 4.4 for potassium. Again the distribution can be considered as the non-relativistic limit of (3.20). Notice that for a temperature of order 1 keV the dispersion velocity of the potassium is much lower than the one for the DM, indeed

$$\langle v^2_{K^*} \rangle = \left( \frac{3T_{K^*}}{m_{K^*}} \right) \approx 7.5 \times 10^{-8} \ll 10^{-6}.$$

We show in figure 4.5 the energy spectrum of the photon emitted via potassium decay. The spectrum is peaked at $x = 1$ meaning that the photon is emitted mostly with the energy it has in the rest frame of the decaying nucleus.

The width is due to the Doppler shift caused by the thermal motion of the initial nuclei in the plasma. Notice though that the line is thinner compared to the sterile neutrino case. This simply reflects the fact that the dispersion velocity of potassium is lower than the sterile neutrino’s one, thus leading to a fainter Doppler effect.

In figure 4.6 we show the Fountain plot for the potassium-like decay, that is the contour plot of the FWHM of the line coming from a nucleus decay in the $(m_A, \delta m_A)$ plane. Remember that each value of $\delta m_A$ corresponds to an energy of the photon in the rest frame of $E^*_\gamma \approx \delta m_A$. 

37
**Figure 4.5:** The energy spectrum of the photon emitted via K XVIII transitions. Here we have again used the variable \( x = \frac{E_\gamma}{E_\gamma^*} \). Notice that the width of this spectrum is smaller than the one in figure 4.2. There is again the peak at \( x = 1 \).

**Figure 4.6:** The *Fountain plot* for the nucleus decay. The figure shows the contour plot of the FWHM of the photon’s energy spectrum in the \((m_A, \delta m_A)\) plane. Each value of \( \delta m_A \) corresponds roughly to the energy of the emitted photon in the rest frame of the excited nucleus. The horizontal red dashed line stands for an energy of 3.5 keV, which in the case of the sterile neutrino decay would correspond to \( \text{FWHM}_N \approx 0.0047 \text{ keV} \). The red line and the grid line of \( m_A = 40 \text{ GeV} \) intersect the contour line corresponding roughly to an FWHM of about 0.0013 keV. In the green boxes are shown the FWHMs expressed in keV corresponding to each contour line.
4.3 The novel DM mechanism

We move now to our new DM mechanism. This is essentially the combination of two processes: an inelastic up-scattering of the DM lighter state $\chi_1$ with the charged particles in the plasma (e.g. electrons and protons) and a DM-like decay of the resulting $\chi_2$ into $\chi_1$ and a photon $\gamma$. The features of the line are different depending on whether the inelastic up-scattering happens with electrons or protons, so we discuss the two cases separately. The parameters of the model are the mass of the lighter state $m_{\chi_1}$ and the mass splitting between the two states $\delta m_{\chi}$.

4.3.1 Inelastic up-scattering with electrons

The distributions for the electrons and the lighter state $\chi_1$ are respectively the same as the ones for the potassium and the sterile neutrino, what changes is only the mass:

\[
\begin{align*}
  f_{\chi_1}(p) &= \left(\frac{2}{27\pi}\right)^{1/2} m_{\chi_1}^{-3} \left\langle v_{DM}^2\right\rangle^{-3/2} p^2 e^{-\frac{3p^2}{2m_{\chi_1}\left\langle v_{DM}^2\right\rangle}}, \\
  f_e(p) &= \left(\frac{2}{\pi}\right)^{1/2} (m_eT_e)^{-3/2} p^2 e^{-\frac{p^2}{2m_eT_e}}.
\end{align*}
\]  

In figure 4.7 we show the two distributions in the same plot. We fix $m_{\chi_1} = 15$ MeV. The velocity dispersion of the electrons for a temperature of 1 keV is given by:

\[
\left\langle v_e^2\right\rangle = \left(\frac{3T_e}{m_e}\right) \approx 6 \times 10^{-3} \gg 10^{-6}.
\]

Starting from the distributions of the initial particles, we can derive the distribution in terms of the variables of the scattering, namely the energy and the momentum of the centre of mass. We did it in the previous chapter for the relativistic extension of the two distributions and we obtained (3.29), that can be interpreted as the distribution of a fictitious particle $Q$ with variable mass $\sqrt{s}$ and momentum $q$, describing the centre of mass. The scattering process can then be thought of as a decay of the object $Q$. We present in figure 4.8 the 3-dimensional plot of (3.29).
Figure 4.8: The 3-dimensional plot for the $f_Q(s,q)$ distribution given by equation (3.29) of the fictitious particle $Q$. As in the previous plot, we fix $m_{\chi_1} = 15$ MeV, $\langle v^2 \rangle^{1/2} = 10^{-3}$, $m_e = 0.511$ MeV and $T_e = 1$. The $s$ axis starts from $(m_{\chi_1} + m_e)^2 \approx 240.60$ MeV$^2$.

The volume subtended by the whole surface extending from $s = (m_{\chi_1} + m_e)^2$ to $+\infty$ is exactly 1 as it should be. It is good to keep in mind though that the integral of the distribution over the $s$ range of $[(m_{\chi_2} + m_e)^2, +\infty]$ is not equal to 1, meaning that there is a region in the momentum space for which values the inelastic up-scattering $\chi_1 e^- \rightarrow \chi_2 e^-$ cannot happen due to the lack of energy.

The probability density in the momentum space to find a $\chi_2$ particle after a collision of the initial particles is given by (3.31). For what we said in the previous paragraph, the integral over the energy of (3.31) is not equal to 1 and it gives instead the probability of an inelastic up-scattering to occur kinematically.

It is interesting to see how the probability distribution (3.31) changes with the mass $m_{\chi_1}$ and the splitting $\delta m_{\chi}$, as shown in figure 4.9. We underline here two important features. The first is that the higher the mass splitting is, the more the probability to find a scattered particle decreases and that is because the minimum energy to produce an inelastic scattering increases. The second is that for masses of about 500 MeV or greater, the zero splitting curve becomes the same as the distribution of the initial $\chi_1$ (black dashed line). This barely reflects the fact that for such great masses the electrons are not able to efficiently transfer momentum to the scattered DM particles. It is like throwing ping pong balls against bowling balls. Hence, in the case of a high $\chi_1$’s mass and a null mass splitting, it is like no scattering occurs. If the mass splitting is different from zero instead, the only consequence of the scattering would be the passage from the lighter state $\chi_1$ to the heavier $\chi_2$. Another way of saying it is that the scattered particle $\chi_2$ keeps a Maxwell-Boltzmann distribution, not normalized if the mass splitting is different from zero. Therefore, for $m_{\chi_1} \gtrsim 500$ MeV, we expect a line with the same width as for a simple DM-like decay, i.e. we could use (3.23) for $\chi_2$ to describe the spectrum. For lower masses, we expect a line with a drastically different width since the scattering becomes very efficient.

We are now able to understand the features of the photon’s energy spectrum in equation (3.33) produced via the novel mechanism with electrons. In figure 4.10 we show the Whale Tail plot, that is the energy spectrum of the photon in terms of the usual variable $x$ and as a function of the mass $m_{\chi_1}$ at a fixed mass splitting of 3.5 keV. The name of the plot comes from its similarity with a killer whale tail. Apart from its aesthetics, the plot in figure 4.10 exhibits directly the features of the line. First of all the peak at $x = 1$ remains, as in the other spectra, for every value of the mass. Secondly, it is clear that for low masses the line is widely spread.
Figure 4.9: The probability density in the momentum space to find a particle $\chi_2$ after a scattering with an electron. Each figure corresponds to a different mass value. The different colours correspond to different mass splittings, as explained in the legend. The red dashed line corresponds to a zero mass splitting, i.e. when the scattering is elastic and $\chi_2 \equiv \chi_1$. The black dashed line represents the Maxwell-Boltzmann distribution function for the initial particle $\chi_1$, which appears to be the same as the zero mass splitting curve for masses near the GeV, meaning that for such great masses the scattering with the electrons is so weak that there is essentially no momentum transfer to the scattered particle. It is clear from the figure that the higher the mass splitting is, the lower the probability density gets.
This reflects the fact that for sufficiently low masses of $\chi_1$, the electrons are able to efficiently transfer momentum to the scattered particle $\chi_2$, which then decays back emitting a photon $\gamma$ with a strong Doppler shift in the medium rest frame. The greater the mass is, the less efficient the scattering and so the spectrum starts to behave like the one for the simple decay.

Figure 4.11 is a list of plots representing the photon’s spectrum at different values of the mass and splitting. The black dotted curve represents the line we would have obtained from a Maxwell-Boltzmann distribution for $\chi_2$ and using (3.23). Starting from $m_{\chi_1} = 100$ MeV we plot also the normalized spectrum for the mass splitting $\delta m_{\chi} = 0.5$ keV (red dashed curve) to show how it behaves exactly like the one produced by a Maxwell-Boltzmann distribution for large masses. Therefore, for masses $m_{\chi_1} \gtrsim 500$ MeV, the line produced via the novel DM mechanism with electrons can be essentially described by (3.23) with mass $m_{\chi_2}$ and temperature $T_{\chi_2} = \frac{1}{3} m_{\chi_2} \langle v_{DM}^2 \rangle$ or by its non-relativistic form (3.24), with $z = 3/\langle v_{DM}^2 \rangle$.

We show in figure 4.12 two more intuitive plots representing the spectrum at different values of masses with fixed splitting and vice-versa.

We conclude this subsection with the *Fountain plot* for the novel DM mechanism with electrons, i.e. the contour plot of the FWHM of the line in the $(m_{\chi_1}, \delta m_{\chi})$ plane, shown in figure 4.13. The widths for typical values of the mass are clearly different from the ones of the two other mechanisms, confirming the fact that an eventual measurement of the FWHM could discriminate the origin of the line.
$\delta m_\chi = 3.5 \text{ keV}$

$\delta m_\chi = 2.0 \text{ keV}$

$\delta m_\chi = 0.5 \text{ keV}$

Normalized $\delta m_\chi$

From MB

$\chi^2$

$0.995$ $1.000$ $1.005$ $1.010$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 5 \text{ MeV}$

$0.995$ $1.000$ $1.005$ $1.010$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 15 \text{ MeV}$

$0.996$ $0.998$ $1.000$ $1.002$ $1.004$ $1.006$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 25 \text{ MeV}$

$0.998$ $1.000$ $1.002$ $1.004$ $1.006$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 50 \text{ MeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$ $1.003$ $1.004$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 100 \text{ MeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$ $1.003$ $1.004$ $1.006$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 500 \text{ MeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$ $1.003$ $1.004$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 1 \text{ GeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 500 \text{ MeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 1 \text{ GeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 500 \text{ MeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 1 \text{ GeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

$m_\chi^1 = 500 \text{ MeV}$

$0.998$ $0.999$ $1.000$ $1.001$ $1.002$

$x = \frac{E_\gamma}{E_{* \gamma}}$

$100$ $200$ $300$ $400$ $500$ $600$ $700$

$dP$ $dx$

Figure 4.11: A series of plots illustrating the different aspects of the line produced via the novel DM mechanism with electrons for different masses. The coloured solid curves represent the spectrum for different mass splittings. The black dotted line is the spectrum from a Maxwell-Boltzmann distribution for $\chi_2$, obtained using (3.23). Starting from $m_\chi^1 = 500 \text{ MeV}$, we plotted the normalized spectrum for a mass splitting of 3.5 keV. Clearly one can notice how it behaves like the Maxwellian spectrum for large masses, hence sharing the same line width.
\[ m_{\chi_1} = 30 \text{ MeV} \]
\[ m_{\chi_1} = 15 \text{ MeV} \]
\[ m_{\chi_1} = 10 \text{ MeV} \]
\[ m_{\chi_1} = 5 \text{ MeV} \]

**Figure 4.12:** Photon’s energy spectra at different values of \( m_{\chi_1} \) with fixed splitting \( \delta m_\chi \) (left) and vice-versa (right). In detail the peak at \( x = 1 \) for \( m_{\chi_1} = 15 \text{ MeV} \) and \( \delta m_\chi = 3.5 \text{ keV} \), still present in every other curve.

\[ \delta m_\chi = 0.5 \text{ keV} \]
\[ \delta m_\chi = 2 \text{ keV} \]
\[ \delta m_\chi = 3.5 \text{ keV} \]
\[ \delta m_\chi = 5 \text{ keV} \]

**Figure 4.13:** The *Fountain plot* for the novel DM mechanism with electrons. The figure shows the contour plot of the FWHM in the \((m_{\chi_1}, \delta m_\chi)\) plane. Each value of \( \delta m_\chi \) corresponds roughly to the energy of the emitted photon in the rest frame of the decaying \( \chi_2 \). The horizontal red dashed line stands for an energy of 3.5 keV, which in the case of a sterile neutrino decay would correspond to \( \text{FWHM}_N \approx 0.0047 \text{ keV} \), while for the K XVIII decay would lead to \( \text{FWHM}_{K^*} \approx 0.0013 \text{ keV} \). A mass of 15 MeV would give an FWHM of roughly 0.025 keV, about ten times greater than the values from the other processes.
Figure 4.14: The Maxwell-Boltzmann momentum distribution for the protons in the plasma ($m_p = 0.938$ GeV). The temperature is set to the value $T_p = 1$ keV.

4.3.2 Inelastic up-scattering with protons

The distribution for the protons in the plasma, plotted in figure 4.14, is peaked at a much greater momentum with respect to the DM's and electrons’ distributions.

At $T_p = 1$ keV, the dispersion velocity for the protons is much smaller than the electrons’ but still greater than the DM's one, indeed:

$$\langle v_p^2 \rangle = \left( \frac{3T_p}{m_p} \right) \approx 3 \times 10^{-6} < 6 \times 10^{-3}.$$

Contrarily to what happens with electrons, it is practically impossible for an inelastic up-scattering to occur if the mass of the DM particle $\chi_1$ is less then hundreds of MeV. This fact is clearly visible in figure 4.15: for masses lower than 500 MeV, the probability densities of the $\chi_2$ for a mass splitting of order keV are essentially null. The probability starts to rise for masses near the proton’s one ($m_p = 938$ MeV).

Based on what we found in the previous section and on the figure 4.15, we can guess a shape of the line similar to the electron case for DM’s masses in the range 500 MeV - 1 GeV.

An interesting thing happens for a mass $m_{\chi_1} = 3$ GeV: the distribution for the scattered particle at zero splitting coincides with the Maxwell-Boltzmann. That is because the temperatures of protons and DM are the same for such mass value, meaning that there is no effective exchange of momentum on average. If this was the case, then the resulting line would be the same as the one produced starting from a Maxwell-Boltzmann distribution and again we could describe it with (3.23). This is always the case for the XDM model ([27, 26]) where the $\chi_2$ particles are produced via the annihilation process $\chi_1 \chi_1 \rightarrow \chi_2 \chi_2$.

For masses greater than 3 GeV, the DM starts to lose energy on average, thus leading to a thinner line in the photon’s spectrum. Anyhow, the zero splitting curve behaves similarly to the Maxwell-Boltzmann distribution when the DM mass is near 2 GeV or above, thus leading to a line width that is not so different from the simple decay case.

We present in figure 4.16 the photon’s spectrum at different masses for mass splittings of 0.5 keV and 3.5 keV. We compute also the width for each line and the values can be read in the legend of the picture. The characteristics we have just anticipated in the previous paragraphs are evident.
Figure 4.15: The probability density in the momentum space to find a particle $\chi_2$ after a scattering with a proton. Each figure corresponds to different mass values. The different colours correspond to different mass splittings. The red dashed line corresponds to a zero mass splitting, i.e. when the scattering is elastic and $\chi_2 \equiv \chi_1$. The black dashed line represents the Maxwell-Boltzmann distribution function for the initial particle $\chi_1$, which is exactly the same as the zero mass splitting curve for a mass of 3 GeV. For masses lower than 3 GeV, the red dashed line is peaked after the Maxwellian, while for mass larger than 3 GeV is peaked before. Furthermore, as expected, the more the mass splitting is high, the lower the probability density gets.
Figure 4.16: The photon's energy spectrum for the novel DM mechanism with protons. The plot on top corresponds to a mass splitting of 0.5 keV and the other one to a mass splitting of 3.5 keV. The different coloured curves are the lines for different values of the mass, as specified in the legend. Here the respective values of the FWHM can be read too. At 3 GeV the line is the same as the one produced by a Maxwell-Boltzmann distribution for $\chi_2$. For masses below 3 GeV the line is wider, while for masses above is thinner, but with a width that is very similar to the Maxwellian case. In detail is shown the peak at $x = 1$ for the red curve.

4.4 Experimental spectra

We show in this section how the line width changes after the convolution of the theoretical spectrum with a Gaussian distribution for the data, as in equation (3.34). We consider different values for the variance $\sigma_E$ and report the changes in the width of a 3.5 keV line for the various cases in table 4.1. We expect resolutions of order 5 eV from the upcoming JAXA’s substitute of Hitomi: the XARM satellite [45, 30, 39]. Figure 4.17 shows the differences between the theoretical and experimental lines for the three different mechanisms.
\[
\sigma = \sigma_z \cdot E_{\gamma}^n
\]

Potassium-like decay

Neutrino-like decay

Novel DM mechanism

\[
\begin{array}{|c|c|c|c|}
\hline
\sigma_E & m_{K^*} = 40 \text{ GeV} & m_N = 7 \text{ keV} & m_{\chi_1} = 15 \text{ MeV, electrons} \\
\text{(eV)} & \text{(keV)} & \text{(keV)} & \\
0 & 0.0013 & 0.0047 & 0.0241 \\
1 & 0.0027 & 0.0053 & 0.0042 \\
5 & 0.0118 & 0.0127 & 0.0268 \\
10 & 0.0235 & 0.0239 & 0.0337 \\
100 & 0.2348 & 0.2348 & 0.2360 \\
\hline
\end{array}
\]

Table 4.1. Changes due to experimental uncertainties in the width of a 3.5 keV line arising from the three different processes. For \( \sigma_E = 100 \text{ eV} \), the experimental uncertainty fully dominates the width, making it practically impossible to discriminate the origin of the line. For values below 10 eV, the potassium-like and neutrino-like lines have comparable widths but still different, while our new mechanism remains clearly distinguishable. The zero value stands for the theoretical spectrum.

4.5 Morphologies from Perseus cluster

In the previous chapter we have found the equation (3.37) describing the morphology of the line, that is an expression for the differential \( J \)-factor assuming spherical symmetry. The function \( \mathcal{N} \) has a different form depending on the process one considers. For what concerns our work, we consider the usual three different cases:

\[
\mathcal{N} = \begin{cases} 
\frac{n_{A^*}}{m_{DM}} & \text{Decay of a nucleus } A^*; \\
\frac{\rho_{DM}}{m_{DM}} & \text{Decay of DM}; \\
\frac{n_{e,p} \cdot \rho_{DM}}{m_{DM}} & \text{Novel DM mechanism.}
\end{cases}
\]

(4.7)

Here \( n_{A^*} \) and \( n_{e,p} \) are respectively the number densities of the decaying nuclei, electrons and protons in the plasma; \( \rho_{DM} \) is the DM mass density in the halo.

We focus our study on the Perseus cluster as the source of the line. Perseus is characterized by:

\[
D \approx 73.6 \text{ Mpc},
\]

\[
R_g \approx 9.24 \text{ Mpc},
\]

where \( D \) is the distance from Earth and \( R_g \) is the radius of the cluster. As in [19], we consider a Navarro-Frenk-White density profile for the DM:

\[
\rho_{DM}(R) = \frac{\rho_0}{\left(\frac{R}{R_s}\right)^2 \left(1 + \frac{R}{R_s}\right)^{\gamma}},
\]

(4.8)

where \( R \) is the distance from the centre of the cluster, \( R_s = 445 \text{ kpc} \) and \( \rho_0 = 0.0217 \text{ GeV/cm}^3 \).

For the nuclei, electrons and protons in the plasma we assume a \( \beta \)-function number density profile:

\[
n_{A^*}(R) = n_{e,p}(R) = 3.9 \times 10^{-2} \text{ cm}^{-3} \left(1 + \left(\frac{R}{80 \text{ kpc}}\right)^2\right)^{1.8} + 4.05 \times 10^{-3} \text{ cm}^{-3} \left(1 + \left(\frac{R}{280 \text{ kpc}}\right)^2\right)^{0.87}.
\]

(4.9)

The resulting morphology of the lines for each process, i.e. \( dJ/dr \), are illustrated in figure 4.18. Clearly, there are differences due to the fact that the potassium-like decay is correlated to the distribution of the plasma only, the neutrino-like decay to the DM distribution in the galactic halos, our novel DM mechanism to both. Therefore another method to discriminate the origin of the line would be to analyze its morphology. Notice though that the \( J \)-factors defined as in our work have different dimensions depending on the process and that is due to
Figure 4.17: Differences between the theoretical (solid) and the experimental (dashed) 3.5 keV lines for a $\sigma_E$ of 1 eV and 5 eV. The masses of the particles are as in table 4.1.
Figure 4.18: The morphology of the lines, i.e. $dJ/dr$, for the three different processes discussed so far, namely the potassium-like decay in red, the neutrino-like decay in blue ($m_N = 7$ keV) and the novel DM mechanism in purple ($m_{\chi_1} = 10$ MeV). Remember that $r \approx \theta \cdot D$, where $\theta$ is the angle from the centre of the cluster and $D$ is the distance from Earth.

the different power of the number density that appears in the integral. In order to have a probability interpretation, that would permit a better comparison between the plots, one first needs to normalize the factors properly.

We compute also the $J$-factor after an integration over an angular region of $1'$ and $12'$ from the centre of the cluster:

$$J_1 \approx 4.23 \times 10^{16} \text{ cm}^{-2},$$

$$J_2 \approx 2.45 \times 10^{22} \left( \frac{7 \text{ keV}}{m_{DM}} \right) \text{ cm}^{-2},$$

$$J_3 \approx 8.42 \times 10^{16} \left( \frac{10 \text{ MeV}}{m_{DM}} \right) \text{ cm}^{-5}.$$
Chapter 5

An explicit realization of the novel DM mechanism

The last part of this thesis is about an explicit realization of our new DM mechanism. In particular, in this last chapter, we study the Effective Field Theory (EFT) described by D’Eramo et al. in [19]. We reproduce the most important results of [19], such as the main features of the EFT (sec. 5.1), the $\chi^2$’s lifetime (sec. 5.2), the annihilation cross-sections (subsec. 5.3.1), the relic density (subsec. 5.3.2) and the cross-section of the inelastic up-scattering (subsec. 5.4.1). We then conclude with the last original result (sec. 5.4.2), namely, we study the spectrum of the photon emitted via the novel DM mechanism for such EFT by applying the results of chapter 3. We make widely usage of two-component spinor notation and rules. Two clear reviews about such notation are [56] and [22].

5.1 Main features of our EFT

We add to the SM two gauge-singlet Weyl fermions described by two-component spinors $\xi$ and $\eta$, taken to be odd under a $Z_2$ symmetry. This implies that the lighter state is stable. We write the most general mass term as:

$$L_{\text{mass}} = \frac{-1}{2} \delta_\xi - \frac{1}{2} \delta_\eta + \text{h.c.}$$

(5.1)

where $\mu$, $\delta_\xi$ and $\delta_\eta$ are mass parameters. Then we introduce an effective interaction of the form

$$L_{\text{EFT}} = -\frac{1}{2\Lambda} \psi_D (c_M + i c_E \gamma^5) \Sigma^{\mu\nu} \psi_D F_{\mu\nu},$$

(5.2)

where the two-component spinors are grouped into one Dirac field, namely:

$$\psi_D = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

The antisymmetric tensor is defined as $\frac{1}{2} \Sigma^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$, while $F_{\mu\nu}$ is the electromagnetic tensor. Our EFT constructed in this way is valid only below a certain cutoff scale described by $\Lambda$, that can be interpret as the mass of some heavy particles. The mass parameters in (5.1) are taken consistently far below the cutoff scale. The operators with $c_M$ and $c_E$ (the Wilson coefficients) are CP-even magnetic and CP-odd electric dipole moments respectively.

5.1.1 Mass spectrum and interaction

To find the mass basis, first we rewrite the mass term in the following way:

$$L_{\text{mass}} = -\frac{1}{2} (\xi \eta) \begin{pmatrix} \delta_\xi & \mu \\ \mu & \delta_\eta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \text{h.c.}$$
From now on we assume the mass parameters to be real.

We need to diagonalize the mass matrix

\[ M \equiv \begin{pmatrix} \delta \xi & \mu \\ \mu & \delta \eta \end{pmatrix}. \]

The eigenvalues \( \lambda_{\pm} \) can be found by solving the characteristic polynomial:

\[ \lambda^2 - \lambda (\delta \xi + \delta \eta) - \mu^2 + \delta \xi \delta \eta = 0, \]

which leads to

\[ \lambda_{\pm} = \frac{\delta \xi + \delta \eta}{2} \pm \mu \sqrt{\frac{(\delta \xi - \delta \eta)^2}{4\mu^2} + 1}. \]

We define

\[ \varepsilon \equiv \frac{\delta \xi - \delta \eta}{2\mu}. \]

Therefore the eigenvalues can be rewritten as:

\[ \lambda_{\pm} = \frac{\delta \xi + \delta \eta}{2} \pm \mu \sqrt{1 + \varepsilon^2}. \]

We are interested in the \( \varepsilon \ll 1 \) limit. Thus we expand around \( \varepsilon = 0 \) getting

\[ \lambda_{\pm} = \frac{\delta \xi + \delta \eta}{2} \pm \mu + O(\varepsilon). \]

Two corresponding orthonormal eigenvectors are:

\[ v_+ = \frac{1}{2\sqrt{1 + \varepsilon^2 \left( \sqrt{1 + \varepsilon^2} - \varepsilon \right)}} \left( 1, \frac{1}{\sqrt{1 + \varepsilon^2} - \varepsilon} \right), \]

\[ v_- = \frac{1}{2\sqrt{1 + \varepsilon^2 \left( \sqrt{1 + \varepsilon^2} - \varepsilon \right)}} \left( \varepsilon - \sqrt{1 + \varepsilon^2}, 1 \right). \]

We can then write the matrix of the change of basis:

\[ O = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \frac{1}{2\sqrt{1 + \varepsilon^2 \left( \sqrt{1 + \varepsilon^2} - \varepsilon \right)}} \left( \begin{array}{cc} 1 & \frac{1}{\sqrt{1 + \varepsilon^2} - \varepsilon} \\ \varepsilon - \sqrt{1 + \varepsilon^2} & 1 \end{array} \right), \]

where \( \theta \) is such that

\[ \tan 2\theta = 1/\varepsilon. \]

Therefore in the mass basis we have:

\[ M' = OMO^T = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \]

and the eigenstates are given by:

\[ O \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \cos \theta + \eta \sin \theta \\ \eta \cos \theta - \xi \sin \theta \end{pmatrix}. \]

We introduce the two mass eigenstates and their limit for \( \varepsilon \to 0 \) as follows:

\[ \chi_1 = i \left( \eta \cos \theta - \xi \sin \theta \right) \xrightarrow{\varepsilon \to 0} \frac{i}{\sqrt{2}} (\eta - \xi), \]

\[ \chi_2 = \left( \xi \cos \theta + \eta \sin \theta \right) \xrightarrow{\varepsilon \to 0} \frac{1}{\sqrt{2}} (\xi + \eta). \]
In the mass basis, the mass term (5.1) in the Lagrangian becomes:

\[ \mathcal{L}_{\text{mass}} = -\frac{1}{2} \left[ -\lambda_- \chi_1 \chi_1 + \lambda_+ \chi_2 \chi_2 \right] + \text{h.c.}, \]

where

\[-\lambda_- \xrightarrow{\varepsilon \to 0} \mu - \frac{\delta \xi + \delta \eta}{2} \equiv m_{\chi_1}, \]
\[\lambda_+ \xrightarrow{\varepsilon \to 0} \mu + \frac{\delta \xi + \delta \eta}{2} \equiv m_{\chi_1} + \delta m_{\chi} = m_{\chi_2}.\]

We can now rewrite (5.2) in terms of the mass eigenstates (5.3). It is convenient to use the Weyl representation for the gamma matrices, which can then be written as:

\[ \gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}. \]

The antisymmetric tensor becomes:

\[ \frac{1}{2} \Sigma^{\mu\nu} = \begin{pmatrix} (\sigma^{\mu\nu})_\beta^\alpha & 0 \\ 0 & (\bar{\sigma}^{\mu\nu})_\beta^\alpha \end{pmatrix}. \]

We can write the interaction eigenstates in terms of the mass ones:

\[ \xi = \frac{1}{\sqrt{2}} (\chi_2 + i \chi_1), \]
\[ \eta = \frac{1}{\sqrt{2}} (\chi_2 - i \chi_1). \]

We plug all these expansions in (5.2) and explicit the spinor indices, getting to the following result:

\[ \mathcal{L}_{\text{EFT}} = -\frac{1}{\Lambda} \left[ i \chi_2^\alpha (c_M - ic_E) (\sigma^{\mu\nu})_\beta^\alpha \chi_1 \beta + i \chi_2^\dagger_\alpha (c_M + ic_E) (\bar{\sigma}^{\mu\nu})_\beta^\dagger \chi_1 \beta \right] F_{\mu\nu}, \]

where we have used the anticommuting properties of the Weyl fermions. We can combine the mass eigenstates into two Majorana four-components spinors as

\[ \psi_1 = \begin{pmatrix} \chi_1 \\ \chi_1^\dagger \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} \chi_2 \\ \chi_2^\dagger \end{pmatrix}. \]

and use them to write in a better form the interaction term for the mass eigenstates:

\[ \mathcal{L}_{\text{EFT}} = -\frac{i}{2\Lambda} \bar{\psi}_2 (c_M + ic_E) \Sigma^{\mu\nu} \psi_1 F_{\mu\nu}. \quad (5.4) \]

The latter form is useful when the effects of the mass splitting are relevant. This is not the case for the relic density, for which we will use (5.2).

The presence of the电磁磁 displacement tensor in (5.2) or (5.4) allows processes which involve both DM and SM particles. The processes we are interested in are the annihilation into SM particles, which are necessary to study the present relic abundance of DM, and the ones that constitute our novel mechanism, namely the scattering with electrons or protons and the decay \( \chi_2 \rightarrow \chi_1 \gamma \). In the following sections, we derive the decay width and the cross-sections for these processes. Everytime the mass splitting is negligible, we use \( m_\chi \) instead of \( m_{\chi_1} \) and \( m_{\chi_2} \), since the latter would be the same.
5.2 Rate of the decay and $\chi_2$’s lifetime

The mass splitting plays a crucial role in the decay of the heavier state $\chi_2$ into the lighter $\chi_1$, hence we use (5.4) to describe the interaction. The process is described by the following Feynman diagram (time flows from left to right):

where the crossed circle denotes the EFT interaction. The diagram is simply the vertex of our EFT. The explicit computation for the decay width is presented in Appendix B. Here we report only the most significative passages.

The amplitude of the process is:

$$M = -\frac{1}{\Lambda} k_\mu e_\nu^\lambda \left[ \overline{u}_1^s \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu\nu} u_2^{s'} \right],$$

which leads to the following averaged Feynman amplitude squared:

$$|M|^2 = \frac{1}{2\Lambda^2} k_\mu k_\nu \sum_{s,s',\lambda} \varepsilon^\lambda e_\sigma^\ast \left[ \overline{u}_1^s \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu\nu} u_2^{s'} \right] \left[ \overline{u}_2^{s'} \left( c_M + ic_E \gamma^5 \right) \Sigma^{\sigma\nu} u_1^s \right].$$

Remembering that

$$\sum_{\lambda} \varepsilon^\lambda e_\sigma^\ast = -g_{\mu\sigma}, \text{ and } \sum_{s,s'} u_1^{s'} \overline{u}_1^{s'} = \left( p_{1,2}^1 + m_{\chi_1,\chi_2} \right),$$

and performing the resulting trace, we get that

$$|M|^2 = \frac{8 \left( c_M^2 + c_E^2 \right)}{\Lambda^2} (k \cdot p_1) (k \cdot p_2).$$

In the CM, all the appearing scalar products between the momenta can be easily computed. Thanks to the formula (3.3) we can get the energy of the photon in the CM in terms of the mass parameters. The integration of (3.8) over the solid angle gives then the decay rate:

$$\Gamma_{\chi_2 \to \chi_1 \gamma} = \frac{c_M^2 + c_E^2}{8\pi \Lambda^2} \left( \frac{\delta m_X + 2m_{\chi_1}}{m_{\chi_2}^3} \right)^3 \delta m_X \delta m_{\chi_2} \approx \frac{c_M^2 + c_E^2}{\pi \Lambda^2} \delta m_X^3. \quad (5.6)$$

The inverse of the decay rate gives the lifetime of the DM heavier state $\chi_2$:

$$\tau_{\chi_2 \to \chi_1 \gamma} \simeq 9.65 \times 10^{-4} \text{ s} \left( \frac{2}{c_M^2 + c_E^2} \right) \times \left( \frac{3.5 \text{ keV}}{\delta m_X} \right)^3 \times \left( \frac{\Lambda}{200 \text{ GeV}} \right)^2. \quad (5.7)$$

In figure 5.1 we show some contour lines of the lifetime in the $(\Lambda, \delta m_X)$ plane.

5.3 Annihilation processes and relic density

The present amount of DM is essentially given by the abundance of $\chi_1$. Indeed, once the heavier state $\chi_2$ forms, it decays back very quickly to the lighter state (see figure 5.1 to have an idea of the $\chi_2$’s typical lifetime). As a consequence, the effects of the mass splitting in the relic abundance are completely negligible, which means that we can confuse the two states as one. Therefore, we use (5.2) to describe the annihilation processes, which are the ones that provide the kinetic equilibrium in the primordial thermal bath.
5.3.1 Annihilation cross-sections

There are three different permitted kinds of annihilation: into leptons, into hadrons and into photons. Let us first consider the annihilation into leptons. Again the full calculation is given in Appendix B. The process is described by the following Feynman diagram:

\[
\begin{align*}
\chi & \rightarrow \ell^- + \ell^+ \\
p_1, s & \rightarrow k_1, r' \\
p_2, s' & \rightarrow k_2, r \\
q & = p_1 + p_2
\end{align*}
\]

The amplitude is

\[
\mathcal{M} = -\frac{e}{\Lambda} \left[ \bar{v}_\ell(k_1) \gamma^\mu v^\mu(k_2) \right] \times \left[ \frac{g_{\mu\nu}}{s} \right] \times \left[ \bar{\chi}(p_1) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho\nu} q_\rho u_\chi'(p_2) \right].
\]

The averaged Feynman amplitude squared reads:

\[
\left| \mathcal{M} \right|^2 = \frac{e^2}{4\Lambda^2 q^4} q_\rho q_\sigma \left| \bar{v}_\ell(k_1) \gamma^\mu v^\mu(k_2) \right| \times \left( k_1 + m_\ell \right) \gamma_\mu \left( k_2 - m_\ell \right) \gamma_\nu \times \left| \bar{\chi}(p_1) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho\nu} \phi_\chi' + m_\chi \right) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho\nu}.\]

All the scalar products between the momenta can be easily computed in the CM, where we have also an expression for \( s \) in terms of the relative velocity between the two initial DM particles (B.8). The differential cross-section times the relative velocity can be obtained from (3.15) with the appropriate substitutions. The integration over the solid angle gives \( \sigma v_{\text{rel}} \). Here we present the result at the first not vanishing order in \( v_{\text{rel}}^2 \), including both the contributions from the magnetic and electric dipole interactions:

\[
\sigma v_{\text{rel}} \bigg|_{\chi \chi \rightarrow \ell^+ \ell^-} \simeq \frac{\alpha_{EM}}{\Lambda^2} \left( \frac{c_M^2 + c_E^2 v_{\text{rel}}^2}{12} \right) \left( 1 - \frac{m_\ell^2}{m_\chi^2} \right)^{1/2} \left( 1 + \frac{m_\ell^2}{2m_\chi^2} \right),
\]

where \( \alpha_{EM} = e^2/4\pi \) is the coupling constant of electromagnetism. In the non-relativistic limit, i.e. \( v_{\text{rel}} \ll 1 \), the magnetic interaction is dominant while the electric one is suppressed by a
factor of $v_{\text{rel}}^2$. The cross-section for the annihilation into hadrons can be evaluated from the ratio
\begin{equation}
\mathcal{R}_h(\sqrt{s}) = \frac{\sigma_{e^+e^-\rightarrow \text{hadrons}}}{\sigma_{e^+e^-\rightarrow \mu^+\mu^-}} = \frac{\sigma_{\chi\chi\rightarrow \text{hadrons}}}{\sigma_{\chi\chi\rightarrow \mu^+\mu^-}}.
\end{equation}

The numerical value for $\mathcal{R}_h(\sqrt{s})$ can be found in the Particle Data Group webpage. Hence the cross-section for the annihilation into hadrons reads:
\begin{equation}
\sigma_{\text{rel}} \big|_{\chi\chi\rightarrow \text{hadrons}} = \mathcal{R}_h(\sqrt{s} = 2m_{\chi}) \times \sigma_{\chi\chi\rightarrow \mu^+\mu^-}.
\end{equation}

The full computation for the cross-section of the annihilation into leptons is presented in Appendix B. Here we show the result, which is suppressed by a factor $1/\Lambda^3$ coming from the two EFT vertices in the Feynman diagram:
\begin{equation}
\sigma_{\text{rel}} \big|_{\chi\chi\rightarrow \ell^+\ell^-} \approx \frac{(c_M^2 + c_E^2)^2}{4\pi\Lambda^4} m_{\chi}^2.
\end{equation}

### 5.3.2 DM relic density

The DM decoupled from the thermal bath through the freeze-out mechanism producing the relic abundance we observe today. The freeze-out occurred when the annihilations were not able to preserve the equilibrium anymore and this happened for $x_f \approx 10$, which means, for masses of $m_{\chi} \approx 10$ MeV $- 1$ GeV, temperatures of $T_{\chi} \approx 1 - 100$ MeV, thus near the QCD phase transitions. As a consequence, we cannot ignore the dependence of the relativistic degrees of freedom $g_*$ and $g_{*, \text{s}}$ upon the temperature.

In order to compute the exact relic density today, we need to solve the Boltzmann equation, which is here given in terms of the variable $Y_{\chi} = n_{\chi}/s$, where $n_{\chi}$ is the number density of DM and $s$ is the entropy density:
\begin{equation}
\frac{dY_{\chi}}{dx} = -\sqrt{\frac{8\pi^3}{45}} M_p m_{\chi} g_{*, \text{s}} \left( \frac{m_{\chi}}{x} \right)^{1/2} \left( \frac{m_{\chi}}{s} \right) \frac{\langle \sigma v \rangle_{\text{rel}}}{x^2} \left[ 1 - \frac{x}{3} g_{*, \text{s}} \left( \frac{m_{\chi}}{x} \right) \frac{d}{dx} g_{*, \text{s}} \left( \frac{m_{\chi}}{x} \right) \right] \left[ Y_{\chi}^2 - (Y_{\chi}^{eq})^2 \right],
\end{equation}

where $x = m_{\chi}/T_{\chi}$. Here the Planck mass is defined as $M_p = \sqrt{1/8\pi G}$. A full derivation of equation (5.11) is given in Appendix C.

We need the thermal average of the cross-section. This can be easily evaluated from (B.10) by using the fact that
\begin{equation}
\langle v_{\text{rel}}^2 \rangle = \langle v_1^2 + v_2^2 - 2v_1v_2\cos\theta \rangle = 2 \langle v_1^2 \rangle = \frac{6T_{\chi}}{m_{\chi}},
\end{equation}

where we have assumed isotropy and used the equipartition theorem. Therefore the averaged cross-section for the annihilation into leptons reads:
\begin{equation}
\langle \sigma_{\text{rel}} \rangle \big|_{\chi\chi\rightarrow e^+e^-} \approx \frac{\alpha_{\text{EM}}}{\Lambda^2} \left( c_M^2 + c_E^2 \right) \left( 1 - \frac{m_{\chi}^2}{m_{\chi}^2} \right)^{1/2} \left( 1 + \frac{m_{\chi}^2}{2m_{\chi}^2} \right).
\end{equation}

The contributions from the annihilation into particles heavier than electrons must be implemented if kinematically allowed. We can ignore the contribution given by the annihilation into photons since it is suppressed by a fourth negative power of $\Lambda$.

Equation (5.11) can be solved numerically and figure 5.2 illustrates the solution for a mass of 10 MeV, at different values of $\Lambda$ and for purely electric (5.2a) and magnetic (5.2b) dipole interactions.

The number density today can be expressed as
\begin{equation}
n_{\chi}^\infty = 2 \cdot s_0 \cdot Y_{\chi}^\infty,
\end{equation}

56
Figure 5.2: Numerical solutions of the Boltzmann equation (5.11) for $Y_\chi(x)$ in a logarithmic scale and for a mass of 10 MeV. The dashed coloured lines correspond to different values of $\Lambda$. Plot (a) is for a purely electric while (b) for a purely magnetic dipole interaction. The vertical black dashed lines correspond to $x = 1$, marking the border between the relativistic and the non-relativistic regimes. The pictures show that the freeze-out occurs for $x_f \sim 10$. The blue line is the equilibrium curve $Y^{eq}_\chi(x)$. 
Figure 5.3: The plot shows the present DM relic abundance $\Omega_\chi h^2$ as a function of the effective parameter $\Lambda$, for different mass values (10 MeV, 100 MeV and 1 GeV), purely electric (dashed) dipole interaction with $c_E = 1$ and magnetic (solid) with $c_M = 1$. The 1 GeV lines are very separated from the others due to the annihilation into hadrons that entered the scene. The black dotted horizontal line corresponds to the DM abundance we observe today: $\Omega_{DM} h^2 = 0.120 \pm 0.001$.

where $s_0 = 2891.2 \text{ cm}^{-3}$ is the current entropy density. The factor 2 accounts for the fact that we are dealing with Dirac fermions. The relic abundance today is given by

$$\Omega_\chi h^2 = \frac{m_\chi n_\chi^\infty}{\rho_c / h^2},$$

(5.14)

where the critical density is

$$\rho_c = 1.05375 \times 10^{-5} h^2 \text{ GeV cm}^{-3}.$$

In figure 5.3 we plot the relic density as a function of $\Lambda$ for different mass values and for purely electric (dashed) and magnetic (solid) dipole interactions and we compare the results with the observed value $\Omega_{DM} h^2 = 0.120 \pm 0.001$ measured by Planck (2018) [3]. We choose $c_E$ and $c_M$ to be equal to 1. However, the result for different values can be obtained simply dividing by $c_{E,M}^2$. 
5.4 The inelastic up-scattering

5.4.1 Scattering cross-section

We are now ready to discuss the essence of our novel mechanism, namely the inelastic up-scattering of DM with the fermions in the plasma. The process is described by the Feynman diagram below:

![Feynman diagram](image)

Again the full computation is given in Appendix B; here we present the most significant results only. First of all it is useful to analyze the case of a vanishing mass splitting, so to have a taste of what the peculiar features are. The differential cross-section in the CM for the electric and magnetic dipole moments separately results in:

\[
\frac{d\sigma}{d\Omega^\ast} \bigg|_{c_M=0} \simeq c^2_E \frac{\alpha_{EM}}{4\pi\Lambda^2} \frac{1}{v^2_{rel} \sin^2(\theta^\ast/2)},
\]

\[
\frac{d\sigma}{d\Omega^\ast} \bigg|_{c_E=0} \simeq c^2_M \frac{\alpha_{EM}}{4\pi\Lambda^2} \frac{1 + \frac{m_\chi - 2m_{e,p}}{m_\chi + m_{e,p}} \sin^2(\theta^\ast/2)}{\sin^2(\theta^\ast/2)}.
\]

The two formulas above tell us that in the non-relativistic limit the electric dipole interaction dominates over the magnetic one, contrarily to what happens for the annihilation processes. Both are divergent at small $\theta$ though and moreover the electric cross-section is also divergent at small velocities. Fortunately, $\delta m_\chi$ regularizes these divergences, which then disappear once the mass splitting is different from zero.

The cross-section times the relative velocity thermically averaged for the case of a not vanishing mass splitting can be expressed in terms of two complicated function $\Sigma_E$ and $\Sigma_M$, respectively referring to the electric and magnetic dipole interactions:

\[
\langle \sigma v_{rel} \rangle \bigg|_{\chi_1 e^-/p^+ \to \chi_2 e^-/p^+} = \frac{e^2}{\Lambda^2} \left[ c^2_E \Sigma_E(m_\chi, \delta m_\chi) + c^2_M \Sigma_M(m_\chi, \delta m_\chi) \right].
\]

Figure 5.4 shows the electric and magnetic functions $\Sigma_{E,M}(m_\chi, \delta m_\chi)$ at a mass splitting of 1 keV, with the temperature of the plasma fixed at 5 keV. Clearly one can see that the electric cross-section dominates at low masses.

Figure 5.5 shows instead the behaviour of $\Sigma_E$ and $\Sigma_M$ as the temperature of the plasma grows and at different mass splittings. We fix the mass of the lighter state at 15 MeV. Interestingly, there is an exponential growth of the cross-section for temperatures near the value of the mass splitting, meaning that the process is enhanced if $T_e \sim \delta m_\chi$.

For each of the last plots we fix the DM dispersion velocity at $v_0^2 \equiv \frac{2}{3} \langle v^2 \rangle = 10^{-6}$. 
\[ \delta m_\chi = 1 \text{ keV} \]

\[ 10^{-10} 10^{-8} 10^{-6} 10^{-4} 0.01 1 100 10^{50} 10^{100} 10^{500} 10^{1000} \]

\( m_\chi \) (MeV)

\( \Sigma \)

\( \Sigma_E, \text{ Electron} \)

\( \Sigma_M, \text{ Electron} \)

\( \Sigma_E, \text{ Proton} \)

\( \Sigma_M, \text{ Proton} \)

\[ \mu^* = \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{p}_1| |\vec{p}_2|} \]

\[ S(s, \mu^*) \equiv \frac{d\sigma_{rel}}{d\mu^*}. \]  

(5.17)

\[ \langle \frac{d\sigma_{rel}}{dE_2} \rangle = \int_{(m_{\chi_2}+m_e)^2}^{+\infty} ds \int_1^{+\infty} d\gamma_q f_Q(s, \gamma_q) S(s, \mu^*(s, \gamma_q, E_2)), \]  

(5.18)

where \( \mu^*(s, \gamma_q) \) can be found by solving the condition imposed by the Dirac delta, namely

\[ \mu^*(s, \gamma_q, E_2) = \frac{E_2 - \gamma_q E_q^2(s)}{\sqrt{\gamma_q^2 - 1} \sqrt{E_2^2(s) - m_{\chi_2}^2}}. \]

The spectrum then becomes:

\[ \frac{1}{J} \frac{d\Phi_\gamma}{dx} = m_{\chi_2} \int_{\frac{1}{2}(x+\frac{1}{2})}^{+\infty} \frac{1}{\sqrt{\gamma_2^2 - 1}} \times \]

\[ \times \int_{(m_{\chi_2}+m_e)^2}^{+\infty} \int_{\gamma_q^-}^{\gamma_q^+} f_Q(s, \gamma_q) S(s, \mu^*(s, \gamma_q, m_{\chi_2} \gamma_2)) \frac{d\gamma_q}{\sqrt{\gamma_q^2 - 1} \sqrt{E_2^2(s) - m_{\chi_2}^2}} d\gamma_2 d\gamma_q ds. \]  

(5.19)

where \( f_Q(s, \gamma_q) \) shall be taken as (3.28). Be aware that (5.19) has the dimensions of an area, contrarily to (3.33), which is dimensionless.

**Figure 5.4:** The electric (solid) and magnetic (dashed) functions \( \Sigma_{E,M}(m_\chi, \delta m_\chi) \) at a fixed mass splitting \( \delta m_\chi = 1 \text{ keV} \). The blue curves are for the scattering with the electrons while the red ones with the protons. The temperature of the plasma is fixed at 5 keV, the DM dispersion velocity instead is such that \( v_0 = 10^{-3} \).

### 5.4.2 Changes in the photon’s energy spectrum

It is now time to evaluate what the changes in the energy spectrum of the photons emitted via the novel DM mechanism are, as a consequence of the dependence of the differential cross-section times the relative velocity upon the energy of the process and the angle of emission. We consider the electric dipole interaction only, since it dominates at small velocities and masses. The differential cross-section times the relative velocity is a function of \( s \) and \( \mu^* \equiv \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{p}_1| |\vec{p}_2|} \):

\[ S(s, \mu^*) \equiv \frac{d\sigma_{rel}}{d\mu^*}. \]  

Therefore, (3.17) after the integration over \( \mu^* \) becomes:

\[ \langle \frac{d\sigma_{rel}}{dE_2} \rangle = \int_{(m_{\chi_2}+m_e)^2}^{+\infty} ds \int_1^{+\infty} d\gamma_q f_Q(s, \gamma_q) S(s, \mu^*(s, \gamma_q, E_2)), \]  

where \( \mu^*(s, \gamma_q) \) can be found by solving the condition imposed by the Dirac delta, namely

\[ \mu^*(s, \gamma_q, E_2) = \frac{E_2 - \gamma_q E_q^2(s)}{\sqrt{\gamma_q^2 - 1} \sqrt{E_2^2(s) - m_{\chi_2}^2}}. \]

The spectrum then becomes:

\[ \frac{1}{J} \frac{d\Phi_\gamma}{dx} = m_{\chi_2} \int_{\frac{1}{2}(x+\frac{1}{2})}^{+\infty} \frac{1}{\sqrt{\gamma_2^2 - 1}} \times \]

\[ \times \int_{(m_{\chi_2}+m_e)^2}^{+\infty} \int_{\gamma_q^-}^{\gamma_q^+} f_Q(s, \gamma_q) S(s, \mu^*(s, \gamma_q, m_{\chi_2} \gamma_2)) \frac{d\gamma_q}{\sqrt{\gamma_q^2 - 1} \sqrt{E_2^2(s) - m_{\chi_2}^2}} d\gamma_2 d\gamma_q ds. \]  

(5.19)
Figure 5.5: The electric (solid) and magnetic (dashed) functions $\Sigma_{E,M}(m_\chi, \delta m_\chi)$ as functions of the temperature of the plasma, for a mass of $m_{\chi_1} = 15 \text{ MeV}$ and at different mass splittings. Below the mass splitting value, the curves die exponentially, meaning that the process is enhanced if $T_e \sim \delta m_\chi$. 

$m_{\chi} = 15 \text{ MeV}$

- Electron, $\delta m_\chi = 0.5 \text{ keV}$
- Electron, $\delta m_\chi = 2.0 \text{ keV}$
- Electron, $\delta m_\chi = 3.5 \text{ keV}$
In figure 5.6 we show the resulting energy spectrum of the photon emitted via the novel DM mechanism with electrons in comparison with the one obtained by assuming a constant and isotropic differential cross-section times relative velocity, as in chapter 3. We fix the mass at 15 MeV, the splitting at 3.5 keV, the temperature of the plasma at 1 keV and the DM dispersion velocity at $10^{-3}$. We compute also the width of the line, which does not differ much from the value obtained without specifying the model.

\[
m_\chi = 15 \text{ MeV}, \delta m_\chi = 3.5 \text{ keV}
\]

**Figure 5.6:** Comparison between the lines from the novel DM mechanism with electrons assuming constant (dashed) and electric dipole (solid) cross-sections times relative velocity. Clearly, the widths of the two lines do not differ much from each other. The solid curve has been made adimensional by rescaling it with the maximum at $x=1$.  

Conclusions

Measuring the total flux of incoming photons is not enough to convincingly determine the origin of X-ray excitation lines from sources like clusters of galaxies. To permit a better classification of the phenomenon that generates lines of this kind one needs to measure other features like the width and the morphology. In this thesis, we have indeed shown that different mechanisms produce lines with peculiar energy spectrum and morphology. After having described a novel DM mechanism to produce such X-ray lines, we have examined also the case of an elemental transition like the K XVIII and the sterile neutrino decay into an active neutrino and a photon. We have found formulas for the differential fluxes of incoming photons for each of the three processes.

Firstly, from these formulas, we have extracted information about the photon’s energy spectrum by assuming a non-relativistic Maxwell-Boltzmann distribution for each initial state. We have then compared the resulting lines for each of the three mechanisms and computed the FWHMs for various mass parameters. We have in this way noticed clear differences in the values of the line widths. The decays produce lines with comparable but still different widths because the nuclei in the plasma and the DM particles in the halos have different velocities. Our novel mechanism instead, which is characterized by a DM inelastic up-scattering with the charged particles in the plasma followed by a DM decay, produces lines that are widely spread for appropriate mass ranges and the FWHMs are typically one order of magnitude greater than the ones from decays.

The analyses up to this point have regarded the theoretical photon’s energy spectrum only and no experimental effects have been taken into account yet. The latter, however, can contribute substantially to the line width. Therefore, we have considered the experimental spectrum to be a convolution of the theoretical one with a Gaussian describing the data of an eventual experiment and then computed again the FWHMs for the different cases. We found that below 10 eV of resolution the lines are still distinguishable, whereas for 100 eV the experimental uncertainties fully dominate the spectra. Future satellites like JAXA’s XARM will probably have the right energy resolution to clearly make such measurements.

From the formulas for the differential fluxes, we have also extrapolated an expression for the morphology that describes how the line is correlated to the spatial distribution of the particles in the initial states. We have seen that, for the Perseus cluster, assumed to be spherical for simplicity, and by considering a Navarro-Frenk-White density profile for the mass density of DM in halos and a β-function number density for the particles in the plasma, the morphologies for the three different mechanisms have different behaviours, as expected.

What has been said so far, when referring to our novel DM mechanism, is a model-independent description. We have in fact considered at first instance an isotropic and energy independent differential cross-section times relative velocity for the inelastic up-scattering, so to have no contribution to the photon’s energy spectrum from it. Such assumption is not usually valid but has permitted us to make an a priori description, in the hope that, once we would have specified the underlying subatomic model, the corrections to the line width would have been small. Indeed, that is what we have found for the EFT described in the last chapter.

In conclusion, our work has confirmed that the width and the morphology of X-ray excitation lines are discriminating factors when one wants to infer about their origin. Further improvements can be made by studying other interesting X-ray production mechanisms with their respective underlying models and by considering different density profiles for various sources.
We hope that experiments scheduled for the near future, thanks also to the results presented here, could probe our scenario, or at least permit to discriminate whether astrophysical X-ray signals can be explained by elemental transitions or if new exotic physics is needed.
Appendix A

Brief review of the SM’s EW sector

The Standard Model of particle physics (SM) is a four-dimensional renormalizable quantum field theory apt to describe three of the four fundamental forces of Nature known so far, namely the electromagnetic, the weak and the strong interactions, gravity excluded. It provides also a classification of all the known subatomic elementary particles. It is a gauge theory whose local symmetry group is

$$SU(3)_C \times SU(2)_L \times U(1)_Y.$$ 

$SU(2)_L \times U(1)_Y$ is the gauge group of the electroweak interaction, i.e. the unification of the electromagnetic and the weak forces. The letter $L$ stands for “left”, meaning that only the left-handed particles, namely the eigenstates of the left chirality operator $P_L = (1 - \gamma^5)/2$, interact with the $SU(2)_L$ gauge bosons. The charges related to the $SU(2)_L$ symmetry are the three components of the weak isospin ($I$). The letter $Y$ stands for the weak hypercharge, which is the charge associated to the $U(1)_Y$ symmetry and it is connected to the third component of the isospin and the electric charge ($Q$) thanks to the relation

$$\frac{Q}{e} = I_3 + Y.$$ \hspace{1cm} (A.1)

The elementary particles which undergo the electroweak interaction are all the elementary fermions: leptons and quarks. The leptons can be divided into three families, each characterized by the same leptonic flavour: the electron ($e$) and the electronic neutrino ($\nu_e$) are the electronic leptons; the muon ($\mu$) and the muonic neutrino ($\nu_\mu$) are the muonic leptons; the tauon ($\tau$) and the tauonic neutrino ($\nu_\tau$) are the tauonic leptons. The left-handed leptons can be grouped into three different $SU(2)_L$ doublets, while the right-handed ones are singlets:

$$L_i^L = \begin{pmatrix} \nu_i^L \\ e_i^L \end{pmatrix}, \begin{pmatrix} \nu_i^L \\ \mu_i^L \end{pmatrix}, \begin{pmatrix} \nu_i^L \\ \tau_i^L \end{pmatrix} \quad (\text{with } i = e, \mu, \tau);$$

$$E_i^R = e^R, \mu^R, \tau^R \quad (\text{with } E_i = e, \mu, \tau).$$

There is no need to consider the right-handed neutrinos in the SM, because they would not interact with any other particle since they would be completely chargeless. The mediators of the electroweak force are the massive $W^\pm$ and $Z$ bosons and the photon $A$.

$SU(3)_C$ is the gauge group of Quantum Chromodynamics (QCD) which describes the strong interaction; here $C$ stands for “colours”, which are the charges associated to the group symmetry. The fundamental particles that interact under the strong force are the quarks: up ($u$), down ($d$), charm ($c$), strange ($s$), top ($t$), bottom ($b$). The quarks are also affected by the electroweak interaction, indeed the left-handed eigenstates can be grouped, just like the leptons, into three different $SU(2)_L$ doublets, while the right-handed ones are singlets:

$$Q_j^L = \begin{pmatrix} u_j^L \\ d_j^L \end{pmatrix}, \begin{pmatrix} c_j^L \\ s_j^L \end{pmatrix}, \begin{pmatrix} t_j^L \\ b_j^L \end{pmatrix} \quad (\text{with } j = 1, 2, 3);$$

$$U_j^R = u^R, t^R, D_j^R = d^R, s^R, b^R \quad (\text{with } U_j = u, c, t, D_j = d, s, b).$$
The mediators of the strong force are 8 different types of massless bosons: the gluons.

Anyway, for what concerns neutrino physics, we are not interested in the strong sector of the SM, so we focus the attention on the electroweak sector only. Table 2.1. is a list of all the elementary particles that interact electroweakly with their respective Q, I$_3$ and Y.

<table>
<thead>
<tr>
<th>Particles</th>
<th>Q/e</th>
<th>I$_3$</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_L^e$, $\nu_L^\mu$, $\nu_L^\tau$</td>
<td>0</td>
<td>+1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>$e^L$, $\mu^L$, $\tau^L$</td>
<td>-1</td>
<td>-1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>$e^R$, $\mu^R$, $\tau^R$</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$u^L$, $c^L$, $t^L$</td>
<td>+2/3</td>
<td>+1/2</td>
<td>+1/6</td>
</tr>
<tr>
<td>$d^L$, $s^L$, $b^L$</td>
<td>-1/3</td>
<td>-1/2</td>
<td>+1/6</td>
</tr>
<tr>
<td>$u^R$, $c^R$, $t^R$</td>
<td>+2/3</td>
<td>0</td>
<td>+2/3</td>
</tr>
<tr>
<td>$d^R$, $s^R$, $b^R$</td>
<td>-1/3</td>
<td>0</td>
<td>-1/3</td>
</tr>
<tr>
<td>$W^\pm$</td>
<td>$\pm 1$</td>
<td>$\pm 1$</td>
<td>0</td>
</tr>
<tr>
<td>$Z$, $A$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$H$</td>
<td>0</td>
<td>-1/2</td>
<td>+1/2</td>
</tr>
</tbody>
</table>

Table 2.1. Table of all the elementary particles that interact under the electroweak force, according to the SM. Q is the electric charge, I$_3$ is the third component of the weak isospin and Y is the hypercharge.

The total Lagrangian of the Standard Model can be written as:

$$\mathcal{L}^{SM} = \mathcal{L}^{EW} + \mathcal{L}^{QCD}. \quad (A.2)$$

The Lagrangian for the electroweak sector is

$$\mathcal{L}^{EW} = \mathcal{L}^{EW}_{K} + \mathcal{L}^{H} + \mathcal{L}^{Y}. \quad (A.3)$$

The SM is constructed from a theory with massless fermions. The interaction term and the gauge bosons are then introduced by substituting the usual derivatives ($\partial_{\mu}$) with the covariant derivatives ($D_{\mu}$), according to the scheme of the Yang-Mills gauge theories. The term for the free vector bosons are obtained by introducing the field strength defined as $F_{\mu\nu} = [D_{\mu}, D_{\nu}]$, which reduces to the usual field strength if the gauge group is abelian. The kinetic term of the Lagrangian for the electroweak sector, including the interactions, can then be written in the form:

$$\mathcal{L}^{EW}_{K} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \sum_{i=1,2,3} \overline{L}_i(i\partial \bar{\phi})L_i + \sum_{i=1,2,3} \overline{E}_i(i\partial \bar{\phi})E_i +$$

$$+ \sum_{j=1,2,3} \overline{Q}_j(i\partial \bar{\phi})Q_j + \sum_{j=1,2,3} \overline{U}_j(i\partial \bar{\phi})U_j + \sum_{j=1,2,3} \overline{D}_j(i\partial \bar{\phi})D_j. \quad (A.4)$$

where $a$ are the SU(2)$_L$ indices, while $g$ and $g'$ are respectively the coupling constants of SU(2)$_L$ and U(1)$_Y$. The covariant derivatives have a form that depends on the field on which they are acting:

$$D_{\mu}L_i^L = \left( \partial_{\mu} + ig_2 \frac{\tau_a W_{\mu}^a}{2} - \frac{1}{2} ig' B_{\mu} \right) L_i^L;$$

$$D_{\mu}Q_j^L = \left( \partial_{\mu} + ig_2 \frac{\tau_a W_{\mu}^a}{2} + \frac{1}{3} ig' B_{\mu} \right) Q_j^L;$$

$$D_{\mu}E_i^R = \left( \partial_{\mu} - ig' B_{\mu} \right) E_i^R;$$

66
\[ D_\mu U_j^R = \left( \partial_\mu + \frac{2}{3} ig'B_\mu \right) U_j^R; \]
\[ D_\mu D_j^R = \left( \partial_\mu - \frac{1}{3} ig'B_\mu \right) D_j^R. \]

The three Ws and the B gauge bosons are not the physical ones. In order to get the physical vector bosons (\(W^\pm, Z\) and \(A\)), one should mix the first two fields and perform a rotation of the remaining two, namely
\[ W^\mu = \frac{1}{\sqrt{2}} (W^1_\mu \pm W^2_\mu), \]
\[ (Z_\mu) = \begin{pmatrix} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix} (W^3_\mu), \]
where \(\theta_W\) is the Weinberg’s angle, such that \(g' \cos \theta_W = g \sin \theta_W = e\): a condition imposed in order to restore Quantum Electrodynamics (QED). By expanding the covariant derivatives one gets to the interaction term of the Lagrangian \(\mathcal{L}_{EW}^I\), which can be written in the following form:
\[
\mathcal{L}_{EW}^I = \mathcal{L}_{EW}^{C.C.} + \mathcal{L}_{EW}^{N.C.},
\]
\[
\mathcal{L}_{EW}^{C.C.} = -\frac{g}{2\sqrt{2}} \left( J^{\mu 1} W^\mu_\mu + J^{\mu 2} W^\mu_\mu \right),
\]
\[
\mathcal{L}_{EW}^{N.C.} = -e J^{\mu}_{EM} A_\mu - \frac{g}{\cos \theta_W} J^0_0 Z_\mu,
\]
where C.C. stands for charged current, while N.C. for neutral current. The currents are
\[
J^{\mu}_{EM} = \sum_{\text{Fermions}} \frac{Q(F)}{e} F^\gamma_{\mu} F,
\]
\[
J^0_0 = \sum_{\text{Fermions}} \left[ g_L(F) F^L \gamma_\mu F^L + g_R(F) F^R \gamma_\mu F^R \right],
\]
\[
J^{\mu} = \sum_{i=1,2,3} N_i \gamma^\mu (1 - \gamma^5) E_i^L + \sum_{j=1,2,3} U_j \gamma^\mu (1 - \gamma^5) D_j,
\]
with \(N_i = \nu_e^L, \nu_{\mu}^L, \nu_{\tau}^L\) and
\[
g_{L,R}(F) = I_3(F_{L,R}) - \frac{Q(F_{L,R})}{e} \sin^2 \theta_W,
\]
where \(F\) stands for fermion. The real world is made of massive particles, so, in order to promote the SM to a theory that describes Nature, the masses of the elementary particles must be implemented somehow in the Lagrangian. The brutal addition of the mass terms is not the right way to do it because they violate the gauge symmetry. One needs a gauge invariant procedure: the Higgs's mechanism. The Higgs’s mechanism consists on the introduction of a SU(2)_L doublet of scalar fields, the Higgs doublet (\(\Phi\)), which has a vacuum expectation value (\(v\)) different from zero. After the electroweak Spontaneous Symmetry Breaking (ew-SSB) process, the interaction terms between the Higgs doublet and the other fields give rise to the mass terms for each particle and to the Higgs boson (\(H\)). The initial Lagrangian remains symmetric under the gauge group SU(2)_L × U(1)_Y, while the vacuum is only symmetric under U(1)_{EM}.

The masses of the bosons are generated by the Higgs terms in the Lagrangian after the SSB:
\[
\mathcal{L}^H = (D_\mu \Phi) \dagger (D^\mu \Phi) - \mu^2 (\Phi \dagger \Phi) - \lambda (\Phi \dagger \Phi)^2, \quad \text{with } D_\mu \Phi = \left( \partial_\mu + ig\frac{\tau_a W_\mu^a}{2} + \frac{1}{2} ig' B_\mu \right) \Phi,
\]
\[ \downarrow \text{ew-SSB} \]
\[ m^2_{W\mu} W^+ W^- \mu, \quad \frac{1}{2} m^2_Z Z \mu, \quad \frac{1}{2} m^2_H H^2, \quad m_A = 0 \]

+ interaction terms.

The masses of the fermions, neutrinos excluded, arise from the Yukawa’s interaction:

\[ L^Y = - \sum_{i,j=1,2,3} \left[ L^L_i (f_\ell)^{ij} E^R_j \Phi \right] - \sum_{i,j=1,2,3} \left[ Q^L_i (f_u)^{ij} U^R_j \Phi \right] - \sum_{i,j=1,2,3} \left[ Q^L_i (f_d)^{ij} D^R_j \Phi \right] + \text{h.c.} \quad (A.8) \]

\[ \downarrow \text{ew-SSB} \]

\[ -\frac{v}{\sqrt{2}} \sum_{i,j=1,2,3} E_i (f_\ell)^{ij} E_j, \quad -\frac{v}{\sqrt{2}} \sum_{i,j=1,2,3} U_i (f_u)^{ij} U_j, \quad -\frac{v}{\sqrt{2}} \sum_{i,j=1,2,3} D_i (f_d)^{ij} D_j \]

+ interaction terms,

where \( \Phi = i \tau_2 \Phi^\ast \), and \( f_\ell, f_d \) and \( f_u \) are matrices acting respectively on the leptonic and quark flavour spaces. To get the exact Dirac mass terms, one should perform a rotation of the fields and obtain the mass eigenstates by diagonalizing such matrices. We do it for the quarks:

\[
\begin{align*}
U^{L_R}_i \rightarrow U^{L_R}_i &= (L, R)^{ij}_a U^{L_R}_j \\
D^{L_R}_i \rightarrow D^{L_R}_i &= (L, R)^{ij}_d D^{L_R}_j
\end{align*}
\]

where the matrices \( L_{u,d} \) and \( R_{u,d} \) are such that \((R^L_{u,d})^{ik}_{ij} (L_{u,d})^{kl}_{ij} = (M_{u,d})^{ij}_{ij}\), with \( M_{u,d} \) being a diagonal matrix. Now \( U^L_i \) and \( U^R_i \) are the mass eigenstates which have the correct Dirac mass terms. By rewriting the interaction term in the charged current, \( L^E_W \) in (A.5), with the mass eigenstates, a matrix comes out and has the following form:

\[ V_{\text{CKM}} = L_u L_d^\dagger. \]

It is the famous Cabibbo - Kobayashi - Maskawa matrix, responsible for the quark mixing phenomenon. The most common interpretation is that the \( W^\pm \) bosons do not see the mass eigenstates alone, which on the contrary are the true physical quarks that are affected by the strong interaction, but a combination of them. Actually, it can be seen as a mixing of the down type quarks only.
Appendix B

Explicit computations

We present here the explicit computations for the width of the decay $\chi_2 \rightarrow \chi_1 \gamma$, the cross-section of the annihilation into leptons $\chi \chi \rightarrow \ell^+ \ell^-$ and into photons $\chi \chi \rightarrow \gamma \gamma$ and finally the cross-section for the inelastic up-scattering with a generic fermion $\chi_1 f \rightarrow \chi_2 f$. Before we proceed, let us write the effective interaction term (5.2) in a more useful form:

$$L_{\text{EFT}} = -\frac{1}{\Lambda} \bar{\psi} D\left( cM + icE\gamma^5 \right) \Sigma^{\mu\nu} \psi D \partial_\mu A_\nu, \quad (B.1)$$

where $A_\nu$ is the 4-potential of electromagnetism and coming from the electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We can do the same also for (5.4):

$$L_{\text{EFT}} = -\frac{1}{\Lambda} \bar{\psi}_1 \left( cM + icE\gamma^5 \right) \Sigma^{\mu\nu} \psi_2 \partial_\mu A_\nu. \quad (B.2)$$

The EFT vertex rule is then straightforward:

$$-\frac{i}{\Lambda} \left( cM + icE\gamma^5 \right) \Sigma^{\mu\nu} (-ik_\mu),$$

where $k_\mu$ is the momentum of the photon.

B.1 Decaying width

The Feynman diagram for the decay is

$$\chi_2 \rightarrow \chi_1 \gamma,$$

where the crossed circle denotes the EFT interaction. The diagram is simply the vertex of our EFT. The Feynman amplitude of the process is

$$M = -\frac{1}{\Lambda} k_\mu \epsilon^\lambda_\mu \left[ \bar{u}_1^\alpha \left( cM + icE\gamma^5 \right) \Sigma^{\mu\nu} u_2^\nu \right]. \quad (B.3)$$

Its complex conjugate is given by

$$M^* = -\frac{1}{\Lambda} k_\mu \epsilon^\lambda_\mu \left[ \bar{u}_1^\alpha \left( cM + icE\gamma^5 \right) \Sigma^{\mu\nu} u_2^\nu \right]^*. \quad (B.4)$$

By expliciting the spinor indeces we get:

$$(B.4) = \frac{1}{\Lambda} k_\mu \epsilon^\lambda_\mu \left( u_1^\alpha \right)^\alpha \left[ \gamma^0 \left( cM + icE\gamma^5 \right) \Sigma^{\mu\nu} \right]_{\alpha\beta} u_2^{\beta}.$$
We show now that \( S^* = S^T \):

\[
S^* = \gamma^0 (c_M - i c_E \gamma^5) \Sigma^{\mu\nu} = -\frac{i}{2} \gamma^0 (c_M - i c_E \gamma^5) (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) =
\]

\[
= -\frac{i}{2} (c_M + i c_E \gamma^5) \gamma^0 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = -\frac{i}{2} (c_M + i c_E \gamma^5) (\gamma^T \gamma^T - \gamma^v \gamma^v) \gamma^0 =
\]

\[
= \frac{i}{2} \left[ (c_M + i c_E \gamma^5)^T (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \gamma^0 \right] = \frac{i}{2} \left[ \gamma^0 ((c_M + i c_E \gamma^5) \Sigma^{\mu\nu})^T = S^T. \right.
\]

Therefore

\[
\mathcal{M}^* = -\frac{1}{A} k e^{\Sigma^{\mu\nu} u^\mu u^\nu} (c_M + i c_E \gamma^5).
\] (B.5)

The Feynman amplitude squared is then

\[
|\mathcal{M}|^2 = \frac{1}{2A^2} k_e k_p e^{\Sigma^{\mu\nu} u^\mu u^\nu} \left[ u_1 (c_M + i c_E \gamma^5) \Sigma^{\mu\nu} u_2 \right] \left[ u_2^* (c_M + i c_E \gamma^5) \Sigma^{\rho\sigma} u_1^* \right].
\]

By averaging over the initial states and summing over the final states we get:

\[
|\mathcal{M}|^2 = \frac{1}{2A^2} k_e k_p \sum_{s, s', \lambda} \varepsilon^s \varepsilon_{s'} \left[ u_1^s (c_M + i c_E \gamma^5) \Sigma^{\mu\nu} u_2^s \right] \left[ u_2^{s'} (c_M + i c_E \gamma^5) \Sigma^{\rho\sigma} u_1^{s'} \right].
\]

Remembering that

\[
\sum_{\lambda} \varepsilon^s \varepsilon_{s'} = -g_{s,s}, \text{ and } \sum_{s, s'} u_1^{s, s'} u_2^{s, s'} = (q_1 + m_{\chi_1} + m_{\chi_2}),
\]

we arrive to

\[
|\mathcal{M}|^2 = -\frac{1}{2A^2} k_e k_p \text{Tr} \left\{ (q_2 + m_{\chi_2}) (c_M + i c_E \gamma^5) \Sigma^{\rho\sigma} (q_1 + m_{\chi_1}) (c_M + i c_E \gamma^5) \Sigma^{\mu\nu} \right\}.
\]

The trace can be computed with FeynCalc. Using the on-shell conditions for the momenta we arrive to:

\[
|\mathcal{M}|^2 = \frac{8}{A^2} (c_M^2 + c_E^2) (k \cdot p_1) (k \cdot p_2).
\] (B.6)

In order to find an expression for the decay rate, we move to the CM where the two scalar products in (B.6) can be easily computed:

\[
k \cdot p_1 = k \cdot p_2 = m_{\chi_2} E^*_{\gamma}.
\]

We use now the formula (3.3) to get the energy of the photon in the CM in terms of the mass parameters and integrate (3.8) over the solid angle to obtain the decay rate:

\[
\Gamma_{\chi_2 \to \chi_1 \gamma} = \frac{c_M^2 + c_E^2}{8 \pi A^2} \left( \frac{\delta m_\chi + 2 m_{\chi_1}^3}{m_{\chi_2}^3} \right) \delta m_\chi^2 \approx \delta m_\chi \to 0 \frac{c_M^2 + c_E^2}{\pi A^2} \delta m_\chi^2.
\] (B.7)

**B.2 Annihilation into leptons**

The process is described by the following diagram

\[
\begin{align*}
\text{\( p_1, s \)} & \quad \chi & \quad \gamma & \quad \ell^- \\
\text{\( p_2, s' \)} & \quad q = p_1 + p_2 & \quad \ell^+
\end{align*}
\]
The amplitude is
\[ \mathcal{M} = -\frac{e}{\Lambda} \left[ \bar{u}_\ell'(k_1) \gamma^\mu v_\ell'(k_2) \right] \times \left( \frac{g_{\mu\nu}}{s} \right) \times \left[ \bar{u}_\chi^s(p_1) \left( c_M + ic_E\gamma^5 \right) \Sigma^\rho \mu u_\chi^s(p_2) \right]. \]

The complex conjugate of the amplitude reads:
\[ \mathcal{M}^* = -\frac{e}{\Lambda} \left[ \bar{u}_\ell'(k_2) \gamma^\mu u_\ell'(k_1) \right] \times \left( \frac{g_{\mu\nu}}{s} \right) \times \left[ \bar{u}_\chi^s(p_2) \left( c_M + ic_E\gamma^5 \right) \Sigma^\rho \mu v_\chi^s(p_1) \right]. \]

Hence the averaged Feynman amplitude squared is
\[ |\mathcal{M}|^2 = \frac{e^2}{4\Lambda^2 q^4} q_\rho q_\sigma \sum_{s, s', r, r'} \left\{ \left[ \bar{u}_\ell'(k_1) \gamma^\mu u_\ell'(k_2) \right] \left[ \bar{u}_\chi^s(p_1) \left( c_M + ic_E\gamma^5 \right) \Sigma^\rho \mu u_\chi^s(p_2) \right] \left[ \bar{u}_\ell'(k_2) \gamma_\rho u_\ell'(k_1) \right] \left[ \bar{u}_\chi^s(p_2) \left( c_M + ic_E\gamma^5 \right) \Sigma^\sigma \nu v_\chi^s(p_1) \right] \right\} = \frac{e^2}{4\Lambda^2 q^4} q_\rho q_\sigma \text{Tr} \left\{ (\bar{k}_1 + m_\ell) \gamma_\mu (\bar{k}_2 - m_\ell) \gamma_\nu \right\} \times \text{Tr} \left\{ (\bar{\phi}_1 - m_\chi) \left( c_M + ic_E\gamma^5 \right) \Sigma^\rho \nu (\bar{\phi}_2 + m_\chi) \left( c_M + ic_E\gamma^5 \right) \Sigma^\sigma \nu \right\}. \]

All the scalar products between the momenta are easily computed in the CM, where the energy is \( E^* = \sqrt{s}/2 \), and can be expressed in terms of the masses, the Mandelstam variable \( s \) and the angle \( \theta^* \) of emission:
\[
\begin{align*}
&\bullet \ q^2 = s, \\
&\bullet \ p_1^2 = p_2^2 = m_\chi^2, \\
&\bullet \ k_1^2 = k_2^2 = m_\ell^2, \\
&\bullet \ p_1 \cdot p_2 = s/2 - m_\chi^2, \\
&\bullet \ k_1 \cdot k_2 = s/2 - m_\ell^2, \\
&\bullet \ p_1 \cdot k_1 = p_2 \cdot k_2 = s/4 - \sqrt{s/4 - m_\chi^2} \sqrt{s/4 - m_\ell^2} \cos(\theta^*) , \\
&\bullet \ p_1 \cdot k_2 = p_2 \cdot k_1 = s/4 - \sqrt{s/4 - m_\ell^2} \sqrt{s/4 - m_\chi^2} \cos(\pi - \theta^*) , \\
&\bullet \ p_1 \cdot q = p_2 \cdot q = k_1 \cdot q = k_2 \cdot q = s/2. 
\end{align*}
\]

In the CM we can easily compute also the relative velocity between the two initial DM particles:
\[ v_{\text{rel}} = \frac{4p^*}{E^*} \Rightarrow v_{\text{rel}}^2 = 4 - \frac{16m_\chi^2}{s}. \]

The latter expression gives \( s \) in terms of \( v_{\text{rel}} \):
\[ s = \frac{16m_\chi^2}{4 - v_{\text{rel}}^2}. \quad (B.8) \]

The differential cross-section for the process times the relative velocity can be obtained from (3.15) with the appropriate substitutions and using FeynCalc. We integrate it over the solid angle to get \( \sigma v_{\text{rel}} \) and we expand the result in terms of \( v_{\text{rel}} \), separating the contributions from the magnetic and electric dipole moments:
\[
\begin{align*}
\sigma v_{\text{rel}} |_{E=0} &= \frac{\alpha_{EM}}{\Lambda^2} c_M^2 \left( 1 - \frac{m_\ell^2}{m_\chi^2} \right)^{1/2} \left( 1 + \frac{m_\ell^2}{2m_\chi^2} \right) + \mathcal{O}(v_{\text{rel}}^2), \\
\sigma v_{\text{rel}} |_{CM=0} &= \frac{\alpha_{EM}}{\Lambda^2} c_E^2 v_{\text{rel}}^2 \left( 1 - \frac{m_\ell^2}{m_\chi^2} \right)^{1/2} \left( 1 + \frac{m_\ell^2}{2m_\chi^2} \right) + \mathcal{O}(v_{\text{rel}}^4), \quad (B.9)
\end{align*}
\]

\(^1\text{We define } \cos \theta^* = \frac{|k_1^* \cdot k_2^*|}{|k_1^*||k_2^*|}.\)
where $\alpha_{EM} = e^2/4\pi$ is the coupling constant of electromagnetism. In the non-relativistic limit, i.e. $v_{\text{rel}} \ll 1$, the magnetic interaction is dominant while the electric one is suppressed by a factor of $v_{\text{rel}}^2$. We can summarize the result in the following formula for the cross-section:

$$\sigma_{\text{rel}} \bigg|_{\chi \rightarrow e^+e^-} \sim \frac{\alpha_{EM}}{\Lambda^2} \left( - \frac{q^2}{m^2_{\chi}} \right)^{1/2} \left( 1 + \frac{m^2_{\chi}}{2m^2_{\chi}} \right). \quad (B.10)$$

### B.3 Annihilation into photons

The process is described by the following diagram:

![Diagram](https://via.placeholder.com/150)

The amplitude is

$$M = \frac{i}{\Lambda^2} \varepsilon^\lambda_\nu (k_1) \varepsilon^\lambda'_{\sigma} (k_2) k_{1\kappa} k_{2\rho} \left[ \overline{\nu_\nu} (p_1) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu\nu} \frac{\gamma^\mu + m_{\chi}}{q^2 - m_{\chi}^2} \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho\sigma} \nu_s (p_2) \right]. \quad (B.11)$$

In order to compute the complex conjugate of the amplitude, we need $B^*$:

$$B^* = u_{s}^\alpha \left[ \gamma^0 \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu\nu} \right]_{\alpha\beta}^* \left( \frac{\gamma^\nu q_a + m_{\chi}}{q^2 - m_{\chi}^2} \right)_{\beta\gamma} \left[ \gamma^0 \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho\sigma} \right]_{\delta\epsilon}^* u_{s'}^\epsilon =$$

$$\equiv u_{s'}^\epsilon S_{\delta\epsilon}^{\gamma\delta} \left( \frac{\gamma^\nu q_a + m_{\chi}}{q^2 - m_{\chi}^2} \right)_{\gamma\beta} S_{\alpha\beta}^{\gamma\alpha} \left( \frac{\gamma^\nu q_a + m_{\chi}}{q^2 - m_{\chi}^2} \right)_{\gamma\beta} \gamma^\delta S_{\delta\alpha} \nu_s^\alpha =$$

$$= \overline{u}_{s'} (c_M + ic_E \gamma^5) \Sigma^{\rho\sigma} \frac{\gamma^\mu + m_{\chi}}{q^2 - m_{\chi}^2} (c_M + ic_E \gamma^5) \Sigma^{\mu\nu} \nu_s. \quad (B.12)$$

Therefore

$$M^* = - \frac{i}{\Lambda^2} \varepsilon^\lambda_\nu (k_1) \varepsilon^\lambda'_{\sigma} (k_2) k_{1\kappa} k_{2\rho} \left[ \overline{\nu_\nu} (p_2) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho\sigma} \frac{\gamma^\mu + m_{\chi}}{q^2 - m_{\chi}^2} \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu\nu} \nu_s (p_1) \right]. \quad (B.13)$$

The Feynman amplitude squared of the process reads:

$$|M|^2 = \frac{1}{\Lambda^4} \varepsilon^\lambda_\nu (k_1) \varepsilon^\lambda'_{\sigma} (k_2) \varepsilon^\lambda_\nu (k_1) \varepsilon^\lambda'_{\sigma} (k_2) k_{1\kappa} k_{2\rho} k_{2\tau} k_{1\lambda} \times$$

$$\times \left[ \overline{\nu_\nu} (p_1) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho\sigma} \frac{\gamma^\mu + m_{\chi}}{q^2 - m_{\chi}^2} \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu\nu} \nu_s (p_2) \right] \times \left[ \overline{\nu_\nu} (p_2) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho\sigma} \frac{\gamma^\mu + m_{\chi}}{q^2 - m_{\chi}^2} \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu\nu} \nu_s (p_1) \right]. \quad (B.13)$$
By averaging over the initial states and summing over the final states we get:

$$|\mathcal{M}|^2 = \frac{k_1 \cdot k_2 \cdot k_2 \cdot k_1 \sigma}{4 \Lambda^4 (q^2 - m^2)} \text{Tr} \left\{ (\not{\phi}_1 - m_\chi) (\not{c}_M + i c \not{E} \gamma^5) \Sigma^{\alpha} (\not{g} + m_\chi) (\not{c}_M + i c \not{E} \gamma^5) \Sigma^{\beta} \times \right.$$  
$$\times (\not{\phi}_2 + m_\chi) (\not{c}_M + i c \not{E} \gamma^5) \Sigma^{\rho} (\not{g} + m_\chi) (\not{c}_M + i c \not{E} \gamma^5) \Sigma^{\sigma} \right\}.$$  
(B.14)

All the scalar products between the momenta are easily computed in the CM, where the energy is

$$E^* = \sqrt{s}/2,$$

and can be expressed in terms of the masses, the Mandelstam variable $s$ and the angle $\theta^*$ of emission:

- $q^2 = t = m^2 - 2 p_1 \cdot k_1,$
- $p_1^2 = p_2^2 = m^2,$
- $k_1^2 = k_2^2 = 0,$
- $p_1 \cdot p_2 = s/2 - m^2,$
- $k_1 \cdot k_2 = s/2,$
- $p_1 \cdot k_1 = p_2 \cdot k_2 = s/4 - \sqrt{s/4 - m^2} \sqrt{s}/2 \cos (\theta^*),$
- $p_1 \cdot k_2 = p_2 \cdot k_1 = s/4 - \sqrt{s/4 - m^2} \sqrt{s}/2 \cos (\pi - \theta^*),$
- $p_1 \cdot q = -p_2 \cdot q = m^2 - s/4 + \sqrt{s/4 - m^2} \sqrt{s}/2 \cos (\theta^*),$
- $k_1 \cdot q = -k_2 \cdot q = s/4 - \sqrt{s/4 - m^2} \sqrt{s}/2 \cos (\theta^*).$

We use (B.8) to express $s$ in terms of the relative velocity between the two initial particles. The differential cross-section for the process times the relative velocity can be obtained from (3.15) with the appropriate substitutions and using FeynCalc. We perform the integration over the solid angle and expand the result around $v_{\text{rel}} = 0$, getting at first order the expression given in (5.10):

$$\sigma v_{\text{rel}} \bigg|_{\chi\chi \rightarrow \gamma\gamma} = \left( \frac{e^2}{4\pi^2} \right)^2 \frac{m^2}{\Lambda^4} + O \left( v_{\text{rel}}^2 \right).$$  
(B.15)
B.4 Inelastic up-scattering

Here is the Feynmann diagram of the process:

\[ \begin{array}{c}
\text{p}_1, s \quad \text{p}_2, s' \\
\text{f} \quad \gamma \quad q = p_1 - p_2 \\
\text{f} \quad \text{k}_1, r \quad \text{k}_2, r' \\
\end{array} \]

where \( f \) stands for a generic fermion, which will then be an electron or a proton in the plasma. Here the mass splitting is crucial, hence we use (B.2) for the interaction term to describe the process.

The Feynman amplitude is

\[ M = -\frac{e}{\Lambda q^2} q_\mu \left[ \bar{u} \gamma^\mu u \left( p_2 \right) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu \nu} q_\nu q_\rho \left( p_1 \right) \right] \left[ \bar{u} \gamma^\rho u \left( k_2 \right) \right], \quad (B.16) \]

and its complex conjugate

\[ M^* = -\frac{e}{\Lambda q^2} q_\mu \left[ \bar{u} \left( p_1 \right) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu \nu} q_\nu q_\rho \left( p_2 \right) \right] \left[ \bar{u} \gamma^\rho u \left( k_1 \right) \right]. \quad (B.17) \]

The averaged Feynman amplitude squared reads:

\[ |M|^2 = \frac{e^2}{4\Lambda^2 q^4} q_\mu q_\rho \text{Tr} \left\{ \left( \bar{\phi}_2 + m_{\chi_1} + \delta m_{\chi} \right) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\mu \nu} \left( \bar{\phi}_1 + m_{\chi_1} \right) \left( c_M + ic_E \gamma^5 \right) \Sigma^{\rho \sigma} \right\} \times \text{Tr} \left\{ \left( \bar{k}_2 + m_f \right) \gamma_\nu \left( \bar{k}_1 + m_f \right) \gamma_\sigma \right\}. \quad (B.18) \]

As for (3.1), in the CM we can find the energy of each particle involved to be

\[ E_1^* (s) = \frac{s - m_2^2 + m_{\chi_1}^2}{2\sqrt{s}}, \quad E_2^* (s) = \frac{s - m_2^2 + (m_{\chi_1} + \delta m_{\chi})^2}{2\sqrt{s}}, \]
\[ E_f^* (s) = \frac{s - m_{\chi_1}^2 + m_f^2}{2\sqrt{s}}, \quad E_f'^* (s) = \frac{s - (m_{\chi_1} + \delta m_{\chi})^2 + m_f^2}{2\sqrt{s}}. \]

The scalar products between the momenta can be written in terms of the Mandelstam variables
\[ s = (p_1 + k_1)^2, \ t = (p_1 - p_2)^2 \text{ and } u = (p_1 - k_2)^2: \]

- \( q^2 = t, \)
- \( p_1^2 = m_{\chi_1}, \)
- \( p_2^2 = (m_{\chi_1} + \delta m_{\chi})^2, \)
- \( k_1^2 = k_2^2 = m_f^2, \)
- \( 2p_1 \cdot p_2 = m_{\chi_1}^2 + (m_{\chi_1} + \delta m_{\chi})^2 - t, \)
- \( 2k_1 \cdot k_2 = 2m_f^2 - t, \)
- \( 2p_1 \cdot k_1 = s - m_{\chi_1}^2 - m_f^2, \)
- \( 2p_2 \cdot k_2 = s - (m_{\chi_1} + \delta m_{\chi})^2 - m_f^2, \)
- \( 2p_1 \cdot k_2 = m_{\chi_1}^2 + m_f^2 - u, \)
- \( 2p_2 \cdot k_1 = (m_{\chi_1} + \delta m_{\chi})^2 + m_f^2 - u, \)
- \( 2p_1 \cdot q = m_{\chi_1}^2 + t - (m_{\chi_1} + \delta m_{\chi})^2, \)
- \( 2p_2 \cdot q = m_{\chi_1}^2 - (m_{\chi_1} + \delta m_{\chi})^2 - t, \)
- \( 2k_2 \cdot q = -2k_1 \cdot q = t. \)

Then
\[
 u = 2m_f^2 + m_{\chi_1}^2 + (m_{\chi_1} + \delta m_{\chi})^2 - s - t \]

and
\[
t = m_{\chi_1}^2 + (m_{\chi_1} + \delta m_{\chi})^2 - 2E_1^*(s)E_2^*(s) + 2\sqrt{E_1^*2(s) - m_{\chi_1}^2 \sqrt{E_2^*2(s) - (m_{\chi_1} + \delta m_{\chi})^2}} \mu^*, \]

where here
\[
\mu^* = \cos \theta^* = \frac{\vec{p}_1^* \cdot \vec{p}_2^*}{|\vec{p}_1^*| |\vec{p}_2^*|}. \]

With all these ingredients we can compute the differential cross-section times the relative velocity through (3.15) with FeynCalc.

There is no dependence upon the azimuthal angle, thus we get
\[
\frac{d\sigma_{\text{rel}}}{d\Omega^*_2} = \frac{1}{2\pi} \frac{d\sigma_{\text{rel}}}{d\mu^*} = \frac{1}{2\pi} S(s, \mu^*), \tag{B.19} \]

where \( S \) is defined as in (5.17). We checked that no divergence appears when \( \delta m_{\chi} \neq 0 \), neither at small angles nor small velocities.

To study the case of a zero mass splitting we can simply set \( \delta m_{\chi} = 0 \) in all the above equations. To get the differential cross-section we need an explicit expression for the relative velocity in terms of \( s \). Such expression in the CM reads:
\[
v_{\text{rel}} = \frac{\sqrt{E_1^*2(s) - m_{\chi_1}^2}}{E_1^*(s)E_2^*(s)} \frac{1}{s}. \tag{B.20} \]

In the non-relativistic limit we can write also \( s \) in terms of \( v_{\text{rel}} \) as
\[
s \approx \left( m_{\chi_1} + m_f + \frac{1}{2} \frac{m_{\chi_1}m_f}{m_{\chi_1} + m_f} v_{\text{rel}}^2 \right)^2, \tag{B.21} \]

where the last term is simply the kinetic energy of the system. The substitution of \( s \) with the above formula enables us to expand the differential cross-section around \( v_{\text{rel}} = 0 \). In the limit of a null mass splitting we get the following formulas for the differential cross-section in the two
cases of electric and magnetic dipole moments respectively:

\[
\frac{d\sigma}{d\Omega} \bigg|_{c_{M}=0} = c_E^2 \frac{\alpha_{EM}}{4\pi \Lambda^2} \left( \frac{1}{v_{\text{rel}}^2 \sin^2(\theta^*/2)} \right) + \mathcal{O}(1),
\]

\[
\frac{d\sigma}{d\Omega} \bigg|_{c_{E}=0} = c_M^2 \frac{\alpha_{EM}}{4\pi \Lambda^2} \frac{1 + \frac{m_s(m_s-2m_f)}{(m_s+m_f)^2} \sin^2(\theta^*/2)}{\sin^2(\theta^*/2)} + \mathcal{O}(v_{\text{rel}}^2).
\]

(B.22)
Appendix C

The Boltzmann equation for DM

We present here the derivation of equation (5.11) starting from the basics, following [47]. The Boltzmann equation is an equation for the evolution of the distribution function of a certain particle species, \( f(p^\mu, x^\mu) \) (distribution function in the phase-space), here normalized to the number of particles \( N \):

\[
\hat{L}[f] = \hat{C}[f],
\]

(C.1)

where \( \hat{L} \) is the Liouville operator and \( \hat{C} \) is the collision operator, which accounts for all the interactions between the different species. For \( \hat{C} = 0 \) we get the Liouville’s theorem. In the non-relativistic limit the Liouville operator is simply the time derivative along the trajectory, namely:

\[
\hat{L}_{NR}[f] \equiv \frac{df}{dt} = \nabla_x f \cdot \frac{\partial \vec{v}}{\partial t} + \nabla_v f \cdot \vec{v} = \left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla_x + \frac{\vec{F}}{m} \cdot \vec{v} \right) f.
\]

(C.2)

The relativistic extension can be obtained by substituting the usual derivative with the covariant derivative with respect to an affine parameter \( \tau \) and using the geodesics equation:

\[
\hat{L}[f] = \left( p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} \right) f,
\]

(C.3)

where the \( \Gamma \)s are the Christoffel symbols of General Relativity.

In the standard model of Cosmology, the flat Universe at large scales is considered to be homogeneous and isotropic and can be described with the Friedmann-Robertson-Walker (FRW) metric \( ds^2 = dt^2 - a^2(t) \left[ dx^2 + dy^2 + dz^2 \right] \), where \( a(t) \) is the time-dependent scale factor. Under these assumptions, the distribution function depends upon energy and time only:

\[ f(p^\mu, x^\mu) = f(E, t). \]

In the FRW metric, the only non vanishing Christoffel symbols are:

\[ \Gamma^0_{ij} = \delta_{ij} \dot{a}a, \quad \text{and} \quad \Gamma^i_{0j} = \Gamma^j_{i0} = \delta^i_j \frac{\dot{a}}{a}. \]

Therefore the Liouville operator applied to the distribution function reads:

\[
\hat{L}_{FRW}[f] = E \frac{\partial f}{\partial t} - H p^2 \frac{\partial f}{\partial E},
\]

(C.4)

where \( H \equiv \dot{a}/a \) is the Hubble’s parameter.

We can obtain an equation for the number density by integrating (C.1) over \( d^3p \) and dividing by \( E \). The number density is:

\[
n(t) = \frac{g}{(2\pi)^3} \int d^3p \, f(E, t),
\]

\[1\] We use the standard notation that roman indices runs from 1 to 3, while greek ones from 0 to 4.
where \( g \) denotes the number of degrees of freedom for a given particle. The left term of the Boltzmann equation then becomes:

\[
\frac{\partial}{\partial t} \left( \frac{g}{(2\pi)^3} \int d^3p \, f \right) - H \frac{g}{(2\pi)^3} \int d^3p \frac{p^2}{E} \frac{\partial f}{\partial E} = \dot{n}(t) - H \frac{g}{(2\pi)^3} \int d^3p \frac{p^2}{E} \frac{\partial f}{\partial E}.
\]

The last term reads:

\[
H \frac{g}{(2\pi)^3} \int d^3p \frac{p^2}{E} \frac{\partial f}{\partial E} = -3\dot{H} n(t),
\]

where the border term arising from the integration by parts disappears if we assume \( f \) to be regular in \( p = 0 \) and going to zero at \( +\infty \). So the Boltzmann equation for the number density is

\[
\dot{n}(t) + 3\dot{H} n(t) = g \frac{\partial}{\partial t} \left( \frac{\hat{C}[f]}{E} \right).
\]

(C.5)

If \( \hat{C} = 0 \) we get that \( \dot{n}(t) = -3\dot{H} n(t) \), thus \( n(t) \propto a^{-3} \), which means that in a comoving volume the number of particles remains constant.

Now we need to write the collisional operator. Let us suppose to have a process like \( 1+2 \leftrightarrow 3 + 4 \). We indicate the respective momenta with \( p_1, p_2, p_3 \) and \( p_4 \). The collisional term for the particle 1 can be written as

\[
\frac{g_1}{(2\pi)^3} \int d^3p \frac{\hat{C}[f_1]}{E_1} = \int d\Pi_1 \, d\Pi_2 \, d\Pi_3 \, d\Pi_4 \frac{(2\pi)^3}{(2\pi)^3} \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \times
\]

\[
\times \left[ |\mathcal{M}_{3+4\rightarrow1+2}|^2 f_3 f_4 (1 \pm f_1)(1 \pm f_2) - |\mathcal{M}_{1+2\rightarrow3+4}|^2 f_1 f_2 (1 \pm f_3)(1 \pm f_4) \right],
\]

(C.6)

where

\[
d\Pi_i = \frac{g_i}{(2\pi)^3} \frac{d^3p_i}{2E_i}
\]

is the phase-space infinitesimal element and

\[
(1 \pm f_i)
\]

are respectively the Bose enhancement and Pauli suppression factors; \(|\mathcal{M}|^2\) is the Feynman amplitude squared of the process.

To get the exact solution one needs to solve all together the coupled equations for the species involved, but some approximations can be made in order to simplify the problem.

1. We assume no CP violation such that \(|\mathcal{M}| = |\mathcal{M}_{1+2\rightarrow3+4}| = |\mathcal{M}_{3+4\rightarrow1+2}|\).

2. We assume that the scattering processes are efficient enough (\(\Gamma_{\text{scatt}} \gg H\)), in order to maintain the kinetic equilibrium. So the \(f_i\)s are the Fermi-Dirac or Bose-Einstein distributions:

\[
f_{FD/BE} = \frac{1}{e^{\frac{E - \mu}{T}} \pm 1},
\]

where \( \mu = \mu(t) \) is the chemical potential. At the thermodynamic equilibrium there is also the chemical equilibrium, i.e. the annihilation processes are efficient too, and the chemical potentials of all the species in the bath are equal to zero [73, 74]. In this case the phase-space distribution is

\[
f_{\text{eq}}^{FD/BE} = \frac{1}{e^{\frac{E}{T}} \pm 1}.
\]

3. We assume that \((E - \mu) \gg T\) so that we can approximate the distribution functions as

\[
f_i \simeq e^{\frac{\mu_i}{T}} e^{-\frac{E_i}{T}}.
\]

This allows us to ignore the Bose and Fermi factors: \(1 \pm f_i \approx 1\). Note also that under this approximation

\[
\frac{f_i}{f_{i\text{eq}}} = \frac{n_i}{n_{i\text{eq}}} = e^{\frac{\mu_i}{T}}.
\]
These assumptions simplify the Boltzmann equation substantially:
\[
\dot{n}_1(t) + 3Hn_1(t) = \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) |\mathcal{M}|^2 [f_3f_4 - f_1f_2].
\] (C.7)

The last factor can be rewritten as
\[
f_3f_4 - f_1f_2 = e^{-\frac{E_1 + E_2}{T} + \frac{n_3 + n_4}{\rho_e} - \frac{n_1 + n_2}{\rho_e}} = e^{-\frac{E_1 + E_2}{T} \left[ \frac{n_3n_4}{n_1 n_2} - \frac{n_1n_2}{n_3 n_4} \right]}.
\]

Notice that
\[
\langle \sigma v_{\text{rel}} \rangle = \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) |\mathcal{M}|^2 e^{-\frac{E_1 + E_2}{T}}.
\]

Hence the Boltzmann equation becomes:
\[
\dot{n}_1 + 3Hn_1 = n_1 e_q e_q \langle \sigma v_{\text{rel}} \rangle \left[ \frac{n_3n_4}{n_1 n_2} - \frac{n_1n_2}{n_3 n_4} \right].
\] (C.8)

Let us consider now a process like $\chi \chi \rightarrow \text{SM SM}$, where SM stands for a generic particle of the Standard Model. This is like the annihilation for the DM candidate discussed in chapter 5. We have now that $n_\chi \equiv n_1 = n_2$ and $n_{\text{SM}} \equiv n_3 = n_4$. We consider also the SM particles to be in thermal equilibrium, thus $n_{\text{SM}} = n_{\text{SM}}^{\text{eq}}$. In this scenario, the Boltzmann equation for $n_\chi$ reads:
\[
\dot{n}_\chi + 3Hn_\chi = -\langle \sigma v_{\text{rel}} \rangle \left[ n_\chi^2 - (n_\chi^{\text{eq}})^2 \right].
\] (C.9)

In order to get (5.11), we need to rewrite the above equation in terms of the variable $Y_\chi \equiv n_\chi/s$, where $s$ is the entropy density. First we notice that:
\[
\frac{d}{dt} (n_\chi a^3) = \dot{n}_\chi a^3 + 3a^2 \dot{a} n_\chi \Rightarrow \dot{n}_\chi + 3Hn = a^{-3} \frac{d}{dt} (n_\chi a^3).
\]

Then, remembering that $d/dt (sa^3) = 0$, we can write
\[
a^{-3} \frac{d}{dt} (n_\chi a^3) = a^{-3} \frac{d}{dt} \left( \frac{n_\chi}{s} sa^3 \right) = s \frac{dY_\chi}{dt}.
\]

Therefore, the equation for $Y_\chi$ is
\[
\frac{dY_\chi}{dt} = -\langle \sigma v_{\text{rel}} \rangle \frac{s}{x} \left[ Y_\chi^2 - (Y_\chi^{\text{eq}})^2 \right].
\]

The time derivative becomes
\[
\frac{d}{dt} = \frac{dx}{dt} \frac{d}{dx} = -x \frac{\dot{T}}{T} \frac{d}{dx}.
\]

The conservation of entropy tells us that $T \propto a^{-1}(t) g_{*,s}^{-1/3}(T)$, leading to
\[
\frac{\dot{T}}{T} = -H \left[ 1 + \frac{1}{3} \frac{T}{g_{*,s}(T)} \frac{d}{dt} g_{*,s}(T) \right]^{-1} = -H \left[ 1 - \frac{1}{3} \frac{x}{g_{*,s}(\frac{m_\chi}{x})} \frac{d}{dx} g_{*,s} \left( \frac{m_\chi}{x} \right) \right]^{-1}.
\]

Therefore the equation for $Y_\chi$ in terms of the variable $x$ is the following:
\[
\frac{dY_\chi}{dx} = -\frac{\langle \sigma v_{\text{rel}} \rangle}{xH} \frac{s}{x} \left[ 1 - \frac{1}{3} \frac{x}{g_{*,s}(\frac{m_\chi}{x})} \frac{d}{dx} g_{*,s} \left( \frac{m_\chi}{x} \right) \right] \left[ Y_\chi^2 - (Y_\chi^{\text{eq}})^2 \right].
\] (C.10)

We use now the Friedmann equation to get $H$ in terms of $x$:
\[
H^2(T) = \frac{1}{3M^2_p} \rho = \frac{1}{3M^2_p} \frac{\pi^2}{30} g_s(T) T^4 \Rightarrow H(x) = \sqrt{\frac{\pi^2}{90M^2_p} g_s^{1/2}(\frac{m_\chi}{x}) \frac{m_\chi^2}{x^2}}.
\]

79
The entropy density is instead

\[ s(T) = \frac{2\pi^2}{45} g_* s(T) T^3 \Rightarrow s(x) = \frac{2\pi^2}{45} g_* s \left(\frac{m_\chi}{x}\right) \frac{m_\chi^3}{x^3}. \]

By combining all these results in (C.10) we get exactly (5.11):

\[
\frac{dY_\chi}{dx} = -\sqrt{\frac{8\pi^3}{45}} M_p m_\chi \frac{g_*, s \left(\frac{m_\chi}{x}\right)}{g_*^{1/2} \left(\frac{m_\chi}{x}\right)} \langle \sigma v_{\text{rel}} \rangle \left[ 1 - \frac{1}{3} \frac{x}{g_* s \left(\frac{m_\chi}{x}\right)} \frac{d}{dx} g_* s \left(\frac{m_\chi}{x}\right) \right] \left[ Y^2 - (Y_{\text{eq}})^2 \right].
\]

(C.11)
References


