On Woronowicz's approach to the Tomita-Takesaki Theory

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The Tomita-Takesaki Theory

The theory of M. Tomita of the standard form of general von Neumann algebras was a turning-point in the theory of Operator Algebras and is up to this day one of the most important tools by working with von Neumann algebras. It became accessible in 1970 in the exposition of M. Takesaki, which contains so many fundamental contributions also of him that the whole theory is usually referred as the "Tomita-Takesaki Theory".

The Tomita-Takesaki Theory is very involved and can be contemplated from different points of view. In the decade 1970 - 1980 several approaches appeared to the theory, each one seeking to attain more transparency (A. Van Daele, U. Haagerup, L. Zsidó, M. Rieffel - A. Van Daele, S. L. Woronowicz). One of them was the paper

S. L. Woronowicz: *Operator Systems and their application to the Tomita-Takesaki Theory*, J. Operator Theory **2** (1979), 169-209.

My goal is to sketch a treatment of the basic facts of the Tomita-Takesaki Theory based on the above work.

The starting point of the Tomita-Takesaki Theory in the case of a von Neumann algebra $\mathcal{M}$ having a bicyclic vector ξ_o is the folloving fundamental theorem:

The "projection" of the involution

$$\mathcal{M} \ni x \longmapsto x^*$$

on the underlying Hilbert space, that is

$$\mathcal{M}\,\xi_o \ni x\;\xi_o \longmapsto x^*\xi_o\,,$$

is a closable antilinear operator and if S stands for its closure, $\Delta = S^*S$, and $S = J\Delta^{1/2}$ is the polar decomposition of S, then

$$\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}\,, \qquad t \in \mathbb{R}$$
$$J\mathcal{M}J = \mathcal{M}'\,.$$

Thus we get the one-parameter automorphism group $\Delta^{it} \cdot \Delta^{-it}, t \in \mathbb{R}$ of $\mathcal{M}$ which plays a similar role in the study of the algebra $\mathcal{M}$ and of the positive linear form

$$\mathcal{M} \ni x \longmapsto \omega_{\xi_o}(x) = (x\,\xi_o \,|\, \xi_o)$$

as the usual modular function of a localy compact group by working with functions on the group and with the Haar measure.

We notice that the Tomita-Takesaki Theory holds in a more general setting in which the positive form ω_{ξ_o} is replaced with a densely defined, not necessarily bounded positive form. However, the treatment of the general case can be reduced to the above one.

Analytic extensions of groups of operators

We present here for further usage some topics concerning analytic extensions of one-parameter groups of linear operators. In the exposition we follow works of I. Ciorănescu - L. Zsidó and U. Haagerup.

The setting is the following:

• H will denote a complex Hilbert space with inner product $(\cdot\,|\,\cdot)$, which is linear in the first variable and antilinear in the second variable.

• A denotes a non-singular, positive, self-adjoint linear operator in H.

• $\alpha_t^{(A)}$ stands for the $*$-automorphism of $B(H)$ implemented by $A^{it}, t \in \mathbb{R}$, that is

$$\alpha_t^{(A)}(x) = \mathrm{Ad}\,(A^{it})(x) = A^{it} x\, A^{-it}\,,\ x \in B(H)\,.$$

The analytic extension of the so-continuous one-parameter group $\alpha^{(A)}$ of $*$-automorphisms of $B(H)$ in the complex number $z \in \mathbb{C}$ is the linear operator $\alpha_z^{(A)}$ in $B(H)$ defined as follows:

(x, y) belongs to the graph of $\alpha_z^{(A)} \iff$

$x, y \in B(H)$ and there is a wo-continuous (or, equivalently, so-continuous) map F from the closed strip $\{\zeta \in \mathbb{C}\,; |\Im\zeta| \leq |\Im z|\,, \Im\zeta \cdot \Im z \geq 0\}$ in $B(H)$, which is analytic in the interior of the strip and for which $F(t) = \alpha_t^{(A)}(x)\,,\ t \in \mathbb{R}$, as well as $F(z) = y$.

Computation rules:

- $z_1, z_2 \in \mathbb{C}\,,\ \Im z_1 \, \Im z_2 \geq 0 \Rightarrow \alpha_{z_1}^{(A)} \alpha_{z_2}^{(A)} = \alpha_{z_1+z_2}^{(A)}$.
  In particular, if $x \in \mathcal{D}(\alpha_z^{(A)})$, then
  $\sup\left\{\|\alpha_\zeta^{(A)}(x)\|; \zeta \in \mathbb{C}, |\Im\zeta| \leq |\Im z|, \Im\zeta \cdot \Im z \geq 0\right\}$
$= \sup\left\{\|\alpha_{i\beta}^{(A)}(x)\|; \beta \in \mathbb{R}, |\beta| \leq |\Im z|, \beta \cdot \Im z \geq 0\right\}$
$< +\infty$.

- $z \in \mathbb{C} \Rightarrow \alpha_z^{(A)}$ injective and $\left(\alpha_z^{(A)}\right)^{-1} = \alpha_{-z}^{(A)}$.

- $z \in \mathbb{C}\,,\, x \in \mathcal{D}(\alpha_z^{(A)}) \Rightarrow x^* \in \mathcal{D}(\alpha_{\overline{z}}^{(A)})$ and
$$\alpha_z^{(A)}(x)^* = \alpha_{\overline{z}}^{(A)}(x^*)\,.$$
- For $x \in B(H)$ and $z \in \mathbb{C}$
  $x \in \mathcal{D}(\alpha_z^{(A)})\,,\, \alpha_z^{(A)}(x)$ self-adjoint $\Leftrightarrow$
  $x \in \mathcal{D}(\alpha_{2z}^{(A)})\,,\, \alpha_{2z}^{(A)}(x) = \alpha_{2\,\Re z}^{(A)}(x)^*$
  and in this case $\|\alpha_z^{(A)}(x)\| \leq \|x\|\,.$

- For $z_1 \in \mathbb{C}\,,\, \Im z_1 \geq 0\,,\, z_2 \in \mathbb{C}\,,\, \Im z_2 \leq 0$ and $x \in \mathcal{D}(\alpha_{z_1}^{(A)}) \cap \mathcal{D}(\alpha_{z_2}^{(A)})$, denoting by $\sigma(y)$ the spectrum of $y \in B(H)$, we have
$$\sigma(x) \subset \text{conv}\Big(\sigma\Big(\alpha_{z_1}^{(A)}(x)\Big) \cup \sigma\Big(\alpha_{z_2}^{(A)}(x)\Big)\Big)\,.$$

- $z \in \mathbb{C}\,,\, x, y \in \mathcal{D}(\alpha_z^{(A)}) \Rightarrow xy \in \mathcal{D}(\alpha_z^{(A)})$ and
$$\alpha_z^{(A)}(xy) = \alpha_z^{(A)}(x)\,\alpha_z^{(A)}(y)\,.$$

A description of he analytic extension $\alpha_z^{(A)}$:

For $0 \neq z \in \mathbb{C}$ *and* $x \in B(H)$ *are equivalent* :
(1) $x \in \mathcal{D}(\alpha_z^{(A)})$;

(2) $A^{iz}xA^{-iz}$ *is defined and bounded on an essential domain of* A^{-iz} ;

(3) $A^{iz}xA^{-iz}$ *is defined and bounded on the whole domain of* A^{-iz} .

If the above conditions are satisfied, then

$$xA^{-iz} \subset A^{-iz}\alpha_z^{(A)}(x)\,,$$

so $A^{iz}xA^{-iz} \subset \alpha_z^{(A)}(x)$ *and* $\alpha_z^{(A)}(x)$ *is equal to the closure* $\overline{A^{iz}xA^{-iz}}$.

The analytic extension $\alpha_i^{(A)}$ (or, if we use an alternative choice, $\alpha_{-i}^{(A)}$) is called the *analytic generator* of the group $\alpha^{(A)}$.

It is easy to see that the point spectrum of $\alpha_i^{(A)}$ is contained in the positive half-line $[0, +\infty)$. However, the spectrum of $\alpha_i^{(A)}$ is equal to $\mathbb{C}$ unless A and A^{-1} are bounded (A. Van Daele and G. A. Elliott - L. Zsidó). Nevertheless :

If $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $\varepsilon > 0$ *and* $x \in \mathcal{D}(\alpha_{i\varepsilon}^{(A)})$, *then* $x \in \mathcal{D}\Big((\lambda\, 1_{B(H)} + \alpha_i^{(A)})^{-1}\Big)$ *and*

$$\Big(\lambda\, 1_{B(H)} + \alpha_i^{(A)}\Big)^{-1}(x)$$
$$= \frac{1}{\lambda}\, x - \frac{1}{2\,\lambda}\, wo - \int_{-\infty+ic}^{+\infty+ic} \frac{\lambda^{i\zeta}}{\sin(i\pi\zeta)}\, \alpha_\zeta^{(A)}(x)\, d\zeta\,,$$

where $0 < c < \min\{\varepsilon, 1\}$ *is arbitrary and* $\lambda^{i\zeta} = |\lambda|^{i\zeta} e^{-\theta\zeta}$ *provided that* $\lambda = |\lambda| e^{i\theta}, -\pi < \theta < \pi$.

It follows that the analytic generator defines the group uniquely :

If B *is another non-singular, positive, self-adjoint linear operator in* H, *then*

$$\alpha_t^{(A)} = \alpha_t^{(B)}\,,\ t \in \mathbb{R} \iff \alpha_i^{(A)} \subset \alpha_i^{(B)}\,.$$

Another consequence is the following invariance result :

If $\varepsilon > 0\,, x \in \mathcal{D}(\alpha^{(A)}_{i\varepsilon})$ *and* $X \subset B(H)$ *is a linear subspace which is closed with respect to the operator norm, then the following conditions are equivalent :*

(1) $\Big(1_{B(H)} + \alpha^{(A)}_i\Big)^{-k}(x) \in X\,,\ k \geq 0\,,$

(2) $\Big(\lambda\, 1_{B(H)} + \alpha^{(A)}_i\Big)^{-1}(x) \in X\,,\ \lambda \in \mathbb{C} \setminus (-\infty, 0]\,,$

(3) $\alpha^{(A)}_\zeta(x) \in X\,,\ \zeta \in \mathbb{C}\,, 0 < \Im\zeta < \varepsilon\,.$

If X *is* wo*-closed, then the above conditions are equivalent also to :*

(4) $\alpha^{(A)}_t(x) \in X\,, \qquad t \in \mathbb{R}\,.$

The Woronowicz polar decomposition

Let A be a non-singular, positive, self-adjoint linear operator in the complex Hilbert space H .

If $x \in B(H)$ and xA is closable, then $\overline{xA}$ is a densely defined, closed linear perator, so we can consider its polar decomposition

$$\overline{xA} = v \, |\overline{xA}| \, .$$

Then there exists a uniquely defined $a \in B(H)$ with $aA \subset |\,\overline{xA}\,|$ and we have $\overline{aA} = |\,\overline{xA}\,|$. Moreover, $x = va \, , a = v^* x$ and the invertibility of x implies the invertibility of a as well as the positivity and the self-adjointness of $a^{-1}A$.

The partial isometry v will be called the *phase of x relative to A* and denoted by $\mathrm{phase}_A(x)$. It seems apropriate to denote a by $|x|_A$ and call

$$x = \mathrm{phase}_A(x) \cdot |x|_A$$

the *polar decomposition of* x *relative to* A.

Computation relative to A:

For $a \in B(H)$:

$$\begin{aligned} & aA \text{ symmetrical} \\ \Longleftrightarrow\ & a \in \mathcal{D}(\alpha^{(A)}_{i/2}),\ \alpha^{(A)}_{i/2}(a) \text{ self-adjoint} \\ \Longleftrightarrow\ & a \in \mathcal{D}(\alpha^{(A)}_{i}),\ \alpha^{(A)}_{i}(a) = a^*, \\ \Longrightarrow\ & a \in \mathcal{D}\Big((1_{B(H)} + \alpha^{(A)}_i)^{-1}\Big), \\ & \alpha^{(A)}_i (1_{B(H)} + \alpha^{(A)}_i)^{-1}(a) \text{ self-adjoint,} \end{aligned}$$

and

$$aA \geq 0 \iff a \in \mathcal{D}(\alpha^{(A)}_{i/2}), \alpha^{(A)}_{i/2}(a) \geq 0 .$$

For an invertible $a \in B(H)$ are equivalent :

(1) $aA \geq 0$ and $a^{-1}A \geq 0$;

(2) aA is self-adjoint and positive ;

(3) aA and $a^{-1}A$ are both self-adjoint and positive ;

(4) $a, a^{-1} \in \mathcal{D}(\alpha^{(A)}_{i/2})$ and $\alpha^{(A)}_{i/2}(a), \alpha^{(A)}_{i/2}(a^{-1}) \geq 0$.

Now we can prove the next result of S. L. Woronowicz, which shows that $\left(1_{B(H)}+\alpha_i^{(A)}\right)^{-1}$ can be expressed in terms of the phase relative to A:

Let $x \in B(H)$ be such that xA is symmetric. Then $x \in \mathcal{D}(\alpha_i^{(A)})$ and

$$\left(1_{B(H)} + \alpha_i^{(A)}\right)^{-1}(x)$$
$$= x - wo - \lim_{0 \neq \varepsilon \to 0} \frac{1}{2\varepsilon i}\left(\text{phase}_A(1_H + i\varepsilon x) - 1_H\right).$$

It follows :

Let $\mathcal{M} \subset B(H)$ be a von Neumann algebra. Then are equivalent :

(1) $\alpha_t^{(A)}(\mathcal{M}) = \mathcal{M}$ *for every* $t \in \mathbb{R}$;
(2) $x \in \mathcal{M}$, xA *closable* $\Rightarrow$ $\text{phase}_A(x) \in \mathcal{M}$;
(3) $x \in \mathcal{M}$, x *invertible* $\Rightarrow$ $\text{phase}_A(x) \in \mathcal{M}$.

Hint :

$$X := \left\{x \in \mathcal{M}\, ;\, xA \ \text{symmetrical}\ \right\} + i\left\{x \in \mathcal{M}\, ;\, xA \ \text{symmetrical}\ \right\}.$$

Foundation of the Tomita-Takesaki Theory

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H having a bicyclic vector ξ_o.

For $a \in \mathcal{M}$ are equivalent :

(1) the linear form $\mathcal{M} \ni x \longmapsto \omega_{\xi_o}(xa)$ is positive ;

(2) there exists $0 \leq a' \in \mathcal{M}'$ with $a\,\xi_o = a'\xi_o$;

(3) $a \in \mathcal{D}\Big(\alpha^{(\Delta)}_{i/2}\Big)$ és $\alpha^{(\Delta)}_{i/2}(a) \geq 0$.

Moreover, if the above equivalent conditions are satisfied, then $a' = J\,\alpha^{(\Delta)}_{i/2}(a)\,J \leq r(a)\,1_H$.

Using the polar decomposition theorem of normal forms of S. Sakai it follows :

For every $x \in \mathcal{M}$ there exists a uniquely defined partial isometry $u \in \mathcal{M}$ such that

$$\omega_{\xi_o}(\cdot\, ux)\,|\,\mathcal{M} \geq 0$$

and u^*u *is the orthogonal projection onto* $\overline{x(H)}$ *. Furthermore,* $x\Delta$ *is closable,* $|x|_{\Delta} = ux$ *and* $\text{phase}_{\Delta}(x) = u^*$.

The above result and Woronowicz' invariance theorem imply immediately the fundamental theorem of the Tomita-Takesaki Theory.

We notice as a curiosity :

$$x \in \mathcal{M} \implies x\,\Delta \text{ is closable,}$$

$$x \in \mathcal{M}\,,\; x\,\Delta \geq 0 \implies x\,\Delta \text{ is essentially self-adjoint.}$$