SURVEY OF MY RESEARCH ACTIVITY

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Contents

1. Overview

My research is at the interface of analysis and mathematical physics. Mathematically, I am primarily involved in functional analysis, operator theory, non-linear partial differential equations, and theoretical numerical analysis. My main focus is on the mathematical methods for quantum mechanics and their applications to models and problems of relevance for solid state physics and theoretical physics.

At present I work mainly on quantum many-body dynamics and effective non-linear PDEs, non-linear (pseudo-)differential Schrödinger equations, dispersive PDEs with also magnetic fields, operator and extension theory, spectral and quantum scattering theory, contact interactions and singular perturbations of elliptic operators and fractional Laplacians, geometric quantum confinement, infinite-dimensional linear inverse problems.

Paragraphs in the following which start with '➠' contain my main achievements.

Date: October 28, 2021.

2. Outline of my main research lines and collaborations

Primary research topics (at present):

(1) Dispersive linear and non-linear PDEs: well-posedness, scattering, ground state

→ Non-linear Schrödinger and Hartree equations: non-relativistic, relativistic, magnetic.
→ Schrödinger-Maxwell and Hartree-Maxwell systems.
→ Magnetic Strichartz estimates. Beyond Strichartz regime with viscosity regularisation.
→ Dispersive and dynamical properties of elliptic differential operators with singular perturbations.
→ Scattering behaviour of Schrödinger equations with singular, zero-range perturbations.

ongoing collaborations and regular scientific links: *P. Antonelli (GSSI L'Aquila), H. Cornean (Aalborg), V. Georgiev (Pisa), R. Scandone (GSSI L'Aquila), K. Yajima (Tokyo Gakushuin)*

(2) Operator theory of few-body and many-body quantum systems with zero-range interactions

→ Singular perturbations of elliptic operators.
→ Self-adjoint extension theories of symmetric operators and quadratic forms theory on Hilbert space.
→ Scaling limits on Schrödinger Hamiltonians with regular potentials yielding contact interactions.
→ Spectral properties of point-interaction Hamiltonians: collapse of the 3-body system (the Thomas effect), relaxation at zero-energy (the Efimov effect), binding properties.
→ Role of zero-energy resonances for finite-range potentials and delta-interactions.
→ Scattering properties of quantum Hamiltonians with zero-range interaction, wave operators, and resonances.
→ Point-interactions in finite domains: spectral geometric optimisation.

ongoing collaborations and regular scientific links: *S. Albeverio (Bonn), S. Aristarkhov (LMU Munich), S. Becker (Cambridge), J. Behrndt (TU Graz), H. Cornean (Aalborg), M. Correggi (Milan), G. Dell'Antonio (Rome La Sapienza and SISSA), D. Finco (UTIU Rome), M. Gallone (Milan), V. Lotoreichik (CAS Prague), D. Noja (Milan Bicocca), A. Ottolini (Stanford), A. Posilicano (Como), R. Scandone (GSSI L'Aquila), A. Teta (Rome La Sapienza), A. Trombettoni (Trieste), T. Turgut (Istanbul Boğaziçi), K. Yajima (Tokyo Gakushuin)*

(3) Derivation of effective non-linear Schrödinger equations from quantum many-body dynamics

→ Effective dynamics of many-body boson systems in the mean-field limit of infinite particles.
→ Short-range, hard-core scaling limits (the Gross-Pitaevski regime).
→ Effective dynamics under small and large external magnetic fields, or point-like perturbations (impurities).
→ Multi-component condensates (mixture, pseudo-spinor, spinor condensates, etc.) and coupled NLS.

ongoing collaborations and regular scientific links: *S. Cenatiempo (GSSI L'Aquila), G. De Oliveira (Minas Gerais), P. T. Nam (LMU Munich), A. Olgiati (CNRS Grenoble), P. Pickl (LMU Munich), G. Pitton (Imperial College London), R. Scandone (GSSI L'Aquila), A. Trombettoni (Trieste);* ***previous collaborations:*** *B. Schlein (Zurich), L. Erdős (IST Vienna)*

(4) Models of geometric quantum confinement and dynamical transmissions on manifolds.

→ Geometric quantum confinement on non-complete Riemannian manifolds
→ Self-adjointness of Laplace-Beltrami operators on Grushin-type manifolds, and identification of Markov extensions
→ Spectral analysis and scattering of the quantum transmission across singularities of the metric

ongoing collaborations and regular scientific links: *U. Boscain (École Polytechnique Paris), M. Gallone (Milan), E. Pozzoli (École Polytechnique Paris)*

(5) Operator-theoretic applications to numerical schemes and to PDEs.

→ Truncation theory for inverse linear problems in infinite-dimensional Hilbert spaces
→ Inverse problems with lack of coercivity and of the other standard assumptions of Petrov-Galerkin methods
→ Theory and applications of Krylov sub-spaces to infinite-dimensional inverse problems.
→ Friedrichs systems of PDE's in a Hilbert space framework: solvability and multiplicity

ongoing collaborations: *N. Antonić (Zagreb), A. N. Caruso (GSSI L'Aquila), M. Erceg (Zagreb), L. Grubišić (Zagreb), P. Novati (Trieste)*

SIDE RESEARCH TOPICS (AT PRESENT):

(6) The mathematics of quantum particles constrained on graphs, curves, or hyper-surfaces

- → Quantum models on graphs as the limiting description for nanotubes and complex molecules.
- → Effective dynamics in the limit of a waveguide shrinking to a graph. Emergence of boundary conditions at vertices.

ongoing collaborations and regular scientific links: *G. Dell'Antonio (Rome La Sapienza and SISSA), V. Lotoreichik (CAS Prague), D. Noja (Milan Bicocca)*

(7) Bose-Einstein condensation vs Superfluidity.

- → Rigorous characterisations of BEC/Superfluidity via CMRDM (centre of mass reduced density matrix) and Drude weights.
- → Finite-size effects of BEC.

ongoing collaboration: *B. Hetenyi (Bilkent)*

(8) Equilibrium/non-equilibrium in Quantum Statistical Mechanics with infinitely many degrees of freedom.

- → C^*-algebraic formulation of Quantum Mechanics and Quantum Statistical Mechanics.
- → Steady states vs Equilibrium (KMS) states.
- → Dissipative dynamics on large spin chains.

ongoing collaborations and regular scientific links: *F. Benatti (University of Trieste), G. Dell'Antonio (Rome La Sapienza and SISSA), G. Morchio (Pisa)*

3. Recent and forthcoming scientific meetings co-organised by me and focused on my research lines

✓ Workshop “Trieste Junior Quantum Days 2020”, Trieste 1-3 July 2020

✓ Workshop “Trieste Junior Quantum Days 2019”, Trieste 24-26 July 2019

✓ INdAM International Meeting “Mathematical Challenges in Zero-Range Physics, III Edition”, Rome, 9-13 July 2018

✓ Workshop “Trieste Junior Quantum Days 2018”, Trieste 11 and 18 May 2018

✓ Conference “Trails in Quantum Mechanics and Surroundings”, Trieste 29-30 January 2018

✓ Symposium “Junior Trieste Quantum Days 2017”, Trieste 12 and 19 May 2017

✓ International Meeting “Trieste Quantum Days 2017”, Trieste 20-24 February 2017

✓ SISSA Seminar “Analysis, Math-Phys, and Quantum” – a year-long regular research seminar series with national and international external speakers – editions 2014-2015, 2015-2016, 2016-2017

✓ International Workshop “Mathematical Challenges in Zero-Range Physics, II Edition”, Trieste 7-11 November 2016

✓ International Workshop “Trieste Quantum Days”, Trieste 21-24 June 2016

✓ INdAM International Meeting “Contemporary Trends in Quantum Mechanics”, Rome 4-8 July 2016

✓ International School and Workshop “Mathematical Challenges in Quantum Mechanics”, Bressanone 8-13 February 2016

✓ International Workshop “Mathematical Challenges of Zero-Range Physics”, CAS-LMU Munich 26-28 February 2014

4. Main research line #1. Derivation of effective non-linear Schrödinger equations for quantum many-body systems and composite condensation

The unmatchable complexity of the many-body Schrödinger equation

$$
\begin{aligned}
\mathrm{i}\partial_t \Psi_N \;&=\; \Big(\sum_{j=1}^{N} \big(-\Delta_{x_j} + U_{\mathrm{trap}}(x_j) \big) + \sum_{1\leqslant j<k\leqslant N} V(x_j - x_k) \Big) \Psi_N \\
\Psi_N &\equiv \Psi_N(x_1,\dots,x_N,t)\,, \qquad x_j \in \mathbb{R}^d\,, \qquad d\in\{1,2,3\}
\end{aligned} \tag{4.1}
$$

for a systems of N interacting particles (confined by a potential U_{trap} and coupled by a two-body potential V) when N is large ($N \sim 10^5 \div 10^{11}$ for a Bose-Einstein condensate, $N \sim 10^{23}$ for a Bose gas, $N \sim 10^{57}$ for a neutron star) can be addressed when the system consists of indistinguishable particles (bosons or fermions) and one deals, as is customary in practice, with one-body or few-body observables only. For a k-body observable $\mathcal{O}_k$ the expectation on the bosonic many-body wave-function Ψ_N is a the number

$$
\langle \Psi_N, \mathcal{O}_k \Psi_N \rangle \;=\; \mathrm{Tr}(\gamma_N^{(k)} \mathcal{O}_k) \tag{4.2}
$$

where one only needs to trace $\mathcal{O}_k$ against the so-called k-body reduced density matrix (or marginal) $\gamma_N^{(k)}$ associated with Ψ_N, namely a positive, self-adjoint, trace-class operator on the Hilbert space of k particles only, which retains an averaged information from Ψ_N, its integral kernel being given by tracing out $N-k$ degrees of freedom from the density matrix of the system, $|\Psi_N\rangle\langle\Psi_N|$.

This is the scenario for the vast mathematical field of the rigorous description of the Bose gas and its condensation [167, 35]. The eigenvalues of $\gamma_N^{(1)}$ having the natural interpretation of *occupation numbers*, Bose-Einstein condensation onto a one-body wave-function φ then corresponds to the occurrence

$$
\gamma_N^{(1)} \;\approx\; |\varphi\rangle\langle\varphi|\,, \tag{4.3}
$$

which, in the sense of the expectations of one-body observables, expresses the property

$$
\Psi_N(x_1,\dots,x_N) \;\sim\; \varphi(x_1)\cdots\varphi(x_N)\,. \tag{4.4}
$$

Of course, for an interacting system Ψ_N is not exactly factorised and has instead a pattern of correlations, to detect which one needs higher order marginals.

➠ In this context, in previous works [186, 182] I proved properties of the reduced marginals and equivalent definitions of BEC that were relevant for the contemporary literature.

On the dynamical side, one general goal is to derive the non-linear Schrödinger equation

$$
\mathrm{i}\partial_t \varphi(t) \;=\; -\Delta\phi(t) + U_{\mathrm{trap}}\phi(t) + 8\pi a_V |\varphi(t)|^2 \varphi(t) \tag{4.5}
$$

as the effective dynamics for $\Psi_N(t)$ given by (4.1) when N is sufficiently large and when the system is prepared at time $t=0$ in a condensate state of the form (4.4) in the sense (4.3), and where, as indicated by the experiments, a_V is the scattering length of the interaction V. The problem then amounts to closing the diagram

$$
\begin{array}{ccccc}
\Psi_N & \xrightarrow{\text{partial trace}} & \gamma_N^{(1)} & \xrightarrow{N\to\infty} & |\varphi\rangle\langle\varphi| \\
\text{many-body} \atop \textbf{linear}\text{ dynamics} \Big\downarrow & & \Big\downarrow & & \Big\downarrow {\textbf{nonlinear} \atop \text{Schrödinger eq.}} \\
\Psi_N(t) & \xrightarrow{\qquad\qquad} & \gamma_N^{(1)}(t) & \xrightarrow{N\to\infty} & |\varphi(t)\rangle\langle\varphi(t)|
\end{array} \tag{4.6}
$$

possibly with a quantitative rate of convergence in N.

In the lack (so far) of a rigorous control of the asymptotics $N \to \infty$ in a genuine thermodynamic limit, what is mathematically doable and physically still meaningful is to mimic the actual thermodynamic limit with some caricature of it realised by scaling the Hamiltonian (4.1) with N in such a way to retain at any N an amount of relevant physical features of the system (among which the property that kinetic and potential energy remain of the same order so that the interaction is still visible in the limit, as well as certain dilution properties of the system and short-range features of the interaction). Typical relevant scaling limits are those in which the two-body potential V that models the interaction among particles is replaced by a N-dependent two-body potential

$$V_N(x) \;=\; N^{3\beta-1}V(N^{\beta}x)\,, \qquad \beta\in[0,1] \tag{4.7}$$

(here $x = x_i - x_j$ is the relative coordinate between particle i and particle j). The regime $\beta = 0$ is the mean-field regime, whereas $\beta = 1$ gives the so-called Gross-Pitaevskii scaling regime.

➠ A more extended discussion of such scaling limits may be found in my works [180, 181].

A massive effort to prove (4.6) in various dimensions and various regimes of scaling, of singularities of the interaction potential, and for modifications of the Hamiltonian so as to account for external magnetic potentials, time-dependent traps, semi-relativistic kinetic energy, has been carried out for decades, boosted more recently by the important progress in the manipulation of cold atoms. Recent surveys are [252, 253, 35]. This required a variety of mathematical techniques for the control of the leading dynamics and its fluctuations (hierarchical, Fock-space methods, projection-counting methods, measure-theoretic, probabilistic methods) and major seminal contributions by Spohn, Bardos, Golse, Mauser, Yau, Erdős, Schlein, Rodnianski, Pickl, among others. A symmetrical scenario for fermions is also under intense investigation [34, 33, 240, 160, 31, 32].

➠ In collaboration with Erdős and Schlein we proved in a quantitative way the dynamical formation of correlations in a three-dimensional Bose-Einstein condensate [86].

➠ With Schlein we derived the effective description for the dynamical collapse of a boson star, up to the first blow-up time of the underlining semi-relativistic NLS [200].

➠ On the one-dimensional setting, in the work [183] I managed to improve the convergence sense of a previous derivation of NLS by Adami, Golse, and Teta, from the weak* L^∞-topology to the more physical trace norm topology. I also supervised a work of Olgiati [235] which provided for the first time the detail of the derivation, by means of the Pickls projection counting method, of the NLS with magnetic fields.

➠ More recently, in collaboration with Olgiati, we started a new line of investigation concerning Bose cases with *composite condensation* – condensate mixtures, pseudo-spinor condensates, spinor condensates, and fragmented condensates.

Condensate mixtures consist of a gas formed by different species of interacting bosons, each of which is brought to condensation, thus with a macroscopic occupation of a one-body orbital for each species, and no inter-particle conversion. They can be prepared as atomic gases of the same element, typically ^{87}Rb, which occupy two hyperfine states with no interconversion between particles of different hyperfine states [224, 176, 129, 130], or also as heteronuclear mixtures such as ^{41}K-^{87}Rb [217], ^{41}K-^{85}Rb [218], ^{39}K-^{85}Rb [175], and ^{85}K-^{87}Rb [238]. A comprehensive review of the related physical properties may be found in [241, Chapter 21]. The initial state is, in the sense of marginals,

$$\Psi_{N_1,N_2}(x_1,\dots,x_{N_1};y_1,\dots,y_{N_2}) \;\sim\; \prod_{j=1}^{N_1} u_0(x_j) \prod_{k=1}^{N_2} v_0(y_j) \tag{4.8}$$

on the Hilbert space $L^2(\mathbb{R}^{N_1 d},\mathrm{d}x_1\cdots\mathrm{d}x_{N_1})\otimes L^2(\mathbb{R}^{N_2 d},\mathrm{d}y_1\cdots\mathrm{d}y_{N_2})$. The effective dynamics observed in the experiments is ruled by a pair of coupled NLS equations,

$$\begin{aligned} \mathrm{i}\partial_t u &= h_1 u + \gamma_1 |u|^2 u + c_2\,\gamma_{12}|v|^2 u \\ \mathrm{i}\partial_t v &= h_2 v + \gamma_2 |v|^2 v + c_1\,\gamma_{12}|u|^2 v\,, \end{aligned} \tag{4.9}$$

where $c_j=\frac{N_j}{N_1+N_2}$, $j\in\{1,2\}$, are the population ratios, and the γ_α's, $\alpha\in\{1,2,12\}$, are the scattering lengths of the interactions, in suitable units.

➠ With Olgiati [190] we derived for the first time the mean-field version of (4.9), thus reproducing a two-component quantitative analogue of the scheme (4.6), from the many-body dynamics of the mixture and for fairly general interaction potentials. Then Olgiati [234] adjusted our scheme to the Gross-Pitaevskii scaling limit, yielding (4.9).

Right after such works, the derivation of effective NLS systems for condensate mixtures has soon gained interest in the community [13].

➠ With De Oliveira [74] we gave an *alternative* proof of the effective mean-field dynamics of a condensate mixture, deriving it for the first time within the other main scheme exploited in the community, namely by means of Fock space methods. In such scheme, designed to analyse the (smallness of the) fluctuation dynamics around the effective equations, we showed that the most natural object to monitor is the (effective+fluctuation) dynamics of coherent states, for which we indeed produced a quantitative convergence theorem as $N\to\infty$.

Pseudo-spinor condensates are the second topical scenario for composite condensation in which I am involved at present. These are gases of ultra-cold atoms that exhibit BEC and possess internal spin degrees of freedom which are often coupled to an external resonant micro-wave or radio-frequency radiation field, however, with no significant spin-spin internal interaction (whence the *pseudo*-spinor terminology). The order parameter of the condensation is therefore a *multi-component vector*, say,

$$\Psi_N(x_1,\dots,x_N,t=0) \;\sim\; \varphi_0(x_1)\cdots\varphi_0(x_N)\,, \tag{4.10}$$

where

$$\varphi_0 \;=\; \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \tag{4.11}$$

unlike scalar condensates such as liquid ^{4}He. The dynamical evolution of these quantum fluids observed in the experiments and predicted theoretically [260, 174, 128, 261, 241] follows a non-linear systems of coupled NLS of the form

$$\begin{aligned} i\partial_t u_t &= (-\Delta + U_{\uparrow}^{\mathrm{trap}})u_t + 8\pi a(|u_t|^2+|v_t|^2)u_t - V_{\mathrm{hf}}\,u_t + (B_1-\mathrm{i}B_2)v_t \\ i\partial_t v_t &= (-\Delta + U_{\downarrow}^{\mathrm{trap}})v_t + 8\pi a(|u_t|^2+|v_t|^2)v_t + V_{\mathrm{hf}}\,v_t + (B_1+\mathrm{i}B_2)u_t \end{aligned} \tag{4.12}$$

with initial data $u_{t=0}\equiv u_0$ and $v_{t=0}\equiv v_0$, where $U_{\uparrow}^{\mathrm{trap}}$ and $U_{\downarrow}^{\mathrm{trap}}$ are the trapping potentials for each hyperfine level, a is the scattering length of the two-body interaction, and $\mathbf{B}\equiv(B_1,B_2,-V_{\mathrm{hf}})$ is the external magnetic field applied to the system, the component V_{hf} being set so as to induce the hyperfine splitting between the two levels.

➠ With Olgiati [189] we recently derived (4.12) in three dimensions with time-dependent magnetic fields, and in the physically relevant Gross-Pitaevskii scaling.

Spinor condensates, the third scenario for composite condensation in which I am involved, are formed when a highly off-resonant magnetic confinement traps the atoms irrespectively of their hyperfine state: in this case the spin becomes a new degree of freedom and this produces interacting Bose gases of ultra-cold atoms where the spatial two-body interaction is mediated by a spin-spin coupling, the order parameter being now a vector in the hyperfine spin space. For spinor condensates, the hallmark of condensation manifests as a reversible spin-changing collisional coherence between particles. The earliest theoretical investigations and observations of spinor condensates appeared some 20 years ago [233, 136, 159]. By now the field has expanded through a vast series of experimental and theoretical studies, for a survey of which we refer to the comprehensive reviews [260, 174, 128, 261].

For typical modern experiments with $F = 1$ ^{87}Rb [58, 59], the experimentally observed evolution for the one-body spinor order parameter

$$\phi \equiv \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in L^2(\mathbb{R}^3, \mathrm{d}x) \otimes \mathbb{C}^3 \cong L^2(\mathbb{R}^3, \mathbb{C}^3) \tag{4.13}$$

is governed by the spinor Gross-Pitaevskii system

$$\begin{aligned} \mathrm{i}\partial_t u &= -\Delta u + 8\pi c_0(|u|^2 + |v|^2 + |w|^2)u \\ &\qquad + 8\pi c_2(|u|^2 + |v|^2 - |w|^2)u + 8\pi c_2(\overline{w}v)v \\ \mathrm{i}\partial_t v &= -\Delta u + 8\pi c_0(|u|^2 + |v|^2 + |w|^2)v \\ &\qquad + 8\pi c_2(|u|^2 + |w|^2)v + 16\pi c_2(\overline{v}w)u \\ \mathrm{i}\partial_t w &= -\Delta u + 8\pi c_0(|u|^2 + |v|^2 + |w|^2)w \\ &\qquad + 8\pi c_2(|v|^2 + |w|^2 - |u|^2)w + 8\pi c_2(\overline{u}\,\overline{v})v\,, \end{aligned} \tag{4.14}$$

where $c_0 = \frac{1}{3}(a_0 + 2a_2)$ and $c_2 = \frac{1}{3}(a_2 - a_0)$ and each a_{f} is the s-wave scattering length for collisions between particles with total two-body spin f.

➠ With Olgiati [191] we derived for the first time the mean-field version of (4.14), in the rigorous sense of spinor reduced density matrices as the number of particles goes to infinity. The difficulty here was to control, in addition to the terms of the fluctuations around the leading dynamics that are technically analogous to those arising from pseudo-spinor Hamiltonians, also new fluctuation terms that involve the genuine spin-spin interaction. The mean-field setting in [191] was just for presentation purposes: in fact, the meticulous analysis we had made in [189] to prove the emergence of the Gross-Pitaevskii dynamics for pseudo-spinor condensates can be extended to such new terms, so as to produce the Gross-Pitaevskii system (4.14). As a further result of [191], we showed that when the resulting control of the dynamical persistence of condensation is quantified with the parameters of modern observations, one obtains a bound that remains quite accurate for the whole typical duration of the experiment (hundreds of msec).

The very difficult description from first principles of *fragmented condensates* remains virtually unexplored and surely deserves a deep investigation with novel ideas. Fragmentation [259, 222] is that feature of BEC in which there occur multiple macroscopic occupations of certain one-body states, which in the language of reduced marginals corresponds to

$$\begin{gathered} \gamma_N^{(1)} \approx c_1|\varphi_1\rangle\langle\varphi_1| + \cdots + c_r|\varphi_r\rangle\langle\varphi_r| \\ c_r \geqslant 0\,, \qquad \sum_{j=1}^{r} c_r = 1\,, \qquad \langle\varphi_j, \varphi_k\rangle = \delta_{j,k}\,. \end{gathered} \tag{4.15}$$

The *no-rank-one* nature of $\gamma_N^{(1)}$ (asymptotically in N) is a source of major complications for the derivation of the systems of NLS equations coupled in various ways and conjectured / expected to provide the correct dynamical description. Also, and most importantly, whereas for *complete condensation* such as (4.3) one has

$$\gamma_N^{(1)} \xrightarrow{N\to\infty} |\varphi\rangle\langle\varphi| \qquad \Leftrightarrow \qquad \gamma_N^{(k)} \xrightarrow{N\to\infty} |\varphi^{\otimes k}\rangle\langle\varphi^{\otimes k}|\,, \tag{4.16}$$

for *fragmented condensation*, instead, it cannot be enough to rely on the sole asymptotic structure (4.15) for $\gamma_N^{(1)}$, and a much more detailed control of higher marginals is needed, as one can see from the two very different many-body states

$$\Psi_N^{(1)} \;=\; \tfrac{1}{\sqrt{2}}\,u^{\otimes N} + \tfrac{1}{\sqrt{2}}\,v^{\otimes N}\,, \qquad \Psi_N^{(2)} \;=\; \Big(u^{\otimes N/2}\otimes v^{\otimes N/2}\Big)_{\mathrm{sym}}\,, \qquad u\perp v\,,$$

which have the same rank-two one-body marginal.

Parallel to the dynamical analysis, and in some sense preliminary to that, is the study of the ground state properties of Bose gases. Under appropriate conditions, suitable energy functionals are expected to capture, through their minimiser, the many-body ground state energy and the orbital for BEC in the many-body ground state. This is a deeply investigated subject for one-component condensates: for the vast, and by now classical literature on the uniqueness of the minimiser of the Hartree or Gross-Pitaevskii functional, the leading order of the ground state energy, and the emergence of condensation in the ground state, in the case of single-component Bose gases, we refer to the monograph [167] and the references therein.

More recently, based on quantum de Finetti methods [61, 163] the ground state energy asymptotics and the proof of condensation in the ground state was re-obtained by Lewin, Nam, and Rougerie [162] in the mean-field scaling, and by Nam, Rougerie, and Seiringer [227] in the GP scaling. Still for one component, the Bogoliubov correction in the MF scaling was investigated by Seiringer, Grech, Lewin, Nam, Serfaty, Solovej, Napiorkowski [255, 119, 164, 226].

➠ For mixtures of Bose-Einstein condensates, in collaboration with Nam and Olgiati [188] we recently proved that, depending on whether one adopts the mean-field or the Gross-Pitaevskii scaling, the correct effective functional is the Hartree functional

$$\begin{aligned}\mathcal{E}^{\mathrm{H}}[u,v] \;:=\; & c_1\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + c_1\int_{\mathbb{R}^3}U^{(1)}_{\mathrm{trap}}|u|^2\,\mathrm{d}x + \frac{c_1^2}{2}\int_{\mathbb{R}^3}(V^{(1)}*|u|^2)\,|u|^2\,\mathrm{d}x \\ & + c_2\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x + c_2\int_{\mathbb{R}^3}U^{(2)}_{\mathrm{trap}}|u|^2\,\mathrm{d}x + \frac{c_2^2}{2}\int_{\mathbb{R}^3}(V^{(2)}*|v|^2)|v|^2\,\mathrm{d}x \\ & + c_1c_2\int_{\mathbb{R}^3}(V^{(12)}*|v|^2)|u|^2\,\mathrm{d}x\end{aligned} \tag{4.17}$$

or the Gross-Pitaevskii functional

$$\begin{aligned}\mathcal{E}^{\mathrm{GP}}[u,v] \;:=\; & c_1\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + c_1\int_{\mathbb{R}^3}U^{(1)}_{\mathrm{trap}}|u|^2\,\mathrm{d}x + 4\pi a_1c_1^2\int_{\mathbb{R}^3}|u|^4\,\mathrm{d}x \\ & + c_2\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x + c_2\int_{\mathbb{R}^3}U^{(2)}_{\mathrm{trap}}|u|^2\,\mathrm{d}x + 4\pi a_2c_2^2\int_{\mathbb{R}^3}|v|^4\,\mathrm{d}x \\ & + 8\pi a_{12}c_1c_2\int_{\mathbb{R}^3}|v|^2|u|^2\,\mathrm{d}x\,,\end{aligned} \tag{4.18}$$

in the sense that under physically relevant assumptions on the potentials the minimisation problems

$$
\begin{aligned}
e_{\mathrm{H}} &:= \inf_{\substack{u,v\,\in\, L^2(\mathbb{R}^3)\\ \|u\|_2=\|v\|_2=1}} \mathcal{E}^{\mathrm{H}}[u,v] \\
e_{\mathrm{GP}} &:= \inf_{\substack{u,v\,\in\, L^2(\mathbb{R}^3)\\ \|u\|_2=\|v\|_2=1}} \mathcal{E}^{\mathrm{GP}}[u,v]
\end{aligned}
\tag{4.19}
$$

have a unique solution, the ground state of the mixture exhibits condensation for each component onto the one-body minimisers u_0 and v_0 of (4.19), and the ground state energy is given asymptotically by

$$
E_N^{\mathrm{GP}} \;=\; N e_{\mathrm{GP}} + o(N) \tag{4.20}
$$

or

$$
E_N^{\mathrm{MF}} \;=\; N e_{\mathrm{H}} + \inf \sigma(\mathbb{H}) + o(N)\,. \tag{4.21}
$$

The operator $\mathbb{H}$ in (4.21) being the Bogoliubov Hamiltonian (the second quantisation of the Hessian of the Hartree functional evaluated at the minimiser): we thus proved the correctness of Bogoliubov's theory for mixtures of BEC in the mean-field limit.

The *two-dimensional* counterpart of the Gross-Pitaevskii system (4.9) for Bose mixtures has been deeply investigated in the mathematical literature over the last decade, in particular in the works [169, 97, 172, 60, 165, 146]. In turn, this subject has been much less studied numerically – unlike the richness of the numerical literature for the two- and three-dimensional single-component case – with an almost exclusive focus on the mixtures of *rotating* condensates [27, 275, 248, 205, 266]

➠ In collaboration with Pitton [198, 197] we developed a systematic numerical study of the two-dimensional Gross-Pitaevskii system in a wide range of relevant regimes of population ratios and intra-species and inter-species interactions. In particular we tested three paradigmatic configurations: the 'one-shot' head-on scattering, the condensate-over-condensate relaxation, and the multiple re-collisions for a harmonically trapped binary condensate.

➠ Our numerical method is based on a Fourier collocation scheme in space combined with a fourth order integrating factor scheme in time. Remarkably, with our numerical code we implement MPI parallel simulations in which we have no difficulties to reach resolutions of magnitude 16384×16384. This is in contrast with the much lower resolution adopted in the previous numerical works (a 96×96 grid in [248], which becomes 200×256 in [275], 256×256 in [27], and 1024×128 in [28]), where the considered phenomena do not involve a *complicated collision interaction* between spatially well-localised components. Previous resolutions were insufficient and parallel computation was required, in order to resolve relevant short-scale details of the dynamics.

5. Main research line #2. Few-body and many-body quantum systems with zero-range interactions

In the 1930's Quantum Mechanics began to be applied to the newly observed nuclear phenomena. At first, the decrease by a factor 10^{-5} from the atomic to the nuclear scale made it plausible to model the interaction among nucleons as a delta-like interaction.

In 1932 Wigner [267] calculated that the nuclear forces interaction must be of very short range and very strong magnitude. This led three years later first Bethe and Peierls [37, 38], then Thomas [264], Fermi, [98], and Breit [44] to describe the neutron-proton scattering, and then some 20 years later, in 1955, Ter-Martirosyan and Skornyakov [258] to describe the three-body problem with zero-range interaction, by means of the Schrödinger equation in the approximation of a two-body potential of very short range, yielding to a boundary condition today known as the celebrated "*Bethe-Peierls contact condition*" or, in momentum space, the "*Ter-Martirosyan–Skornyakov condition*".

Such a constraint, still today ubiquitous in many formal physical treatments, prescribes on the basis of physical heuristics that the wave-function $\Psi(x_1,\dots,x_N)$ of N three-dimensional particles subject to a two-body zero-range interaction of scattering length a_{ij} among particles i and j behaves asymptotically as

$$\Psi(x_1,\dots,x_N) \;\approx\; \Big(\frac{1}{|x_i-x_j|}-\frac{1}{a_{ij}}\Big) \qquad \text{as } |x_i-x_j|\to 0\,. \tag{5.1}$$

Clearly, what makes this approximation appealing, and computationally advantageous, is its dependence on few parameters only (the a_{ij}'s), instead of the complete knowledge of the interaction.

The first rigorous mathematical analysis of few-body quantum systems with zero-range interaction was initiated in the early 1960's by Berezin and Faddeev [36] for the two-body problem and by Minlos and Faddeev [214, 213] for the three-body problem. Physically it soon became clear that the assumption of zero range was only a crude simplification of no fundamental level, yet the use of formal delta-like potentials remained for some decades as a tool for a formal first-order perturbation theory in application to atomic physics [81]. Albeverio, Gesztesy, and Høegh-Krohn, and their collaborators (among whom, Streit and Wu), in the end of the 1970's and throughout the 1980's, unified an amount of previous investigations and established a proper mathematical branch on rigorous models of point interactions, with a systematic study of *two-body* Hamiltonians and of *one-body* Hamiltonians with finite or infinitely many *fixed centres* of point interaction. The monograph [4] provides a comprehensive overview.

The main tools in this new mainstream were: self-adjoint extension theory to construct point interaction Hamiltonians as extensions of the restriction of the free Laplacian to functions that vanish in a neighbourhood of the point where the interaction is supported; resolvent identities (of Kreĭn and of Konno-Kuroda type, see [4, Appendices A and B]) by which such self-adjoint extensions were recognised to be finite-rank perturbations of the free Laplacian, in the resolvent sense, and were also re-obtained by resolvent limits of Schrödinger Hamiltonians with shrinking potentials; plus an amount of additional methods (Dirichlet quadratic forms, non-standard analysis methods, renormalisation methods) for specific problems.

The three-body problem with point interaction re-gained mathematical centrality too (while physically a stringent experimental counterpart was still lacking) around the end of the 1980's and throughout the 1990's. This was first due to Minlos and his school [206, 215, 219, 177, 178, 207, 256] (among which Melnikov, Mogilner, and Shermatov), by means of the operator-theoretic approach used for three identical bosons by Minlos and Faddeev, and slightly later due to Dell'Antonio and his school [263, 79, 80] (among which Figari and Teta), with an approach based on quadratic forms, where the "physical" energy form is first regularised by means of an ultra-violet cut-off and a suitable renormalisation procedure, and then is shown to be realised by a self-adjoint Hamiltonian. An alternative direction was started further later by Pavlov and his school (Kuperin, Makarov, Melezhik, Merkuriev, and

Motovilov) [157, 173], by introducing internal degrees of freedom, i.e., a spin-spin contact interaction, so as to realise semi-bounded below three-body Hamiltonians.

More recently the subject has been experiencing a new boost, due to the rapid progress in the manipulation techniques for ultra-cold atoms and, in particular, for tuning the effective s-wave scattering length by means of a magnetically induced Feshbach resonance [239, Section 5.4.2]. This has made it possible to prepare and study ultra-cold gases in the so-called "*unitary regime*" [56], i.e., the case of negligible two-body interaction range and huge, virtually infinite, two-body scattering length (both lengths being compared to a standard reference length such as the Bohr radius). In such a regime, unitary gases show properties, including superfluidity, that have the remarkable feature of being universal in several respects [42], and are under active experimental and theoretical investigation. Experimentally (as reviewed in the introductory sections of [202, 196]), zero-range interactions in ultra-cold atom physics are today far from being just an idealisation of real-world two-body potentials with small support and in many realisations the delta-like character of the interaction turns out to be an extremely realistic and in fact an unavoidable description.

In turn, all this has brought new impulse and motivations to the already developing mathematical research on the subject, with a series of fundamental contributions in the last few years [208, 209, 210, 99, 64, 212, 211, 65, 193, 194], many of which provide rigorous ground to experimental or numerical evidence on the physical side.

The first natural problem one encounters is the construction of an unambiguous (i.e., self-adjoint) and stable (i.e., semi-bounded from below) Hamiltonian of point interaction for the system. As was realised since the early works of Minlos and Faddeev [214, 213], two-body boundary conditions of Bethe-Peierls/Ter-Martirosyan-Skornyakov type (5.1) at the coincidence hyperplanes $\{x_i = x_j\}$ with fixed inverse scattering length $\alpha \equiv -(4\pi a)^{-1}$ only identify a self-adjoint operator under special regimes of particle masses and symmetry, whereas in general they only select a symmetric operator that admits self-adjoint extensions. Thus, in general,

$$H_\alpha^{\mathrm{TMS}} \;\subsetneq\; (H_\alpha^{\mathrm{TMS}})^* \,. \tag{5.2}$$

The class of such extensions (even when it consists of one element only) is customarily referred to as the *TMS Hamiltonians of contact interactions*.

➠ The identification and classification of such extensions is best understood within the Kreĭn-Višik-Birman (KVB) self-adjoint extension theory. Due to this circumstance, together with Gallone and Ottolini [108], I recently revisited the KVB theory in application to contact interaction Hamiltonians.

➠ This allowed us to realise that the KVB theory is particularly informative (and considerably more than the modern language of boundary triplets [254, Chapters 13 and 14]) also for other quantum problems of multiplicity of self-adjoint realisations, the primary example of which is the Dirac operator with Coulomb interaction

$$H \;:=\; -\mathrm{i}c\hbar\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 - \frac{cZ\alpha_{\mathrm{f}}}{|x|}\mathbb{1}\,. \tag{5.3}$$

For the Hamiltonian (5.3), in collaboration with Gallone [104, 106, 105] we classified all extensions, and their spectra, in the 'critical' regime (atomic number $118 < Z < 136$), qualifying each extension by the asymptotics as $|x| \to 0$ of the generic function in the operator domain.

➠ Back to contact interaction, for the prototypical '*2+1 fermionic system*' (two identical fermions in contact interaction with a third particle of different nature and relative mass m as compared to the fermion mass), a rigorously constructed TMS Hamiltonian H_α for $m > m^* \approx (13.607)^{-1}$ (the Efimov mass), together with the precise determination of m^* and the proof of the self-adjointness and the semi-boundedness from below of H_α, was done in the work [64] by Correggi, Dell'Antonio, Finco, Michelangeli, and Teta, by means of quadratic form techniques for contact interactions [263, 99], an approach started by Dell'Antonio, Figari, and Teta [79].

A previous thorough investigation by Minlos [206, 210, 212, 211] and Minlos and Shermatov [215], based instead on the KVB theory, had the virtue of showing the emergence of a multiplicity of TMS extensions for the 2+1 fermionic system when m is not too large, say, $m < m_{\mathrm{Minlos}}$. However, Minlos's analysis contained a flaw in the treatment of the so-called 'space of charges', thus resulting in the wrong threshold m_{Minlos}, as compared to observations and numerical simulations from physicists.

➠ Such a flaw was only recently identified and explained by Michelangeli and Ottolini [193] with the correct application of the KVB theory.

➠ In a further work by Correggi, Dell'Antonio, Finco, Michelangeli, and Teta [65], in the special case $\alpha = 0$ (the '*unitary regime*', i.e., infinite scattering length) we derived from first principles the rigorous threshold m^{**} for the multiplicity ($m < m^{**}$) or the uniqueness ($m > m^{**}$) of TMS realisations, in complete agreement with the physical evidences.

➠ For generic α, the identification of H_α as the highest (Friedrichs-type) among all the Ter-Martirosyan–Skornyakov self-adjoint Hamiltonians was finally made by Michelangeli and Ottolini in [194]. The rigorous derivation of the correct threshold m^{**} (for generic α) is still missing and is a major open problem.

➠ Analogous questions (self-adjointness and semi-boundedness) are under investigation for other few-body and many-body systems of $M + N$ identical particles with contact interactions. The $N + 1$ fermionic system was studied by Dell'Antonio, Figari, and Teta [79], Minlos [207, 208, 209], Correggi, Dell'Antonio, Finco, Michelangeli, and Teta [64], Moser and Seiringer [220], Griesemer and Linden [120].

➠ The 2+2 fermionic system is at an early stage of mathematical understanding. In the work [196] by Pfeiffer and myself we completed the construction of a self-adjoint TMS Hamiltonian and we found numerically that the system is stable. An ongoing research with Ottolini is aiming at providing rigorous grounds to the findings of [196]. Meanwhile, refining the estimates of the quadratic form approach of Finco and Teta [99], Moser and Seiringer have recently shown stability [221].

➠ Next to the construction of the contact interaction Hamiltonian is the study of its spectral properties. In a work with Schmidbauer [202], which opened this line of investigation, we qualified in part analytically and for the remaining part numerically the ground state energy of the 2+1 fermionic H_α as a function of m, exploring in particular the regime $m \downarrow m^*$.

➠ With Becker and Ottolini [30], we continued the spectral analysis of the 2+1 fermionic model at zero range, and for arbitrary magnitude of the interaction, and arbitrary value of the mass parameter (the ratio between the mass of the third particle and that of each fermion) above the stability threshold, we identified the essential spectrum, localised the discrete spectrum and proved its finiteness, qualified the angular symmetry of the eigenfunctions, and proved the increasing monotonicity of the eigenvalues with respect to the mass parameter. We also demonstrated the existence or absence of bound states in the physically relevant regimes of masses.

➠ In the recent work [185] I obtained and discussed the complete mathematical construction of the physically relevant Hamiltonians for *bosonic trimers*, both as operators and as quadratic forms, together with their spectral analysis (such Hamiltonians display both the Efimov and the Thomas effect), the study of regularised models, and the analysis of related ill-posed models in the literature.

Three-body or more generally many-body systems with contact interactions still represent an ample territory of mathematical challenges. As compared to that, the *two-body* problem with contact interaction, as well as the one-particle problem with many fixed point interaction centres, is by now completely understood [4, 10], yet there still emerge new questions related with contemporary problems.

➠ With Monaco [187] we considered the Kronig-Penney model for a quantum crystal with equispaced periodic delta-interactions of alternating strength [156, 4, 138, 10, 158]. For this model all spectral gaps at the centre of the Brillouin zone are known to vanish, although this noticeable property was proved through a very delicate analysis of the discriminant of the corresponding ODE and the associated monodromy matrix [274, 232]. We found a new, alternative proof by showing that this model can be approximated, in the norm resolvent sense, by a model of regular periodic interactions with finite range for which Michelangeli and Zagordi [203] had previously proved that all gaps at the centre of the Brillouin zone are still vanishing. In particular this shows that the vanishing gap property is stable in the sense that it is present also for the "physical" approximants and is not only a feature of the idealised model of zero-range interactions.

Other sources of interesting modern problems are surely quantum metric graphs. In order to define the Schrödinger dynamics on a metric graph it is necessary to impose suitable (self-adjoint) boundary conditions at the vertices [153, 154]. The boundary conditions used in chemical physics are seldom motivated theoretically, and are usually chosen so as to fit experimental data [12]. With Dell'Antonio we study how the vertex boundary conditions for the dynamics of a quantum particle constrained on a graph emerge in the limit of the dynamics of a particle in a tubular region around the graph ("fat graph") when the transversal section of this region shrinks to zero. (The limit of the Laplacian in shrinking tubular domains was attacked previously by [247, 133, 45, 242, 243, 78, 121].)

➠ With Dell'Antonio [75] we gave evidence of the fact that if the limit dynamics exists and is induced by the Laplacian on the graph with certain self-adjoint boundary conditions, such conditions are determined by the possible presence of a zero energy resonance on the fat graph. Pictorially, one may say that in the shrinking limit the resonance acts as a bridge connecting the boundary values at the vertex along the different rays.

➠ An analogous mechanism on the half-line with shrinking potentials at the origin was studied in collaboration with Dell'Antonio in [76]: we found that in the singular limit the dynamics is generated by a self-adjoint negative Laplacian on the half-line, with a possible preservation or modification of the boundary condition at the origin, depending on the magnitude of the scaling and of the strength of the potential.

6. Main research line #3. Linear and non-linear PDEs of relevance for quantum systems

There is a vast class of linear and non-linear partial differential equations that arise in connection with the class of problems described in Sections 4 and 5 and thus describe quantum systems that are relevant in modern mathematical physics. Their study requires a combination (and often a non-trivial adaptation) of the general PDE methods, especially for Schrödinger-type equations and semi-linear dispersive PDEs, with specific features dictated by the quantum problem itself.

As discussed in Section 4, non-linear, time-dependent Schrödinger equations with cubic non-linearity emerge as the effective dynamical equations for quantum many-body dynamics – the non-linearity is cubic as an effect of the *two-body* interaction; a generalisation to *three-body* interactions produces, at the effective level, a quintic non-linearity. In a different context, NLS equations also emerge as the effective dynamics of quantum plasmas: for densely charged plasmas, the pressure term in the degenerate (i.e., zero-temperature) electron gas is effectively given by a non-linear function of the electron charge density [127], which in the wave-function dynamics corresponds to a power-type non-linearity (further details, e.g., in [14]).

Within the well-established, systematic theory of non-linear Schrödinger equations [57], relevant generalisations are the inclusion of external magnetic potentials, pseudo-relativistic Laplacian, time-dependent fields, etc. In fact, the very derivation techniques of NLS from the many-body linear dynamics require that the underlying NLS is well-posed for all the times for which the reduced marginal $\gamma_N^{(1)}(t)$ is controlled in the limit $N \to \infty$, as the above-mentioned work [200] with Schlein shows (the many-body collapse of the boson star is shown to coincide with the NLS blow-up time).

A key case is the Cauchy problem for the non-linear Hartree equation with external, singular magnetic fields,

$$
\begin{aligned}
&\mathrm{i}\partial_t u \;=\; -(\nabla-\mathrm{i}\mathbf{A})^2 u + Vu + (W*|u|^2)\,u \\
&(t,x)\in\mathbb{R}\times\mathbb{R}^{d+1} \qquad \mathbf{A}:\mathbb{R}^d\to\mathbb{R}^d\,, \qquad V:\mathbb{R}^d\to\mathbb{R}\,, \qquad W:\mathbb{R}^d\to\mathbb{R}\,.
\end{aligned}
\tag{6.1}
$$

The standard well-posedness scheme for (6.1) [57, Corollary 6.1.2] requires suitable *magnetic Strichartz estimates* for $-(\nabla-\mathrm{i}\mathbf{A})^2$ or for $-(\nabla-\mathrm{i}\mathbf{A})^2+V$, clearly combined with diamagnetic inequality [166, Theorem 7.21]

$$
|\nabla|f||\;\leqslant\;|(\nabla-\mathrm{i}\mathbf{A})f| \qquad \begin{array}{l}\text{for a.e. } x\in\mathbb{R}^d\\ \forall f\in H^1_{\mathbf{A}}(\mathbb{R}^d)\end{array} \qquad \mathbf{A}\in L^2_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)\,.
\tag{6.2}
$$

The needed magnetic Strichartz estimates are available for certain classes of $\mathbf{A}$'s and V's: one the one hand, smooth electric and magnetic potentials $V\in C^\infty(\mathbb{R}^d,\mathbb{R})$ and $\mathbf{A}\in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$ with a growth at spatial infinity that is at most quadratic for V and linear for $\mathbf{A}$, as proved by Yajima [269, 216]); on the other hand, rough $\mathbf{A}$'s and V's up to the critical scaling $|\mathbf{A}(x)|\sim|x|^{-1}$ and $|V(x)|\sim|x|^{-2}$, as proved by Erdoğan, Goldberg, and Schlag [88, 89], D'Ancona and Fanelli [68], D'Ancona, Fanelli, Visciglia, and Vega [69], Fanelli, Felli, Fontelos, and Primo [95], and others. (In fact, counterexamples are known to the validity of Strichartz estimates for certain $\mathbf{A}$-fields when $d\geqslant 3$ and certain V-fields when $d\geqslant 3$ which decay as $|x|\to\infty$ less than the critical behaviour [116, 96].)

➠ To go *beyond Strichartz-controllable magnetic fields* one approach is to exploit energy methods: to set up the appropriate energy space that identifies, as a form domain, the self-adjoint realisation of the map $u\mapsto-(\nabla-\mathrm{i}\mathbf{A})^2u+Vu$ corresponding to the linear part of the Hartree equation; then, to prove local existence and uniqueness by means of a contraction argument based on the fact that the Hartree non-linearity $\mathcal{N}(u)=(W*|u|^2)u$ is locally Lipschitz in the energy space (no conservation rules are used in this step); last, to prove that this solution is global by means of the charge and energy conservation. In the work [184] I carried on this programme proving global well-posedness for (6.1) for $L^2_{\mathrm{loc}}(\mathbb{R}^d)$-magnetic fields, $d\geqslant 2$, and fairly singular potentials W.

The latter approach of course constrains (the negative part of) V to be Δ-form bounded with relative bound <1 (say, $V(x)\sim|x|^{-2+\varepsilon}$, $\varepsilon>0$) and cannot accommodate the physically relevant *local* non-linearities. The typical NLS one would like to control is rather

$$
\begin{aligned}
\mathrm{i}\partial_t u \;&=\; -(\nabla-\mathrm{i}\,\mathbf{A})^2u+\mathcal{N}(u) \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^3\\
\mathcal{N}(u)\;&=\;\lambda_1|u|^{\gamma-1}u+\lambda_2(|\cdot|^{-\alpha}*|u|^2)u \qquad \begin{array}{l}\gamma\in(1,5]\\ \alpha\in(0,3)\\ \lambda_1,\lambda_2\geqslant 0\,.\end{array}
\end{aligned}
\tag{6.3}
$$

➠ With Antonelli and Scandone [15] we proved the existence of global-in-time, finite energy, weak solutions to (6.3) in the presence of an external, rough, time-dependent magnetic potential, say,

$$\mathbf{A} \in L^a_{\mathrm{loc}}(\mathbb{R}, L^b(\mathbb{R}^3, \mathbb{R}^3))\,, \qquad a \in (4, +\infty], \quad b \in (3,6), \quad \tfrac{2}{a} + \tfrac{3}{b} < 1$$

or

$$\mathbf{A} \in L^a_{\mathrm{loc}}(\mathbb{R}, W^{1,\frac{3b}{3+b}}(\mathbb{R}^3, \mathbb{R}^3))\,, \qquad a \in (2, +\infty], \quad b \in (3, +\infty], \quad \tfrac{2}{a} + \tfrac{3}{b} < 1\,,$$

with also

$$\partial_t \mathbf{A} \in L^1_{\mathrm{loc}}(\mathbb{R}, L^b(\mathbb{R}^3, \mathbb{R}^3))\,.$$

Global well-posedness of (6.3) and stability results in the case of suitable smooth potentials were proved by De Bouard [73], Nakamura and Shimomura [225], Michel [179], but with our assumptions above energy methods or magnetic Strichartz estimates based approach cannot work. We then exploited a 'viscosity' regularisation, introducing a dissipation term in the equation

$$i\partial_t u \;=\; -(1 - \mathrm{i}\,\varepsilon)(\nabla - \mathrm{i}\,\mathbf{A})^2 u + \mathcal{N}(u) \tag{6.4}$$

and studying the approximated problem, in the same spirit as Guo, Nakamitsu, and Strauss for the study of the Maxwell-Schrödinger system [126]. We first obtained suitable Strichartz-type and smoothing estimates for the viscous magnetic evolution semi-group $t \mapsto e^{(i+\varepsilon)t\Delta}$, which allow to prove the global well-posedness of (6.4) (in the energy critical case $\gamma = 5$ too), then, thanks to uniform-in-ε mass and energy a priori bounds on the approximated solutions it is possible to remove the regularisation and to show the existence of a finite energy weak solution to our original problem (6.3) at the obvious price of loosing the uniqueness, as well as its continuous dependence on the initial data.

➠ In a follow-up project with Antonelli and Scandone [16] we managed to cover the endpoint case of a (constant-in-time) magnetic potential $\mathbf{A} \in L^3(\mathbb{R}^3), L^3(\mathbb{R}^3))$ by means of endpoint magnetic Strichartz estimates for the propagator defined by $\mathrm{i}\partial_t u = -(1 - i\varepsilon)\Delta u$, which in turn are obtained through an adaptation to the magnetic case of the Keel-Tao argument [150].

One further class of previously unexplored Schrödinger equations of physical relevance is the one where the Laplacian $-\Delta$ is replaced with its singular perturbation, the point interaction Hamiltonian centred at a number of given fixed points $y_1, \dots, y_N \in \mathbb{R}^3$ and realised self-adjointly with boundary condition of Bethe-Peierls/ TMS-type when $|x - y_j| \to 0$, as in (5.1).

The construction of such operator is by now standard [4]. The class of self-adjoint extensions in $L^2(\mathbb{R}^3)$ of the positive and densely defined symmetric operator $-\Delta|_{C^\infty_0(\mathbb{R}^3\setminus\{0\})}$ is a one-parameter family of operators $-\Delta_\alpha$, $\alpha \in (-\infty, +\infty]$, defined by

$$\begin{aligned} \mathcal{D}(-\Delta_\alpha) \;&=\; \Big\{ \psi \in L^2(\mathbb{R}^3) \,\Big|\, \psi = \phi_\lambda + \frac{\phi_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\, G_\lambda \text{ with } \phi_\lambda \in H^2(\mathbb{R}^3) \Big\} \\ (-\Delta_\alpha + \lambda)\,\psi \;&=\; (-\Delta + \lambda)\,\phi_\lambda\,, \end{aligned} \tag{6.5}$$

where $\lambda > 0$ is an arbitrarily fixed constant and

$$G_\lambda(x) \;:=\; \frac{e^{-\sqrt{\lambda}\,|x|}}{4\pi|x|} \tag{6.6}$$

is the Green function for the Laplacian, that is, the distributional solution to $(-\Delta + \lambda)G_\lambda = \delta$ in $\mathcal{D}'(\mathbb{R}^3)$. The above decomposition of a generic $\psi \in \mathcal{D}(-\Delta_\alpha)$is unique and is valid for every chosen λ. The extension $-\Delta_{\alpha=\infty}$ is the Friedrichs extension and is precisely the self-adjoint $-\Delta$ on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$. Moreover, each $\psi \in \mathcal{D}(-\Delta_\alpha)$ satisfies indeed the short range asymptotics

$$\psi(x) \;=\; c_\psi\Big(\frac{1}{|x|}-\frac{1}{a}\Big)+o(1) \qquad \text{as } x\to 0\,, \qquad a:=(-4\pi\alpha)^{-1} \tag{6.7}$$

for some $c_\psi \in \mathbb{C}$.

Analogously, fixed $Y := \{y_1,\dots,y_N\}$ (N distinct points in $\mathbb{R}^3$), the operator

$$\mathring{H}_Y \;:=\; -\Delta \upharpoonright C^\infty_0(\mathbb{R}^3\backslash Y) \tag{6.8}$$

is densely defined, real symmetric, and non-negative on $L^2(\mathbb{R}^3)$, with deficiency indices (N,N), and hence it admits a N^2-parameter family of self-adjoint extensions. The most relevant sub-class of them is the N-parameter family

$$\{H_{\alpha,Y}\,|\,\alpha\equiv(\alpha_1,\dots,\alpha_N)\in(-\infty,\infty]^N\} \tag{6.9}$$

of so-called 'local' extensions, namely extensions of $\mathring{H}_Y$ whose domain of self-adjointness is only qualified by the asymptotics

$$\psi(x) \underset{x\to y_j}{\sim} \frac{1}{|x-y_j|}-\frac{1}{a_j}\,, \qquad a_j:=-(4\pi\alpha_j)^{-1}\,. \tag{6.10}$$

The perturbations $H_{\alpha,Y}$ of $-\Delta$ have a long history of investigation. In the mathematical literature they were introduced and characterised for the case $N=1$ by Berezin and Faddeev [36], Albeverio, Høegh-Krohn, and Streit [6], Nelson [229], Albeverio, Fenstad, and Høegh-Krohn [3], and Albeverio and Høegh-Krohn [5]. For generic $N\geqslant 1$ centres, $H_{\alpha,Y}$ was rigorously studied first by Albeverio, Fenstad, and Høegh-Krohn [3], and subsequently characterised by Zorbas [276], Grossmann, Høegh-Krohn, and Mebkhout [122, 123], Dąbrowski and Grosse [66], and more recently by Arlinskiĭ and Tsekanovskiĭ [24], and by Goloshchapova, Malamud, and Zastavnyi [117, 118].

➠ In collaboration with Dell'Antonio, Scandone, and Yajima [77] we proved, for arbitrary centres and strengths, that the wave operators

$$W^{\pm}_{\alpha,Y} \;=\; \lim_{t\to\pm\infty} e^{\mathrm{i}tH_{\alpha,Y}}e^{\mathrm{i}t\Delta}\,. \tag{6.11}$$

associated to the pair $(H_{\alpha,Y},-\Delta)$ exist and are complete in $L^2(\mathbb{R}^3)$, they are *bounded* in $L^p(\mathbb{R}^3)$ for $1<p<3$, and *unbounded* for $p=1$ and for $p\geqslant 3$. The two main ingredients were: an explicit representation that we determined for the (kernel of the) wave operators $W^{\pm}_{\alpha,Y}$, based on the explicit resolvent difference $(H_{\alpha,Y}-z^2\mathbb{1})^{-1}-(H_0-z^2\mathbb{1})^{-1}$, and tools from harmonic analysis, especially Calderón-Zygmund operators and the Muckenhaupt weighted inequalities, for the $L^p\to L^p$ estimate of $W^{\pm}_{\alpha,Y}$. Such result is the first of its kind, as opposed to the vast literature on the L^p-boundedness of wave operators for the pair $(-\Delta+V,-\Delta)$ with sufficiently regular $V:\mathbb{R}^d\to\mathbb{R}$ vanishing at spatial infinity [270, 271, 25, 265, 142, 67, 100, 143, 29, 272, 273]: there, the problem is well known to depend crucially on the spectral properties of $-\Delta+V$ at the bottom of the absolutely continuous spectrum, that is, at energy zero, and in fact $H_{\alpha,Y}$ is zero-energy resonant.

➠ As a consequence of our result for (6.11), by means of the intertwining property

$$f(H_{\alpha,Y})P_{\mathrm{ac}}(H_{\alpha,Y}) \;=\; W^{\pm}_{\alpha,Y}\,f(-\Delta)\,(W^{\pm}_{\alpha,Y})^* \tag{6.12}$$

(f Borel function on $\mathbb{R}$) of the wave operators and of the dispersive estimates for the free Schrödinger propagator, we could deduce dispersive estimates

$$\|e^{-\mathrm{i}tH_{\alpha,Y}}P_{\mathrm{ac}}(H_{\alpha,Y})u\|_p \;\lesssim\; |t|^{-3(\frac{1}{2}-\frac{1}{p})}\|u\|_{p'}\,, \qquad t\neq 0\,, \quad p\in[2,3) \tag{6.13}$$

and, by means of a well-known argument [115, 268], also Strichartz estimates

$$(6.14)\qquad \begin{aligned} \|e^{-\mathrm{i}tH_{\alpha,Y}}P_{\mathrm{ac}}(H_{\alpha,Y})u\|_{L^q(\mathbb{R}_t,L^p(\mathbb{R}^3_x))} \;&\lesssim\; \|u\|_{L^2(\mathbb{R}^3)} \\ \Big\|\int_0^t e^{-\mathrm{i}(t-s)H_{\alpha,Y}}P_{\mathrm{ac}}(H_{\alpha,Y})\,u(s)\,\mathrm{d}s\Big\|_{L^q(\mathbb{R}_t,L^p(\mathbb{R}^3_x))} \;&\lesssim\; \|u\|_{L^{s'}(\mathbb{R}_t,L^{r'}(\mathbb{R}^3_x))} \\ (q,p)\text{ and }(s,r)\text{ satisfying}\qquad p\in[2,3)\,,\quad 0\leqslant \tfrac{2}{q} \;=\; 3\Big(\tfrac{1}{2}-\tfrac{1}{p}\Big) \;<\; \tfrac{1}{2} \end{aligned}$$

for the point interaction Hamiltonian $H_{\alpha,Y}$.

A weighted version of (6.13) had been previously proved by D'Ancona, Pierfelice, and Teta [70] directly from the explicit kernel of the propagator $e^{-\mathrm{i}tH_{\alpha,Y}}$, a kernel found by Scarlatti and Teta [251] and by Albeverio, Brzeźniak, and Dąbrowski [2]. Then Iandoli and Scandone [139], in a work supervised by me, had proved the non-weighted dispersive estimate (6.13) by means of a simpler and more direct arguments (i.e., without using any result from the scattering theory for $H_{\alpha,Y}$) in the special case $N = 1$.

Very recently, Erdoğan, Goldberg, and Green proved the L^p-boundedness, $p \in (1,+\infty)$, of the wave operators for the pair $(-\Delta+V,-\Delta)$ in $d = 2$ dimensions when $-\Delta + V$ is zero-energy resonant [87]. The fact that (the two-dimensional analogue of) $H_{\alpha,Y}$ too is zero-energy resonant suggested immediately a strong evidence for the validity of the two-dimensional analogue of the result proved in [77].

➠ This led Cornean, Yajima, and myself to produce a complete study of the threshold behaviour of two dimensional Schrödinger operators with finitely many local point interactions [63]. We showed that the resolvent can either be continuously extended up to the threshold, in which case we say that the operator is of regular type, or it has singularities associated with s or p-wave resonances or even with an embedded eigenvalue at zero, for whose existence we give necessary and sufficient conditions. An embedded eigenvalue at zero may appear only if we have at least three centres. When the operator is of regular type we prove that the wave operators are bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. With a single centre we always are in the regular type case.

A typical obstacle to the dispersive and scattering properties of the time evolution group associated with the Schrödinger equation

$$\mathrm{i}\partial_t u \;=\; -\Delta u + Vu \tag{6.15}$$

in the unknown $u \equiv u(t,x)$, where $t \in \mathbb{R}$, $x \in \mathbb{R}^d$ ($d \in \mathbb{N}$), and $V : \mathbb{R}^d \to \mathbb{R}$ is a given measurable potential, is the existence of non-trivial solutions to

$$-\Delta u + Vu \;=\; \mu u \tag{6.16}$$

for some $\mu \in \mathbb{R}$. In those cases, relevant in a variety of contexts, where V is sufficiently localised ('with short range') and/or is a suitably small perturbation of the Laplacian, the existence of non-trivial $L^2(\mathbb{R}^d)$- solutions to (6.16) are interpreted as bound states of the associated Schrödinger operator, and if $\mu > 0$ one refers to it as an eigenvalue embedded in the continuum. Solutions to (6.16) in weaker L^2-weighted spaces are generally known instead as *resonances* (a notion that we shall explicitly define in due time for the purposes of the present analysis), and they too affect the dispersive and scattering behaviour of the propagator defined by (6.15).

When $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and in (6.16) $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$, it was first proved by Kato [148] that positive eigenvalues are absent, and by Agmon [1] and by Alshom and Smith [11] that positive resonances are absent too. For rougher (non-L^2_{loc}) potentials, positive eigenvalues were excluded by Ionescu and Jerison [140] and by Koch and Tataru [152] by means of suitable Carleman-type estimates which imply, owing to a unique continuation principle [145, 151], that the corresponding eigenfunctions must be compactly supported and hence vanish. Absence of positive resonances whose associated resonant state u (solution to (6.16)) satisfies appropriate radiation conditions at infinity, was proved by Georgiev and Visciglia [113] for $L^{d/2}_{\mathrm{loc}}$-potentials decaying as $|x|^{-(1+\varepsilon)}$ or faster.

A closely related and equally challenging context, which this work is part of, is the counterpart problem of existence or non-existence of spectral obstructions (eigenvalues or resonances) when the potential V in (6.15)-(6.16) is formally replaced by a finite number of delta-like bumps localised at certain given points in space, and hence the operators of the type $H_{\alpha,Y}$ considered in (6.9) above.

The analysis of the dispersive and scattering properties of the Schrödinger propagator $e^{\mathrm{i}t\Delta_{\alpha,Y}}$, $t \in \mathbb{R}$, has been an active subject as well. I already discussed above that a class of $L^p \to L^q$ dispersive estimates were established by D'Ancona, Pierfelice, and Teta [70] (in weighted form), and by Iandoli and Scandone [139] (removing the weights used in [70] in the largest regime possible of the (p, q)-indices). I also discussed above that the L^p-boundedness of the wave operators for the pair $(H_{\alpha,Y}, -\Delta)$ in the regime $p \in (1, 3)$ (from which dispersive and Strichartz estimates can be derived by intertwining $-\Delta$ and $H_{\alpha,Y}$), as well as the L^p-unboundedness of the wave operators when $p = 1$ or $p \in [3, +\infty]$, was proved by Dell'Antonio, Scandone, Yajima, and myself [77] (with counterpart results by Duchêne, Marzuola, and Weinstein [84] in $d = 1$ and Cornean, Yajima, and myself [63] in $d = 2$ dimensions).

In analogy with the ordinary Schrödinger equation (6.15), also the dispersive features of the singular point-perturbed Schrödinger equation

$$\mathrm{i}\partial_t u \;=\; H_{\alpha,Y}\, u \tag{6.17}$$

strictly depend on the possible presence of eigenvalues or resonances for $H_{\alpha,Y}$, and indeed in the above-mentioned works [70, 139, 77] special assumptions on the choice of α and Y are often made so as to ensure that no spectral obstruction occurs. In fact (see [276, 122, 123], as well as [4, Sect. II.1.1]), the spectrum $\sigma(H_{\alpha,Y})$ only consists of an absolutely continuous component $[0, +\infty)$ which is also the whole essential spectrum, plus possibly a number of non-positive eigenvalues. Thus, as usual, for the purposes of the dispersive analysis, one considers $P_{\mathrm{ac}}\, e^{\mathrm{i}t\Delta_{\alpha,Y}}$, namely the action of the singular Schrödinger propagator on the sole absolutely continuous subspace of $L^2(\mathbb{R}^3)$, and additionally one has to decide whether possible resonances are present.

When $N = 1$ the picture is completely controlled: $H_{\alpha,Y}$ has only one negative eigenvalue if $\alpha < 0$, and has only a resonance, at zero, if $\alpha = 0$; correspondingly the integral kernel of the propagator $e^{\mathrm{i}t\Delta_{\alpha,Y}}$ is explicitly known, as found by Scarlatti and Teta [251] and Albeverio, Brzeźniak, and Dąbrowski [2], from which $L^p \to L^q$ dispersive estimates are derived directly, as found in [70]. In certain regimes of p, q slower decay estimates do emerge in the resonant case $\alpha = 0$, as opposed to the non-resonant one. For generic N perturbation centres, it is again well understood that at most N non-positive eigenvalue can add up to the absolutely continuous spectrum $[0, +\infty)$ of $H_{\alpha,Y}$. In particular, as discussed by one of us in [250, Sect. 3], a zero-energy eigenvalue may occur (see also [4, page 485]).

The study of resonances for generic N has been attracting a considerable amount of attention. It is known since the already mentioned work [123] by Grossmann, Høegh-Krohn, and Mebkhout (see also [4, Sect. II.1.1]), that resonances and eigenvalues z^2 of $H_{\alpha,Y}$ are detected, on an equal footing, by the singularity of an auxiliary $N \times N$ square matrix $\Gamma_{\alpha,Y}(z)$ depending on $z \in \mathbb{C}$. *Real negative resonances* (thus $z = \mathrm{i}\lambda$ with $\lambda > 0$) are excluded by the arguments of [123]. A *zero resonance* may occur, and one of us [250] qualified this possibility in terms of a convenient low-energy resolvent expansion for $\Gamma_{\alpha,Y}(z)$. *Complex resonances* ($\mathfrak{Im} z < 0$) have been investigated by Albeverio and Karabash [7, 8, 9] and Lipovský and Lotoreichik

[170], using techniques on the localisation of zeroes of exponential polynomials, and turn out to lie mostly within certain logarithmic strips in the complex z-plane.

➠ In the work [201] in collaboration with Scandone, we finally excluded *real positive (non-zero) resonances* (thus, $z \in \mathbb{R} \setminus \{0\}$), which essentially completes the picture of the spectral theory for three-dimensional Schrödinger operators with finitely many point interactions. Our argument has the two-fold virtue of being particularly compact as compared to the general setting of [117, 118], and exploiting the explicit structure of the matrix $\Gamma_{\alpha,Y}(z)$, unlike the abstract reasoning of [111] which also indirectly excludes positive resonances: as such, the approach that we present here has its own autonomous interest. Moreover, we have already mentioned that the absence of positive resonances for an ordinary Schrödinger operator $-\Delta + V$ is typically proved with Carleman's estimate, whereas for the singular version $-\Delta_{\alpha,Y}$ it appears to be very hard to use those classical techniques – and indeed our proof relies on a direct analysis based on the explicit formula for the resolvent: this makes any proof of absence of resonances surely valuable.

Another crucial question on $H_{\alpha,Y}$, above all in view of the associated non-linear Schrödinger equation, is the qualification of the corresponding fractional Sobolev spaces. Let us consider $-\Delta_\alpha$ (i.e., $Y = \{0\}$) for simplicity, and non-restrictively $\alpha > 0$ (repulsive point interaction). In this case one would like to qualify the '*perturbed*' or '*singular Sobolev space*' of order s, namely the Hilbert space

$$\widetilde{H}^s_\alpha(\mathbb{R}^3) := \mathcal{D}((-\Delta_\alpha)^{s/2}) \tag{6.18}$$

equipped with the 'fractional singular Sobolev norm'

$$\|\psi\|_{\widetilde{H}^s_\alpha} := \|(\mathbb{1} - \Delta_\alpha)^{s/2}\psi\|_2 . \tag{6.19}$$

➠ In collaboration with Georgiev and Scandone [112], based on functional calculus and interpolation methods, we studied $\widetilde{H}^s_\alpha(\mathbb{R}^3)$ in the relevant regime $s \in [0, 2]$ ($s = 0$ yields the whole $L^2(\mathbb{R}^3)$, $s = 2$ yields the operator domain (6.5), whereas $s = 1$ yields the form domain), finding

$$\begin{aligned} \widetilde{H}^s_\alpha(\mathbb{R}^3) &= H^s(\mathbb{R}^3) \\ \|\psi\|_{\widetilde{H}^s_\alpha} &\approx \|\psi\|_{H^s} \qquad \text{if } s \in [0, \tfrac{1}{2}) , \end{aligned} \tag{6.20}$$

$$\begin{aligned} \widetilde{H}^s_\alpha(\mathbb{R}^3) &= H^s(\mathbb{R}^3) \dotplus \mathrm{span}\{G_\lambda\} \\ \|\phi_\lambda + \kappa_\lambda\, G_\lambda\|_{\widetilde{H}^s_\alpha} &\approx \|\phi_\lambda\|_{H^s} + (1+\alpha)|\kappa_\lambda| \qquad \text{if } s \in (\tfrac{1}{2}, \tfrac{3}{2}) , \end{aligned} \tag{6.21}$$

$$\begin{aligned} \widetilde{H}^s_\alpha(\mathbb{R}^3) &= \Big\{\psi \in L^2(\mathbb{R}^3) \,\Big|\, \psi = \phi_\lambda + \frac{\phi_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\, G_\lambda \text{ with } \phi_\lambda \in H^s(\mathbb{R}^3)\Big\} \\ \|\phi_\lambda + \kappa_\lambda\, G_\lambda\|_{\widetilde{H}^s_\alpha} &\approx \|\phi_\lambda\|_{H^s} , \qquad \text{if } s \in (\tfrac{3}{2}, 2) , \end{aligned} \tag{6.22}$$

where here '$\approx$' denotes the equivalence of norms. The transition cases $s = \frac{1}{2}$ and $s = \frac{3}{2}$ are also covered, with less explicit formulas. Thus, for $s > \frac{1}{2}$, $\widetilde{H}^s_\alpha(\mathbb{R}^3)$ decomposes into a regular H^s-component and a singular component (with local singularity $|x|^{-1}$, precisely as the domain of $-\Delta_\alpha$ itself. And for $s > \frac{3}{2}$, regular and singular parts are constrained by a local boundary condition among them of the same type as in (6.5).

➠ The works [77] and [112] provided the main ingredients (Strichartz estimates and fractional Sobolev spaces) to study the Cauchy problem of the '*singular Hartree equation*'

$$\mathrm{i}\partial_t u = -\Delta_\alpha u + (w * |u|^2)u . \tag{6.23}$$

In collaboration with Olgiati and Scandone [192] we proved local well-posedness

- in $L^2(\mathbb{R}^3)$ for $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$ with $\gamma \in [0, \frac{3}{2})$,
- in $\widetilde{H}^s_\alpha(\mathbb{R}^3)$, $s \in (0, \frac{1}{2})$, for $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$ with $\gamma \in [0, 2s]$,
- in $\widetilde{H}^s_\alpha(\mathbb{R}^3)$, $s \in (\frac{1}{2}, \frac{3}{2})$, for $w \in W^{s,p}(\mathbb{R}^3)$ with $p \in (2, +\infty)$,
- $\widetilde{H}^s_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$, $s \in (\frac{3}{2}, 2)$, for $w \in W^{s,p}_{\mathrm{rad}}(\mathbb{R}^3)$ with $p \in (2, +\infty)$ (the requirement of spherical symmetry turning out to be a natural one for high regularity),

then we prove global well-posedness

- in $L^2(\mathbb{R}^3)$ for $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$ with $\gamma \in [0, \frac{3}{2})$ or $w \in L^\infty(\mathbb{R}^3) \cap W^{1,3}(\mathbb{R}^3)$
- in $\widetilde{H}^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$ for $w \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^3)$ with $p \in (2, +\infty)$ and $w \geqslant 0$,

and we prove existence of a unique strong solution in $\widetilde{H}^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$ for small initial data. In order to exploit the norm equivalences previously proved in [112] we needed to us tools from fractional calculus, especially the fractional Leibniz rule by Kato and Ponce [149], both in the generalised version by Gulisashvili and Kon [125] and with a more versatile re-distribution of the derivatives, as recently established by Fujiwara, Georgiev, and Ozawa [103].

Recently, especially for the solution theory of non-linear Schrödinger equations whose linear part is governed by singular Hamiltonians of point interactions (as, e.g., in [112, 192]), as well as linear Schrödinger-like equations for singular perturbations of fractional powers of the Laplacian [223, 236, 50, 161, 249, 262, 141, 228], the interest has increased around various ways of *combining two natural constructions* for a pseudo-differential operators: the singular perturbations and the fractional powers. In the prototypical case of the Laplacian $-\Delta$, this amounts to consider the operators

$$\begin{aligned} \mathfrak{h}_\tau^{s/2} &= (-\Delta + \text{singular perturbation at } x_0)^{s/2} = (-\Delta_\alpha)^{s/2} \\ (\mathfrak{h}_\tau + \mathbb{1})^{s/2} &= (-\Delta + \mathbb{1} + \text{singular perturbation at } x_0)^{s/2} \\ \mathsf{k}_\tau^{(s/2)} &= (-\Delta)^{s/2} + \text{singular perturbation at } x_0 \\ \mathfrak{d}_\tau^{(s/2)} &= (-\Delta + \mathbb{1})^{s/2} + \text{singular perturbation at } x_0\,. \end{aligned} \tag{6.24}$$

(Observe that $\mathfrak{h}_\tau^{s/2}$ and $(\mathfrak{h}_\tau + \mathbb{1})^{s/2}$ are genuine fractional powers of a non-negative self-adjoint operator on $L^2(\mathbb{R}^3)$, whereas the different notation for the superscript $s/2$ in $\mathsf{k}_\tau^{(s/2)}$ and in $\mathfrak{d}_\tau^{(s/2)}$ is to indicate that the latter operators are instead singular perturbations of $s/2$-th powers, and not fractional powers of singular perturbations.)

➠ In the work [195] with Ottolini and Scandone we completed the rigorous *construction* and the *spectral analysis* of the operators (6.24), making qualitative and quantitative comparisons, qualifying in particular

- the nature of the perturbation in the resolvent sense (finite rank vs infinite-rank perturbations);
- the natural decomposition of the domain of the considered operators into a regular component and a singular component, and to determine the boundary condition constraining such two components.

Our construction went through a natural 'restriction-extension' procedure: first one restricts the operator $(-\Delta)^{s/2}$ (initially defined, e.g., as a Fourier multiplier) to smooth functions vanishing in neighbourhoods of x_0, and then one builds all the operator extensions of such restriction that are self-adjoint on $L^2(\mathbb{R}^d)$. Moreover, in application to the linear and non-linear Schrödinger equations for the corresponding operators we outlined a long-term programme of relevant questions that deserve being investigated.

The above approach is surely satisfactory from the point of view of the interpretation of the output operator, which by construction is to be regarded as a point-like perturbation of the fractional Laplacian through an interaction supported only at x_0, say, "$(-\Delta)^{s/2}+\delta(x-x_0)$". However, it obfuscates an amount of physical meaning, since it does not provide information, as the intuition would make one expect instead, on how the actual singular perturbation $\mathsf{k}_\tau^{(s/2)}$ in (6.24) is approximately realised as a genuine pseudo-differential operator $(-\Delta)^{s/2}+V(x-x_0)$ with a regular potential V centred around $x=0$, with sufficiently short range and strong magnitude.

➠ In the follow-up investigation [199], with Scandone we indeed reconstructed the rank-one, singular (point-like) perturbations of the d-dimensional fractional Laplacian in the physically meaningful norm-resolvent limit of fractional Schrödinger operators with regular potentials centred around the perturbation point and shrinking to a delta-like shape. We analysed both the possible regimes, the resonance-driven and the resonance-independent limit, depending on the power of the fractional Laplacian and the spatial dimension. To this aim, we also qualify the notion of *zero-energy resonance* for Schrödinger operators formed by a fractional Laplacian and a regular potential.

The singular perturbation $-\Delta_\alpha$ of the differential operator Δ has recently found new relevance in another context for differential equation, the so-called *Friedrichs systems*.

The latter are a wide variety of differential equations of mathematical physics, including classical elliptic, parabolic, and hyperbolic equations, which can be rewritten in a suitable form, originally identified by Friedrichs in his research on symmetric positive systems [102]. For a given open and bounded set $\Omega\subseteq\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, let the matrix functions $\mathbf{A}_k\in W^{1,\infty}(\Omega)$ and $\mathbf{C}\in L^\infty(\Omega)$ satisfy $\mathbf{A}_k=\mathbf{A}_k^*$ and

$$\exists\,\mu_0>0\quad \mathbf{C}+\mathbf{C}^*+\sum_{k=1}^d\partial_k\mathbf{A}_k\geqslant 2\mu_0\mathbf{I}\qquad\text{a.e. on }\Omega\,.$$

Then the first-order differential operator $T:L^2(\Omega)\longrightarrow\mathcal{D}'(\Omega)$ defined by

$$T\mathsf{u}\;:=\;\sum_{k=1}^d\partial_k(\mathbf{A}_k\mathsf{u})+\mathbf{C}\mathsf{u}\tag{6.25}$$

is called *the (classical) Friedrichs operator*, while (for given $\mathsf{f}\in L^2(\Omega)$) the first-order system of partial differential equations $T\mathsf{u}=\mathsf{f}$ is called *the (classical) Friedrichs system.* The general problem for such systems is the well-posedness in a suitable regularity class and for suitable boundary conditions. Recently, this has become of particular relevance in numerical analysis [137, 144, 91], as Friedrichs systems turned out to provide a conveniently unified framework for numerical solutions to partial differential equations of different types. This aim of ample versatility has also naturally led to formulate the differential problem relative to classical Friedrichs systems in an abstract form on a Hilbert space [94, 19], in order to exploit powerful and general operator-theoretic methods, applicable to each concrete version. Important recent results concern well-posedness results [94, 17, 19], the representations of boundary conditions [17], the connection with the classical theory [18, 19, 22, 20], applications to various initial or boundary value problems of elliptic, hyperbolic, and parabolic type [21, 48, 82, 91, 204], and the development of different numerical schemes [47, 46, 49, 91, 92, 93].

Ern, Guermond and Caplain [94] and Antonić and Burazin [17] introduced an abstract, thus unified, formulation of Friedrichs systems by means of duality arguments on suitable Banach spaces, with a sufficient condition on the abstract boundary condition for the system to be well-posed.

➠ In collaboration with Antonić and Erceg [23] we presented a purely Hilbert-space operator-theoretic description of such abstract Friedrichs systems and, by means of Grubb's operator extension theory of Grubb [124] we proved that any pair of abstract Friedrichs operators admits bijective extensions with a 'signed boundary map'. This is in fact a classification of those abstract boundary conditions that ensure the well-posedness of the problem.

➠ Then, in collaboration with Erceg [85], we showed how to realise the Hamiltonians of contact interaction $-\Delta_\alpha$ on $L^2(\mathbb{R}^d)$ for $d = 3$, as well as its lower-dimensional analogues for $d = 1, 2$, within the framework of abstract Friedrichs systems. In particular, we proved that the construction of the self-adjoint (or even only closed) operators of contact interaction supported at a fixed point can be associated with the construction of the bijective realisations of a suitable pair of abstract Friedrichs operators. In this respect, the Hamiltonians of contact interaction provide *novel examples* of abstract Friedrichs systems.

The study of a quantum particle on degenerate Riemannian manifold, and the problem of the purely geometric confinement away from the singularity of the metric, as opposite to the dynamical transmission across the singularity, has recently attracted a considerable amount of attention in relation to Grushin structures on sphere, cylinder, cone, and plane [39, 41, 244, 40, 101, 109].

In the works [109], in collaboration with M. Gallone and E. Pozzoli we focussed on the paradigmatic class of quantum models on Grushin plane, namely a two-dimensional manifold with an incomplete Riemannian metric both on the right and the left open half-plane, and a singularity of the metric along the separation line among the two halves. That is, the manifold $(M, \mathrm{d}\mu_{g_\alpha})$ with

$$M^\pm \;:=\; \mathbb{R}^\pm \times \mathbb{R}\,, \qquad \mathcal{Z} \;:=\; \{0\} \times \mathbb{R}\,, \qquad M \;:=\; M^+ \cup M^- \tag{6.26}$$

and

$$g_\alpha \;:=\; \mathrm{d}x \otimes \mathrm{d}x + \frac{1}{|x|^{2\alpha}}\,\mathrm{d}y \otimes \mathrm{d}y\,, \tag{6.27}$$

where

$$\mu_\alpha \;:=\; \mathrm{vol}_{g_\alpha} \;=\; \sqrt{\det g_\alpha}\,\mathrm{d}x \wedge \mathrm{d}y \;=\; |x|^{-\alpha}\,\mathrm{d}x \wedge \mathrm{d}y \tag{6.28}$$

is the Riemannian volume form.

Our analysis covers also the mathematically simpler case of the Grushin cylinder, namely the compactified version of the Grushin plane along the direction of the metric singularity.

For such models, the geometric quantum confinement in each side of the plane (or of the cylinder) corresponds to the essential self-adjointness of the Laplace-Beltrami operator on its minimal domain away from the singularity, and the quantum transmission between the two half-planes (or half-cylinders) is the lack of essential self-adjointness of the associated Laplace Beltrami. In the study case, the Laplace-Beltrami operator has the form

$$\Delta_{\mu_\alpha} \;=\; \frac{\partial^2}{\partial x^2} + |x|^{2\alpha}\frac{\partial^2}{\partial y^2} - \frac{\alpha}{|x|}\,\frac{\partial}{\partial x}\,. \tag{6.29}$$

➠ In [109], together with Gallone and Pozzoli, we fully characterised the presence or lack of essential self-adjointness of the minimally defined Δ_{μ_α}, complementing and generalising the analysis of previous approaches [39, 41, 244, 40, 101], with a novel approach based on a constant-fibre direct integral scheme, in combination with Weyl's analysis in each fibre, which allowed for a complete control of both regimes.

➠ In the subsequent work [110], again with Gallone and Pozzoli, we studied the family of inequivalent self-adjoint realisations of such differential operator by means of the general extension theory of Kreĭn, Višik, and Birman. By combining this general theory with the constant-fibre direct sum / direct integral scheme, we classified a large sub-class of self-adjoint realisations of the Laplace-Beltrami operator, in fact the physically most relevant ones, namely those characterised by local boundary conditions at the singularity region. Our classification highlighted the role of the confining vs transmitting extensions, and includes the special extensions (Friedrichs, bridging) previously singled out by [40].

➠ More recently [107], together with Gallone, we characterised positivity, spectrum, ground state, and scattering properties of the local protocols of transmission of a quantum particles constrained on a Grushin cylinder and colliding with the singularity of the metric. Relevant phenomena of embedded eigenvalues, energy filters in the transmission, and perfect transmission/reflection have been identified.

7. Main research line #4. Analysis of inverse linear problems on infinite-dimensional Hilbert space

An *inverse linear problem* on Hilbert space is the problem, given a Hilbert space $\mathcal{H}$, a linear operator A acting on $\mathcal{H}$, and a vector $g \in \mathcal{H}$, to determine the solution(s) $f \in \mathcal{H}$ to the linear equation

$$Af \;=\; g\,. \tag{7.1}$$

One says that: (7.1) is *solvable* if a solution f exists, namely if $g \in \mathrm{ran}A$; (7.1) is *well-defined* if additionally the solution f is unique, i.e., if A is also injective (in which case one refers to f as the '*exact*' solution); (7.1) is *well-posed* if there exists a unique solution that depends continuously (i.e., in the norm of $\mathcal{H}$) on the datum g, equivalently, that $g \in \mathrm{ran}A$ and A has bounded inverse on its range.

Numerical solutions to inverse linear problems are customarily constructed by means of truncation schemes in the framework the celebrated *Petrov-Galerkin projection methods.* that in their essence can be summarized as follows [155, Chapter 4], [246, Chapter 5], [26, Chapter 9]. Two a priori known orthonormal bases $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ are considered, with the feature that for each truncation size $N \in \mathbb{N}$ the reduced problem

$$Q_N(A\widehat{f^{(N)}} - g) = 0 \tag{7.2}$$

admits a unique solution $\widehat{f^{(N)}}$ belonging to the 'solution space' $\mathrm{span}\{u_1,\dots,u_N\}$, where Q_N is the orthogonal projection onto the 'trial space' $\mathrm{span}\{v_1,\dots,v_N\}$; moreover, suitable hypotheses on A and on the truncation bases are assumed, so as to guarantee the convergence $\widehat{f^{(N)}} \xrightarrow{N\to\infty} f$. Both the solvability of the truncated problems and the assumptions ensuring an efficient convergence theory are integral parts of Petrov-Galerkin methods. In particular, when $u_n = v_n$ for all n one simply speaks of *Galerkin* projection methods.

All this is very familiar and already under control for relevant classes of boundary value problems on $L^2(\Omega)$ for some domain $\Omega \subset \mathbb{R}^d$, the typical playground for Galerkin and Petrov-Galerkin finite element methods [90, 245]. In these cases A is an *unbounded* operator, say, of elliptic type [90, Chapter 3], [245, Chapter 4], of

Friedrichs type [90, Sect. 5.2], [94, 19, 23], of parabolic type [90, Chapter 6], [245, Chapter 5], of 'mixed' (i.e., inducing saddle-point problems) type [90, Sect. 2.4 and Chapter 4], etc. Such A's are assumed to satisfy (and so they do in applications) some kind of coercivity, or more generally one among the various classical conditions that ensure the corresponding problem (7.1) to be well-posed, such as the Banach-Nečas-Babuška Theorem or the Lax-Milgram Lemma [90, Chapter 2].

For the above-mentioned classes of inverse linear problems, the finite-dimensional truncation and the infinite-dimensional error analysis are widely studied and well understood. In that context, in order for the finite-dimensional solutions to converge strongly, one requires stringent yet often plausible conditions [90, Sect. 2.2-2.4], [245, Sect. 4.2] both on the truncation spaces, that need to approximate suitably well the ambient space $\mathcal{H}$ ('*approximability*', thus the interpolation capability of finite elements), and on the behaviour of the reduced problems, that need admit solutions that are uniformly controlled by the data ('*uniform stability*'), and that are suitably good approximate solutions of the original problem ('*asymptotic consistency*'), together with some suitable boundedness of the problem in appropriate topologies ('*uniform continuity*').

As plausible as the above conditions are, they are *not* matched by several other types of inverse problem of applied interest. Mathematically this is the case whenever A does not have a 'good' inverse, for instance when A is a compact operator on $\mathcal{H}$ with arbitrarily small singular values, or when the exact solution of the inverse problem does not belong to the corresponding Krylov space used for the finite-dimensional truncations.

This leads to the more general scenario, outside the Petrov-Galerkin framework, of *general projection methods*, where

(i) (7.1) is only assumed to be solvable;
(ii) the orthonormal systems $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are not necessarily *complete* in $\mathcal{H}$;
(iii) the truncated problem (7.2) is not guaranteed to be well-defined (it may thus have a multiplicity of solutions), let alone solvable (it may have no solution at all);
(iv) the standard conditions for the convergence theory of Petrov-Galerkin schemes, which in that context guarantee the vanishing in the Hilbert norm of the error and/or the residual along the sequence of approximate solutions, are not assumed a priori.

➠ In [55], in collaboration with Caruso and Novati, we analysed such a scenario (clearly when it is *genuinely infinite-dimensional*, that is, as customary [254, Sect. 1.4], when A is *not* reduced to $A = A_1 \oplus A_2$ by an orthogonal direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\dim \mathcal{H}_1 < \infty$, $\dim \mathcal{H}_2 = \infty$, and $A_2 = \mathbb{O}$). In particular, we presented a clean abstract formulation of a general projection method and we identified generic convergence mechanisms for the error and the residual, i.e.,

$$\mathscr{E}_N \;:=\; f - \widehat{f^{(N)}}\,, \qquad \mathfrak{R}_N \;:=\; g - A\,\widehat{f^{(N)}}\,, \tag{7.3}$$

possibly in weaker topologies than the Hilbert norm, for compact operators and for general bounded operators. Through an elucidative series of model examples and numerical tests we discussed the occurrence when the truncated problems (7.2) are singular at every size N, and the genericity of *weak* vanishing of the error (7.3).

In many applications, the finite-dimensional truncation of the problem (7.1) is performed so that approximate solutions to (7.1) are sought among the linear combinations of the vectors $g, Ag, A^2g, \dots$ which span the so-called '*Krylov subspace*'

$$\mathcal{K}(A,g) \;:=\; \mathrm{span}\{A^k g \,|\, k \in \mathbb{N}_0\} \tag{7.4}$$

associated with A and g.

Krylov subspace methods constitute a wide class of efficient numerical schemes for finite-dimensional inverse linear problems, even counted among the 'Top 10 Algorithms' of the 20th century [83, 62].

The infinite-dimensionality of the underlying Hilbert space $\mathcal{H}$ comes with a load of new issues, starting from the very definition of the Krylov vectors $A^k g$ if A is unbounded [52]. Even when A is everywhere defined and bounded, and hence $\mathcal{K}(A,g)$ is well-defined, it may well happen that $\mathcal{K}(A,g)$ is not dense in $\mathcal{H}$, thus preventing the truncation spaces to have that approximability feature which, as mentioned above, is a typical assumption for (Petrov-)Galerkin schemes.

Among such potential difficulties, the first crucial question is whether the solution(s) to (7.1) can be well approximated by vectors in $\mathcal{K}(A,g)$, say, whether they belong to the closure $\overline{\mathcal{K}(A,g)}$ taken in the $\mathcal{H}$-norm topology. In the affirmative case, the Krylov subspace is a reliable space for the approximants of the exact solution(s): such an occurrence is referred to by saying that the problem (7.1) is '*Krylov-solvable*' and a solution f to (7.1) such that $f \in \overline{\mathcal{K}(A,g)}$ is referred to as a '*Krylov solution*'. Additional relevant questions then arise, for example in the presence of a multiplicity of solutions some may be Krylov and others may not.

➠ In [54], in collaboration with Caruso and Novati, we showed that the triviality of the *Krylov intersection* for the given A and g, namely the subspace

$$\overline{\mathcal{K}(A,g)} \cap (A\,\mathcal{K}(A,g)^{\perp})\,, \tag{7.5}$$

provides the 'intrinsic' operator-theoretic mechanism for the Krylov-solvability of the problem. The comprehension of such an intrinsic mechanism is most valuable, because, as we showed with a large series of examples and counter-examples, even stringent assumptions on A such as the simultaneous occurrence of compactness, normality, injectivity, and density of $\mathrm{ran}A$ do *not* ensure, in general, that the solution f to $Af = g$ ($g \in \mathrm{ran}A$) is a Krylov solution. For bounded self-adjoint operators A, Krylov solvability (and hence the triviality of (7.5)) boils down to the reducibility of A with respect to the 'Krylov decomposition'

$$\mathcal{H} \;=\; \overline{\mathcal{K}(A,g)} \oplus \mathcal{K}(A,g)^{\perp}\,, \tag{7.6}$$

which is however not sufficient to ensure Krylov solvability for non-self-adjoint A's.

➠ Still in [54] we produced various necessary and sufficient conditions for a solution to (7.1) to be a Krylov solution, for general bounded operators and for special classes of them (self-adjoint, unitary, normal), with well-posed and ill-posed problems. In the current lack (to our knowledge) of a complete characterisation of all Krylov-solvable inverse problems on infinite-dimensional Hilbert space, it is of interest to identify special sub-classes of them. We showed that a whole class of paramount relevance, the self-adjoint inverse problems, are certainly Krylov-solvable (and the Krylov solution is unique), and so too are those for which A^{-1} exists as a polynomial approximation $\|p_n(A) - A^{-1}\|_{\mathrm{op}} \to 0$, besides various other examples of inverse problems with lack of well-posedness.

A popular algorithm for the numerical solution to (7.1) in the framework of Krylov subspace methods is the *method of conjugate gradients* (also referred to as CG). It was first proposed in 1952 by Hestenes and Stiefel [135] and since then, together with its related derivatives (e.g., conjugate gradient method on the normal equations (CGNE), least-square QR method (LSQR), etc.), it has been widely studied in the finite-dimensional setting (see the monographs [246, 257, 168]) and also, though to a lesser extent, in the infinite-dimensional Hilbert space setting with bounded operators.

This algorithm applies to inverse problems of the type (7.1) when A is self-adjoint and non-negative, i.e., $A \geqslant \mathbb{O}$ in the sense of expectations. Assuming for the sake of generality that A may be unbounded, let

$$\mathcal{S} \;:=\; \{f \in \mathcal{D}(A) \,|\, Af = g\} \tag{7.7}$$

be the 'solution manifold' relative to the problem (7.1) when $g \in \mathrm{ran}A$.. By assumption $\mathcal{S}$ is a convex, non-empty set in $\mathcal{H}$ which is also closed, owing to the fact that A, being self-adjoint, is in particular a closed operator. As a consequence, the projection map $P_{\mathcal{S}} : \mathcal{H} \to \mathcal{S}$ is unambiguously defined and produces, for generic $x \in \mathcal{H}$, the closest to x point in $\mathcal{S}$.

In its iterative implementation, the conjugate gradient algorithm starts with an initial guess $f^{[0]} \in \mathcal{H}$ and produces iterates $f^{[N]}$ according to a prescription that can be described in various equivalent ways [246, 168], the most convenient of which for our purposes is

$$f^{[N]} \;:=\; \underset{h\in\{f^{[0]}\}+\mathcal{K}_N(A,\mathfrak{R}_0)}{\arg\min} \|A^{1/2}(h - P_{\mathcal{S}} f^{[0]})\|\,, \qquad N \in \mathbb{N}\,. \tag{7.8}$$

More generally, one defines conjugate gradient style algorithms with iterates

$$f^{[N]} \;:=\; \underset{h\in\{f^{[0]}\}+\mathcal{K}_N(A,\mathfrak{R}_0)}{\arg\min} \|A^{\theta/2}(h - P_{\mathcal{S}} f^{[0]})\|\,, \qquad N \in \mathbb{N} \tag{7.9}$$

for some parameter $\theta \geqslant 0$ (the case $\theta = 1$ being *the* conjugate gradient method).

In (7.8)-(7.9) the vector $\mathfrak{R}_0$ is the zero-th order of the *residuals* defined by

$$\mathfrak{R}_N \;:=\; Af^{[N]} - g\,, \qquad N \in \mathbb{N}_0 \tag{7.10}$$

in terms of each iterate, and the vector space

$$\mathcal{K}_N(A,\mathfrak{R}_0) \;:=\; \mathrm{span}\{\mathfrak{R}_0, A\mathfrak{R}_0, \dots A^{N-1}\mathfrak{R}_0\}\,, \qquad N \in \mathbb{N} \tag{7.11}$$

is the *N-th order Krylov subspace* associated to A and $\mathfrak{R}_0$.

For (7.11) and hence (7.8)-(7.9) to make sense for any N when A is *unbounded*, additional technical assumptions are needed in order to avoid possible domain issues. Clearly, the above definitions are all well-posed if A is bounded.

The convergence of $f^{[N]}$ to the exact solution f is by now a classical and deeply understood theory (see, e.g., the monographs [246, 168]).

The convergence theory of CG has been markedly less explored in the setting of *infinite-dimensional* $\mathcal{H}$, a line of investigation in which yet important works have been produced over the last five decades, both in the scenario where A is bounded with everywhere-defined bounded inverse [71, 72, 134], or at least with bounded inverse on its range [147], and in the scenario where A is bounded with possible unbounded inverse on its range [147, 230, 231, 171, 43].

In contrast, the scenario where A is *unbounded* has been only recently considered from special perspectives, in particular in view of existence [237] (for GMRES algorithms), or convergence when A is regularised and made invertible with everywhere-defined bounded inverse [114], whereas the general convergence theory (that is, including the case where (7.1) is ill-posed) was virtually unexplored.

➡ In [52], in collaboration with Caruso, we demonstrated a class of convergence result $f^{[N]} \to f$ in the most general case where the self-adjoint, non-negative operator is unbounded and with minimal, technically unavoidable assumptions on the initial guess of the iterative algorithm. The convergence is monitored in the sense

$$\lim_{N\to\infty} \big\| A^{\sigma/2}(f^{[N]}-f)\big\| \;=\; 0 \tag{7.12}$$

(where $A^{\sigma/2}$ is understood as a power of the inverse of A on its range, if $\sigma<0$), and it is is proved to always hold in the Hilbert space norm (error convergence: $\sigma=0$), as well as at other levels of regularity (energy norm, $\sigma=1$; residual, $\sigma=2$, etc.) depending on the regularity of the iterates.

➠ The work [52] generalises (and to some extent simplifies), through a delicate combination of measure-theoretic analysis and modern theory of orthogonal polynomials, the very profound and intricate works [230, 231] of Nemirovskiy and Polyak from the 1980s, where the convergence theory for conjugate gradients is developed for bounded operators on infinite-dimensional Hilbert space. In [52] we also discuss, both analytically and through a selection of numerical tests, the main features and differences of our convergence result as compared to the bounded case: in particular, we show that the optimal rate of convergence found by Nemirovskiy and Polyak in the bounded case is *violated* in the unbounded case: virtually, depending on the choice of the initial guess, the vanishing (7.12) may occur at an arbitrarily slow pace.

Pushing the investigation programme on Krylov solvability of the inverse problem (7.1) further to *unbounded* A's, of course an additional operational restriction must be imposed on g for the very notion of Krylov subspace to make sense, that is, one now assumes that

$$g \;\in\; \mathrm{ran}A\cap C^\infty(A) \tag{7.13}$$

where $C^\infty(A)$ is the space of elements of $\mathcal{H}$ simultaneously belonging to all the domains of the natural powers of A,

$$C^\infty(A) \;:=\; \bigcap_{k\in\mathbb{N}}\mathcal{D}(A^k)\,. \tag{7.14}$$

In this respect, the above-mentioned work [52] with Caruso can be regarded as a first step to study Krylov solvability in the unbounded case, however with the two-fold limitation that A has to be self-adjoint and non-negative (as required in conjugate gradient methods), and that Krylov solvability emerges only as a by-product result with no explicit insight on the operator-theoretic mechanism for it.

➠ In the work [51], in collaboration with Caruso, we extensively discussed Krylov solvability when A is densely defined and closed on $\mathcal{H}$. First, we demonstrate that the inverse problem (7.1) is indeed Krylov-solvable when A is generically (unbounded and) self-adjoint or skew-adjoint, as a by-product of our previous analysis [52] on conjugate gradients.

➠ Moreover, in [51] we identified new obstructions in the issue of Krylov solvability, which are not present in the bounded case. A most serious one is the somewhat counterintuitive phenomenon of 'Krylov escape', namely the possibility that vectors of $\overline{\mathcal{K}(A,g)}$ that also belong to the domain of A are mapped by A *outside* of $\overline{\mathcal{K}(A,g)}$, whereas obviously $A\mathcal{K}(A,g)\subset\mathcal{K}(A,g)$. We also determined that if the closures of $\mathcal{K}(A,g)$ in the Hilbert space norm and in the stronger A-graph norm are the same (up to intersection with $\mathcal{D}(A)$), an occurrence that we named 'Krylov-core condition', then Krylov escape is actually prevented. We demonstrated that under assumptions like the Krylov-core condition (and, more generally, lack of Krylov escape) the intrinsic mechanisms of Krylov reducibility and triviality of the Krylov intersection play a completely analogous role as compared to the bounded case.

➠ In addition to that, in [51] we re-considered the (unbounded) self-adjoint scenario, that from the practical point of view we already solved by conjugate-gradients-based arguments, investigating Krylov solvability from the perspective of the abstract operator-theoretic mechanisms mentioned above. Noticeably, this is also a perspective that rises up interesting open questions. Indeed, whereas we can prove that self-adjoint operators do satisfy the Krylov-core condition and are Krylov-reducible for a distinguished dense set of A-smooth vectors g's, and that for the same choice of g the subspace $\overline{\mathcal{K}(A,g)}$ is naturally isomorphic to $L^2(\mathbb{R}, \mathrm{d}\mu_g^{(A)})$ (here $\mu_g^{(A)}$ is the scalar spectral measure), yet we cannot decide whether Krylov escape is prevented for *any* self-adjoint A and A-smooth g (which is remarkable, as by other means we know that $Af = g$ is Krylov solvable). This certainly indicates a future direction of investigation.

Krylov solvability of the inverse problem allows for solution approximations that, in applications, correspond to the very efficient and popular Krylov subspace methods. One natural line of further investigation is to consider *perturbations* of the original problem $Af = g$ of the form $A'f' = g'$, where A and A', as well as g and g' are close in a controlled sense, and we study the *effect of the perturbation on the Krylov solvability*. This context is clearly connected with the general framework of "ill-posed" inverse linear problems [131, 132], where only the perturbed quantities A' or g' are accessible, due for instance to measurement errors, and ill-posedness manifests for instance through the fact that $g' \notin \mathrm{ran}A$, the goal being to approximate the actual solution f in a controlled sense.

Yet, there is also a different spirit of the perturbation. One may keep regarding A and g as exactly known or, in principle, exactly accessible, but with the idea that close to the problem $Af = g$ there is a perturbed problem $A'f' = g'$ that serves as an auxiliary one, possibly more easily tractable, say, with Krylov subspace methods, in order to obtain conclusions on the Krylov solvability of the original problem. Or, conversely, one may inquire under which conditions the nice property of Krylov solvability for $Af = g$ is stable enough to survive a small perturbation (that in applications could arise, again, from experimental or numerical uncertainties), or when instead Krylov solvability is washed out by even small inaccuracies in the precise knowledge of A or g – an occurrence in which Krylov subspace methods would prove to be unstable. And, more abstractly, one may pose the question of a convenient notion of vicinity between the subspaces $\overline{\mathcal{K}(A,g)}$ and $\overline{\mathcal{K}(A',g')}$ when A and A' (respectively, g and g') are suitably close.

➠ In the recent work [53], in collaboration with Caruso, we studied the possible behaviours of persistence, gain, or loss of Krylov solvability under suitable small perturbations of the inverse problem – the underlying motivations being precisely the stability or instability of Krylov methods under small noise or uncertainties, as well as the possibility to decide a priori whether an inverse problem is Krylov solvable by investigating a potentially easier, perturbed problem. We mapped all typical phenomena that may occur to the Krylov solvability of an inverse linear problem $Af = g$ in terms of the Krylov solvability, or lack of thereof, of auxiliary inverse problems where A or g or both are perturbed in a controlled sense. Such survey indicates that the sole control of the operator or of the data perturbation, in the respective operator and Hilbert norm, still leaves the possibility open to all phenomena such as the persistence, gain, or loss of Krylov solvability in the limit $A_n \to A$ or $g_n \to g$, where A_n (and so g_n) is the generic element of a sequence of perturbed objects. The implicit explanation is that an information like $A_n \to A$ or $g_n \to g$ is not enough to account for a suitable vicinity of the corresponding Krylov subspaces – as mentioned above, from our previous work [54] we learnt the Krylov solvability of the inverse problem $Af = g$ corresponds to certain structural properties of the subspace $\mathcal{K}(A, g)$ (like the triviality of the Krylov intersection (7.5)(, therefore one implicitly needs to monitor how the latter properties are preserved or altered under the perturbation. This also suggests that the additional constraint of performing the perturbation within certain subclasses of operators may supplement further information on Krylov solvability: this is in principle a vast programme, and in [53] we focussed on the operators of $\mathscr{K}$-class we had previously considered in [54], and we discussed the robustness and fragility of this class from the perturbative perspective of the induced inverse problems.

➠ Additionally, in [53], we addressed more systematically the issue of vicinity of Krylov subspaces in a sense that be informative for the Krylov solvability of the corresponding inverse problems. What shows encouraging properties, next to some serious limitations, though, is the comparison of (the closures of) two Krylov subspaces in terms of the Hausdorff distance between the respective unit balls, considered as closed subset of the Hilbert unit ball when the latter is metrised with respect to the *weak* Hilbert topology. (Had we used the *norm* topology, that would have not even controlled the very intuitive convergence of the finite-dimensional Krylov subspaces, namely with iterates up to some $A^{N_0}g$, to its infinite-dimensional counterpart, as $N_0 \to \infty$.) This framework leads to appealing approximation results, as the inner approximability of Krylov subspaces, or also prototypes of perturbative results in which the perturbation is controlled in the appropriate distance between Krylov subspaces, and this *predicts* the persistence of Krylov solvability when the perturbation is removed.

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