Dissipative dynamics on large spin chains

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Outline I

Introduction

- Infinite quantum spin chains: transition from micro to macro in the large N limit
- Microscopic description level: finitely localized spin observables
- Macroscopic description level: collective observables
  - mean field quantities, $1/N$ scaling
  - commutative behaviour
- At the interface between micro and macro: mesoscopic description level
  - quantum fluctuations, scaling $\frac{1}{\sqrt{N}}$
  - Bosonic behaviour
- Mesoscopic Entanglement: two clouds of 10^{11} Caesium atoms entangled by light

Julsgaard, Kozhekin, Polzik, Lett. Nature **413** (2001)

Narnhofer, Thirring, PRA **66** (2002)

Quantum spin chains

- Quantum spin chain: a one-dimensional lattice supporting a same finite-dimensional matrix algebra $\mathcal{A}^{(j)} = M_d(\mathbb{C})$
- Local sub-algebras: $\mathcal{A}_{[q,p]} = \bigotimes_{j=p}^{q} \mathcal{A}^{(j)}$
- Quasi-local-algebra: $\mathcal{A} = \overline{\bigcup_{q \leq p} \mathcal{A}_{[q,p]}}^{norm}$
- Embedding of single-site spin operators $x \in M_d(\mathbb{C})$

$$x^{(j)} = \mathbf{1}_{j-1]} \otimes x \otimes \mathbf{1}_{[j+1}$$

- Asymptotic Abelianess: $\lim_{j \to \infty} \|[x^{(j)}, b]\| = 0$ for all $b \in \mathcal{A}_{[p,q]}$.

Clustering states

- Translation automorphism: $\tau : \mathcal{A} \mapsto \mathcal{A}$: $\tau(x^{(j)}) = x^{(j+1)}$.
- Translation-invariant states: positive, normalised linear functionals $\mathcal{A} \ni a \mapsto \omega(a) \in \mathbb{C}$

$$\omega(x^{(j)}) = \omega(x^{(j+1)}) = \omega(x) = \mathrm{Tr}(\rho\, x)$$

ϱ: on-site spin density matrix

- Clustering states: translation-invariant and such that

$$\lim_{n \to +\infty} \omega\Big(a^\dagger \tau^n(b) c\Big) = \omega(a^\dagger\, c)\, \omega(b) \quad \forall a, b, c \in \mathcal{A}\ .$$

Collective spin chain observables

- Local observables: microscopic description level
- Collective description level: proper scaling needed

Mean-field observables

- Averages of microscopic observables:

$$X_N = \frac{1}{N} \sum_{k=0}^{N-1} x^{(k)}$$

Averages: Macroscopic Commutative Algebras

- Averages commute: $[x^{(j)}\,,\, y^{(k)}] = \delta_{jk} z^{(j)}$,

$$\lim_N \left\| \left[X_N\,,\ Y_N \right] \right\| \leq \lim_N \frac{1}{N^2} \left\| \sum_{k=0}^{N-1} \left[x^{(k)}\,,\, y^{(k)} \right] \right\| = 0$$

- and weakly converge to scalars: ω clustering implies

$$\lim_{N\to\infty} \omega\left(b^\dagger X_N c\right) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \omega\left(b^\dagger\, x^{(k)}\, c\right)$$

$$= \lim_{N\to\infty} \omega\left(b^\dagger\, x^{(N-1)}\, c\right) = \omega(b^\dagger c)\,\omega(x)\ , \quad \forall b, c \in \mathcal{A}_{[p,q]}$$

Quantum fluctuations

D. Goderis et al., CMP **128** (1990)
T. Matsui, Ann. Henri Poincaré **4** (2002)
A. Verbeure, Many-Body Boson Systems (Springer, 2011)

- Local quantum fluctuations:

$$F_N(x) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(x^{(k)} - \omega(x) \right)$$

- Commutators of fluctuations are mean-field quantities:

$$\left[F_N(x)\,,\, F_N(y) \right] = \frac{1}{N} \sum_{j,k=0}^{N-1} \left[x^{(j)}\,,\, y^{(k)} \right] = \frac{1}{N} \sum_{k=0}^{N-1} \left[x^{(k)}\,,\, y^{(k)} \right]$$

Large N Quantum Fluctuations

- Weak convergence of commutators of fluctuations to scalars:

$$w - \lim_{N\to+\infty} \Big[F_N(x)\,,\,F_N(y)\Big] = \omega\left([x,y]\right)$$

- $w - \lim_{N\to\infty} F_N(x)$ does not exist
- How can collective fluctuations be defined?
- Mesoscopic limit:

$$\begin{aligned} F(x) &= m - \lim_{N\to+\infty} F_N(x) \\ [F(x)\,,\,F(y)] &= \omega\left([x,y]\right) \qquad \text{Bosonic behaviour} \end{aligned}$$

Large N Quantum Fluctuations

- Local Weyl-like operators:

$$F_N(x) \mapsto W_N(r) = \exp\left(i\, r\, F_N(x)\right)\ , \qquad r \in \mathbb{R}$$

- Characteristic functions: $\omega\Big(W_N(r)\Big)$
- Large N Weyl-like commutation relations:

$$\lim_{N\to\infty} \omega\left(\mathrm{e}^{iF_N(x)}\,\mathrm{e}^{iF_N(y)}\right) = \lim_{N\to\infty} \omega\left(\mathrm{e}^{iF_N(x+y)}\right)\ \exp\left(-\frac{i}{2}\omega([x\,,\,y])\right)$$

Large N

Quantum Fluctuations

- Baker-Campbell-Hausdorff: $\mathrm{e}^{iF_N(x)}\,\mathrm{e}^{iF_N(y)} = \mathrm{e}^{G_N(x,y)}$,

$$G_N(x,y) \simeq i\Big(F_N(x)+F_N(y)\Big) - \frac{1}{2}\Big[F_N(x)\,,\,F_N(y)\Big] + \frac{1}{12}\Big(\Big[F_N(x)\,,\,\Big[F_N(x),F_N(y)\Big]\Big] - \Big[F_N(y),\Big[F_N(x)\,,\,F_N(y)\Big]\Big]\Big)$$

- Contributions with more commutators vanish in norm:

$$\Big[F_N(x)\,,\,\Big[F_N(x)\,,\,F_N(y)\Big]\Big] = \frac{1}{N^{3/2}}\sum_{k=0}^{N-1}\Big[x^{(k)},\Big[x^{(k)}\,,\,y^{(k)}\Big]\Big]$$

Quantum Central Limit Theorem

- Finite set of single site observables: $\chi = \{x_j\}_{j=1}^{p} \subset M_d(\mathbb{C})$
- Restricted class of clustering states:

$$\sum_{k=0}^{\infty} \left|\omega(x_i x_j^{(k)}) - \omega(x_i)\omega(x_j)\right| < +\infty \quad \forall x_i, x_j \in \mathcal{X}$$

- normal multivariate quantum fluctuations w.r.t ω if

$$\lim_{N\to\infty} \omega\left(F_N^2(x_j)\right) = \lim_{N\to\infty} \frac{1}{N} \sum_{\ell,k=0}^{N-1} \left(\omega\left(x_j^{(\ell)}\, x_j^{(k)}\right) - \omega^2(x_j)\right) =: \Sigma_{jj}^{\omega}$$

$$\lim_{N\to\infty} \omega(e^{itF_N(x_j)}) = \exp\left(-\frac{t^2}{2}\Sigma_{jj}^{\omega}\right) \qquad \forall t \in \mathbb{R}\,.$$

Quantum Central Limit Theorem

- $(d \times d)$ Covariance matrix Σ^ω:

$$\Sigma^\omega_{ij} = \lim_{N\to\infty} \frac{1}{2}\omega\Big(\Big\{F_N(x_i)\,,\,F_N(x_j)\Big\}\Big)$$

- $(d \times d)$ Symplectic matrix σ^ω:

$$\sigma^\omega_{ij} = -i \lim_{N\to\infty} \omega\Big(\Big[F_N(x_i)\,,\,F_N(x_j)\Big]\Big)$$

Quantum Central Limit Theorem

- Local Weyl-like operators:

$$W_N(r) = \exp\left(i\sum_{j=1}^{p} r_j F_N(x_j)\right) \quad , \qquad r = \{r_j\}_{j=1}^{p} \in \mathbb{R}^p$$

- Theorem Goderis et al. Comm. Math. Phys. **128** (1990)

$$\lim_{N\to+\infty} \omega\left(W_N(r_1)\, W_N(r_2)\right) = \exp\left(-\frac{(r_1+r_2, \Sigma^\omega\,(r_1+r_2))}{2}\right) \times$$

$$\times \exp\left(-\frac{i}{2}\,(r_1\,, \sigma^\omega\, r_2)\right)$$

Mesoscopic Limit

- Weyl algebra $\mathcal{W}$ of Weyl operators $W(r)$

$$W(r_1)W(r_2) = W(r_1 + r_2)\, \exp\left(-\frac{i}{2}(r_1, \sigma^\omega r_2)\right)$$

- Mesoscopic Gaussian state on $\mathcal{W}$:

$$\Omega(W(r)) = \exp\left(-\frac{1}{2}(r, \Sigma^\omega\, r)\right)$$

Quantum Central Limit

- Mesoscopic limit: $m - \lim_{N\to\infty} W_N(r) = W(r)$

$$\lim_{N\to\infty} \omega\left(W_N(r_1)\, W_N(r)\, W_N(r_2)\right) = \Omega\left(W(r_1)\, W(r)\, W(r_2)\right)$$

- Mesoscopic limit: $m - \lim_{N\to\infty} F_N(x_j) = F(x_j)$

$$\lim_{N\to\infty} \omega\left(W_N(r_1)\, F_N(x_j)\ \, W_N(r_2)\right) = \lim_{N\to\infty} \partial_{r_j}\omega\left(W_N(r_1)\, W_N(r)\ \, W_N(r_2)\right)$$
$$= \partial_{r_j}\Omega\left(W(r_1)\, W(r)\, W(r_2)\right) = \Omega\left(W(r_1)\, F(x_j)\, W(r_2)\right)$$

- Weyl operators: $W(r) = \exp\left(i \sum_{j=1}^{p} r_j F(x_j)\right)$

Example: Double Spin Chain

- Double spin chain: single site j algebra $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$,

$$\begin{array}{ccccccccc} \cdots\cdots & \uparrow & \rightarrow & \downarrow & \nearrow & \searrow & \uparrow & \nwarrow & \cdots\cdots \\ \cdots\cdots & \uparrow & \searrow & \nwarrow & \downarrow & \rightarrow & \uparrow & \searrow & \cdots\cdots \\ \cdots\cdots & -3 & -2 & -1 & 0 & +1 & +2 & +3 & \cdots\cdots \end{array}$$

- Factorized thermal state:

$$A \mapsto \omega_\beta(A) = \mathrm{Tr}_{[q,p]}\left(\bigotimes_{k=p}^{q} \rho_\beta^{(k)}\, A\right) \quad , \qquad \rho_\beta^{(k)} := \frac{\mathrm{e}^{-\beta H^{(k)}}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H^{(k)}}\right)} \ ,$$

$A \in \mathcal{A}_{[p,q]} \otimes \mathcal{A}_{[p,q]}$ and

$$H^{(k)} = \frac{\eta}{2}\left(\sigma_3^{(k)} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_3^{(k)}\right) \qquad \forall k \in \mathbb{Z}$$

Double Spin Chain

- Set of microscopic observables $\mathcal{X} = \{x_j\}_{j=1}^{8}$:

$$x_1 = \sigma_1 \otimes \mathbf{1} \ , \ x_2 = \sigma_2 \otimes \mathbf{1} \ , \ x_3 = \mathbf{1} \otimes \sigma_1 \ , \ x_4 = \mathbf{1} \otimes \sigma_2$$
$$x_5 = \sigma_1 \otimes \sigma_3 \ , \ x_6 = \sigma_2 \otimes \sigma_3 \ , \ x_7 = \sigma_3 \otimes \sigma_1 \ , \ x_8 = \sigma_3 \otimes \sigma_2 \ .$$

- Microscopic state: (infinite) tensor product of on-site thermal density matrices

$$\omega_\beta(x_j) = \mathrm{Tr}\left(\rho_\beta\, x_j\right) = 0 \qquad \forall\, j = 1, 2, \ldots, 8$$

- Local fluctuations

$$F_N(x_j) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N-1} x_j^{(k)}$$

Double Spin Chain

- Symplectic matrix:

$$\begin{aligned}\omega_\beta\left(\left[F_N(x_i)\,,\;F_N(x_j)\right]\right) &= \frac{1}{N}\sum_{k,\ell=0}^{N-1}\omega_\beta\left(\left[x_i^{(k)}\,,\;x_j^{(\ell)}\right]\right)\\ &= \mathrm{Tr}\left(\rho_\beta\left[x_i\,,\;x_j\right]\right)\end{aligned}$$

$$\sigma_\beta = 2\epsilon\begin{pmatrix} S & 0 & -\epsilon S & 0\\ 0 & S & 0 & -\epsilon S\\ -\epsilon S & 0 & S & 0\\ 0 & -\epsilon S & 0 & S\end{pmatrix}\,,\qquad S = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$$

$$\epsilon = \tanh\left(\frac{\beta\eta}{2}\right)$$

Double Spin Chain

- Covariance matrix:

$$\omega_\beta\left(F_N(x_i)F_N(x_j)\right) = \frac{1}{N}\sum_{\ell,k=0}^{N-1}\omega_\beta(x_i^{(\ell)}x_j^{(k)}) = \mathrm{Tr}\left(\rho_\beta\, x_i\, x_j\right)$$

$$\Sigma_\beta = \begin{pmatrix} \mathbf{1} & 0 & -\epsilon\mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & -\epsilon\mathbf{1} \\ -\epsilon\mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & -\epsilon\mathbf{1} & 0 & \mathbf{1} \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Double Spin Chain

- Passage to creators and annihilators: $c = \sqrt{1-\epsilon^2}$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \\ a_4^\dagger \end{pmatrix} = \frac{1}{2c\sqrt{\epsilon}} \begin{pmatrix} c & -ic & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & -i\epsilon & 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & c & -ic & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & -i\epsilon & 0 & 0 & 1 & -i \\ c & ic & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & i\epsilon & 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & c & ic & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & i\epsilon & 0 & 0 & 1 & i \end{pmatrix} \begin{pmatrix} F(x_1) \\ F(x_2) \\ F(x_3) \\ F(x_4) \\ F(x_5) \\ F(x_6) \\ F(x_7) \\ F(x_8) \end{pmatrix}$$

- $F(x_{1,2})$ and $a_1, a_1^\dagger$ first chain mesoscopic degrees of freedom
- $F(x_{3,4})$ and $a_3, a_3^\dagger$ second chain mesoscopic degrees of freedom

Double Spin Chain

- From Weyl operators to displacement operators

$$W(r) = \exp\left(\sum_{j=1}^{4}\left(z_j a_j^\dagger - z_j^* a_j\right)\right) =: D(z)$$

- Mesoscopic state: thermal state at the inverse temperature β of the microscopic state

$$\Omega_\beta\left(W(r)\right) = \left(1 - \mathrm{e}^{-\beta\eta}\right)^4 \mathrm{Tr}\left(\mathrm{e}^{-\beta\eta\sum_{j=1}^{4} a_j^\dagger a_j}\, D(z)\right)$$

Open Quantum Systems R.Alicki, K.Lendi LNP 717 (2007)

- Open quantum system $\mathcal{S}$ in weak interaction with its environment
- Master equation:

$$\begin{aligned} \partial_t \varrho_t &= \mathbb{L}[\varrho_t] = -i[H, \varrho_t] + \mathbb{K}[\varrho_t] \\ \mathbb{K}[\varrho_t] &= \sum_{ij} K_{ij} \Big(V_i \varrho_t V_j^\dagger - \frac{1}{2} \Big\{ V_j^\dagger V_i, \varrho_t \Big\} \Big) , \qquad K = \big[K_{ij} \big] \geq 0 \end{aligned}$$

- Solution: Completely Positive, Trace-Preserving maps $\gamma_t = \mathrm{e}^{t\mathbb{L}}$

$$\varrho \mapsto \varrho_t = \mathrm{e}^{t\mathbb{L}}[\varrho] , \quad \mathrm{Tr}\varrho_t = 1$$

- One-parameter semigroup:

$$\gamma_s \circ \gamma_t = \gamma_t \circ \gamma_s = \gamma_{s+t} \quad \forall\, s, t \geq 0$$

Dissipative effects

- Dissipation: $\mathbb{D}[\varrho_t] = -\frac{1}{2}\sum_{ij} K_{ij}\left\{V_j^\dagger V_i, \varrho_t\right\}$
- Statistical Mixing: $\mathbb{M}[\varrho_t] = \sum_{ij} K_{ij} V_i \varrho_t V_j^\dagger$
- Pure states get transformed into mixtures: decoherence

Bipartite Entanglement

- Entangled superpositions of two spin vector states

$$|\Psi_{12}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\otimes\downarrow\rangle + |\downarrow\otimes\uparrow\rangle)$$

$$|\Psi_{12}\rangle\langle\Psi_{12}| = \frac{1}{2}\Big(|\uparrow\rangle\langle\uparrow|\otimes|\downarrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\downarrow|\otimes|\uparrow\rangle\langle\uparrow|$$
$$+ |\uparrow\rangle\langle\downarrow|\otimes|\downarrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow|\otimes|\downarrow\rangle\langle\uparrow|\Big)$$

Entanglement usually destroyed by a dissipative dynamics:

$$|\Psi_{12}\rangle\langle\Psi_{12}| \longmapsto \frac{1}{2}\,(|\uparrow\rangle\langle\uparrow|\otimes|\downarrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\downarrow|\otimes\uparrow\rangle\langle\uparrow|)$$

Two-spin Entanglement

- Separable states:

$$\rho_{sep} = \sum_j \lambda_j \, \rho_j^{(1)} \otimes \rho_j^{(2)} \,, \qquad \sum_j \lambda_j = 1 \,,\ \lambda_j \geq 0$$

- Positive under Partial Transposition:

$$\mathrm{id} \otimes \mathbb{T}[\rho_{sep}] = \sum_j \lambda_j \, \rho_i^{(1)} \otimes \mathbb{T}[\rho_j^{(2)}] \geq 0$$

- Before partial transposition:

$$|\Psi_{12}\rangle\langle\Psi_{12}| = \frac{1}{2}\Big(|\uparrow\rangle\langle\uparrow| \otimes |\downarrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\downarrow| \otimes |\uparrow\rangle\langle\uparrow|$$

$$+|\uparrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\uparrow|\Big) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- After partial transposition:

$$\mathrm{id} \otimes \mathbb{T}\left[|\Psi_{12}\rangle\langle\Psi_{12}|\right] = \frac{1}{2}\Big(|\uparrow\rangle\langle\uparrow| \otimes |\downarrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\downarrow| \otimes |\uparrow\rangle\langle\uparrow|$$

$$+|\uparrow\rangle\langle\downarrow| \otimes |\uparrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\downarrow| \otimes |\uparrow\rangle\langle\downarrow|\Big) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Partial Transposition: Entanglement Witness

Horodecki's, PLA 223, 1996

Two-spin states are entangled IFF NOT PPT

Dissipative entanglement generation

F.B., R. Floreanini, M.Piani PRL 91 (2003)

- NO spin-spin interactions: $[H, \varrho_t] = 0$, ONLY statistical mixing:

$$\mathbb{M}[\varrho_t] = \sum_{\mu,\nu=1}^{3} D_{\mu\nu}\Big(\sigma_\mu \otimes \mathbf{1}\,\varrho_t\,\sigma_\nu \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_\mu\,\varrho_t\,\mathbf{1} \otimes \sigma_\nu$$

$$+ \quad \sigma_\mu \otimes \mathbf{1}\,\varrho_t\,\mathbf{1} \otimes \sigma_\nu + \mathbf{1} \otimes \sigma_\mu\,\varrho_t\,\sigma_\nu \otimes \mathbf{1}\Big)$$

- Kossakowski matrix: $K = \begin{pmatrix} D & D \\ D & D \end{pmatrix}$, $D = [D_{\mu\nu}]$,

$$D = \begin{pmatrix} 1 & -i\epsilon & 0 \\ i\epsilon & 1 & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad 0 \leq \epsilon \leq 1\,, \quad \gamma \geq 0$$

Microscopic dissipative time-evolution

- Local generator: $\partial_t X(t) = \mathbb{L}_N[X(t)]$, $\mathbb{L}_N[X] = \mathbb{H}_N[X] + \mathbb{K}_N[X]$

$$\mathbb{H}_N[X] = i\Big[\sum_{k=0}^{N-1} h^{(k)}\,,\,X\Big]\,,\qquad h^{(k)} = h = h^\dagger$$

$$\mathbb{K}_N[X] = \sum_{k,\ell=0}^{N-1} J_{k\ell} \sum_{\mu,\nu=1}^{p} D_{\mu\nu}\Big(v_\mu^{(k)}\,X\,(v_\nu^\dagger)^{(\ell)} - \frac{1}{2}\Big\{v_\mu^{(k)}\,(v_\nu^\dagger)^{(\ell)})\,,\,X\Big\}\Big)$$

- Translation invariant, fast decaying mixing coefficients:

$$J_{k\ell} = J(|k-\ell|)\ ,\ J(0) =: J_0 > 0\ ;\ \lim_{N\to+\infty}\frac{1}{N}\sum_{k,\ell=0}^{N-1}|J_{k\ell}| < +\infty\ .$$

- Local dynamics: $\Phi_t^N = \mathrm{e}^{t\mathbb{L}_N}$ completely positive if $K = J\otimes D \geq 0$

Which mesoscopic dynamics on the Weyl algebra emerges in the large N limit?

Does there exist the mesoscopic limit of the dynamics:

$$\Phi_t := m - \lim_{N\to\infty} \Phi_t^N \qquad ?$$

What one ought to show

$$\lim_{N\to+\infty} \omega_\beta\left(W_N(r_1)\Phi_t^N[W_N(r)]\,W_N(r_2)\right) = \Omega_\beta\left(W(r_1)\,\Phi_t[W(r)]\,W(r_2)\right)$$

Working conditions

- Time-invariant microscopic state: $\omega \circ \Phi_t^N = \omega$, otherwise time-dependent mesoscopic commutation relations
- Linear span of χ mapped left invariant by $\mathbb{L}_N$:

$$\mathbb{L}_N[x_j^{(k)}] = \mathbb{H}_N[x_j^{(k)}] + \mathbb{D}_N[x_j^{(k)}] = \sum_{\ell=1}^{p} \left(\mathcal{H}_{j\ell} + \mathcal{D}_{j\ell}\right) x_\ell^{(k)}$$

$$x_j^{(k)} \mapsto \mathbb{L}_N[x_j^{(k)}] = \sum_{\ell=1}^{p} \mathcal{L}_{j\ell}\, x_\ell^{(k)} , \qquad \mathcal{L} = \mathcal{H} + \mathcal{D}$$

Theorem

F.B., F. Carollo, R. Floreanini, PLA 378 (2014)

F.B., F. Carollo, R. Floreanini, Ann.Phys. (Berlin) 527 (2015)

The emergent mesoscopic dynamics is dissipative and Gaussian

$$W(r) \mapsto W_t(r) := \Phi_t[W(r)] = \mathrm{e}^{f_r(t)}\, W(r_t)$$

$$r_t = \mathrm{e}^{\,t\,\mathcal{L}^{tr}}\, r\;, \quad f_r(t) = -\frac{1}{2}\,(r, \mathcal{Y}_t\, r)\;, \quad \mathcal{Y}_t \;=\; \Sigma_\beta \,-\, \mathrm{e}^{t\mathcal{L}}\,\Sigma_\beta\, \mathrm{e}^{t\mathcal{L}^{tr}}$$

Mesoscopic generator

Gaussian mesoscopic generator: $\Phi_t = \exp(t\mathbb{L})$

$$\mathbb{L}[W_t(r)] = \frac{i}{2}\sum_{i,j=1}^{p} H_2^{ij}\Big[F(x_i)F(x_j)\,,\ W_t(r)\Big]$$

$$+\sum_{i,j=1}^{p} D_{ij}\left(F(x_i)\,W_t(r)\,F(x_j)\ -\ \frac{1}{2}\Big\{F(x_i)F(x_j)\,,\ W_t(r)\Big\}\right)$$

$$H_2 = -i(\sigma^\omega)^{-1}\left(\mathcal{L}\,C\ -\ C\,\mathcal{L}^{tr}\right)(\sigma^\omega)^{-1} = H_2^\dagger$$

$$D = (\sigma^\omega)^{-1}\left(\mathcal{L}\,C\ +\ C\,\mathcal{L}^{tr}\right)(\sigma^\omega)^{-1} \geq 0$$

$$C = \left[C_{ij}\right]\ ,\quad C_{ij} = \lim_{N\to\infty}\omega_\beta\Big(F_N(x_i)F_N(x_j)\Big)$$

Algebraic setting and sketch of proof

- Quantum spin chain: quasi-local algebra $\mathcal{A}$
- Microscopic state: translation-invariant, clustering KMS state ω_β
- Selected microscopic observables: self-adjoint set of on-site observables χ
- Microscopic dissipative dynamics: semigroup of local completely positive, unital maps $\Phi_t^N = \exp(t\mathbb{L}_N)$ such that $\omega_\beta = \omega_\beta \circ \Phi_t^N$
- Locality condition: linear span of χ left invariant by $\mathbb{L}_N$.

Sketch of proof

- Step 1: one can substitute $\Phi_t[W(r)]$ with

$$W_N^t(r) = \mathrm{e}^{f_r(t)}\, W_N(r_t) = \mathrm{e}^{f_r(t)}\, \mathrm{e}^{i(r_t, F_N)}$$

- Step 2: one studies

$$\begin{aligned} W_N^t(r) - \Phi_t^N[W_N(r)] &= \int_0^t \mathrm{d}s\, \frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi_{t-s}^N\left[W_N^s(r)\right]\right) \\ &= \int_0^t \mathrm{d}s\, \Phi_{t-s}^N\left[\frac{\mathrm{d}}{\mathrm{d}s} W_N^s(r) - \mathbb{L}_N[W_N^s(r)]\right] \end{aligned}$$

Comparison of terms scaling as $1/\sqrt{N}$ and $1/N$ in

- Time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} W_N^t(r) \simeq \left(\frac{\mathrm{d}f_r(t)}{\mathrm{d}t} + i(\dot{r}_t\,,\,F_N) - \frac{1}{2}\Big[(r_t, F_N)\,,\,(\dot{r}_t, F_N)\Big]\right) W_N^t(r)$$

- Action of the generator:

$$\begin{aligned}
\mathbb{L}_N[W_N(r)] \quad &\simeq \quad i\,\mathbb{L}_N\left[(r, F_N)\right] W_N(r) \\
&- \quad \frac{1}{2}\Big[(r, F_N)\,,\,\mathbb{L}_N\left[(r, F_N)\right]\Big] W_N(r) \\
&+ \quad \frac{1}{2}\Big(\mathbb{L}_N\left[(r, F_N)\right]\,(r, F_N) + (r, F_N)\,\mathbb{L}_N\left[(r, F_N)\right] \\
&- \quad \mathbb{L}_N\left[(r, F_N)^2\right]\Big) W_N(r)
\end{aligned}$$

Gaussianity preserved by Φ_t

- Gaussian states are identified by theor covariance matrix:

$$\Omega(W(r)) = \exp\left(-\frac{1}{2}(r\,,\Sigma\,r)\right)$$

- Creation and annihilation operator formalism:

$$\Omega(D(z)) = \exp\left(-\frac{1}{2}\Big((z^*,z)\,,\Sigma\,(z,z^*)\Big)\right)$$

$$\Sigma = \frac{1}{2}\left[\Omega\left(\left\{A_i\,,\,A_j^\dagger\right\}\right)\right]\,,\qquad A = \left(\{a_j\}_j,\{a_j^\dagger\}\right)^{tr}$$

- Positivity condition:

$$\Sigma + \frac{\Sigma_3}{2} \geq 0\,,\quad \Sigma_3 = \begin{pmatrix}\sigma_3 & 0 & 0 & 0\\ 0 & \sigma_3 & 0 & 0\\ 0 & 0 & \sigma_3 & 0\\ 0 & 0 & 0 & \sigma_3\end{pmatrix}\,,\quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Entanglement of two-mode Gaussian states

Simon criterion: Partial transposition identifies entanglement

R. Simon, PRL **84** (2000)

- two modes: $\{a_1\,,\,a_1^\dagger\}$, $\{a_3\,,\,a_3^\dagger\}$
- Similarly to partial tansposition: $a_3^{\#}$ fixed

$$a_1\mapsto a_1^\dagger\ ,\quad a_1\mapsto a_1^\dagger\ ,\quad a_1^\dagger a_1\mapsto a_1^\dagger a_1$$

- Induced transformation of the covariance matrix: $\Sigma\mapsto\widetilde{\Sigma}$
- Simon criterion: 2-mode Gaussian states Ω separable **IFF**

$$\widetilde{\Sigma}+\frac{\Sigma_3}{2}\geq 0$$

Open double spin chains: microscopic dynamics

- Local dissipative dynamics:

$$\partial_t X_t = \mathbb{L}_N[X_t] \ , \qquad X_t \in \mathcal{A}_{[0,N-1]}$$

- Generator: $\mathbb{L}_N[X_t] = i \sum_{k=0}^{N-1} \left[H^{(k)} \, , \, X_t \right] + \sum_{k=0}^{N-1} \mathbb{D}_N^{(k)}[X_t]$
- Hamiltonian term: $H^{(k)} = \sigma_3^{(k)} \otimes \mathbf{1}^{(k)} \, + \, \mathbf{1}^{(k)} \otimes \sigma_3^{(k)}$
- Dissipative term:

$$\mathbb{D}_N^{(k)}[X_t] = J_0 \sum_{\mu,\nu=1}^{6} D_{\mu\nu} \left(v_\mu^{(k)} \, X_t \, v_\nu^{(k)} - \frac{1}{2} \left\{ v_\mu^{(k)} \, v_\nu^{(k)} \, , \, X_t \right\} \right)$$

- Kraus operators: $v_{1,2,3} = \sigma_{1,2,3} \otimes \mathbf{1}, \ v_{4,5,6} = \mathbf{1} \otimes \sigma_{1,2,3}$

- Statistically mixing term: $w_\mu = \sigma_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_\mu$

$$\begin{aligned}\mathbb{M}_N^{(k)}[X] &= w_1^{(k)}\,X\,w_1^{(k)} + w_2^{(k)}\,X\,w_2^{(k)} + \gamma\,w_3^{(k)}\,X\,w_3^{(k)} \\ &- i\epsilon\,w_1^{(k)}\,X\,w_2^{(k)} + i\epsilon\,w_2^{(k)}\,X\,w_1^{(k)}\end{aligned}$$

- Schrödinger picture: local states ϱ_N, dual generator $\mathbb{L}_N^T$:

$$\mathrm{Tr}\left(\varrho_N \mathbb{L}_N[X]\right) = \mathrm{Tr}\left(\mathbb{L}_N^T[\varrho_N]X\right)$$

- Local invariant state: ϱ_N^β such that $\mathbb{L}_N^T[\rho_N^\beta] = 0$,

$$\varrho_N^\beta = \bigotimes_{k=0}^{N-1} \frac{\mathrm{e}^{-\eta\beta w_3^{(k)}/2}}{4\cosh^2(\frac{\eta\beta}{2})}$$

Lindblad generator: annihilation and creation operators

- Action of $\mathbb{L} = \mathbb{H} + \mathbb{D}$ on $D(z) = \exp(\sum_{j=1}^{4} z_j a_j^\dagger - z_j^* a_j)$

$$\mathbb{H}[D(z)] = i\,\omega \left[\sum_{j=1}^{4} a_j^\dagger a_j\,,\, D(z)\right]$$

$$\mathbb{D}[D(z)] = \sum_{i,j=1}^{8} K_\beta^{ij} \left(V_i^\dagger\, D(z)\, V_j - \frac{1}{2}\left\{ V_i^\dagger\, V_j\,,\, D(z)\right\}\right)$$

where

$$V = (a_1, a_2, a_1^\dagger, a_2^\dagger, a_3, a_4, a_3^\dagger, a_4^\dagger)^{tr}$$

- Kossakowski matrix

$$K_\beta = \frac{2}{\epsilon}\begin{pmatrix}(1+\epsilon)A_\beta & 0 & (1+\epsilon)B_\beta & 0\\ 0 & (1-\epsilon)A_\beta & 0 & (1-\epsilon)B_\beta\\ (1+\epsilon)B_\beta & 0 & (1+\epsilon)A_\beta & 0\\ 0 & (1-\epsilon)B_\beta & 0 & (1-\epsilon)A_\beta\end{pmatrix}$$

$$A_\beta = \begin{pmatrix}1+\gamma & 0\\ 0 & 3+\gamma\end{pmatrix}, \quad B_\beta = \begin{pmatrix}\epsilon^2 & -\epsilon c\\ -\epsilon c & 1+c^2\end{pmatrix},$$

$\epsilon = \tanh(\eta\beta/2)$, $c = \sqrt{1-\epsilon^2}$

- Mixes mesoscopic degrees of freedom $a_1^\#$, $a_3^\#$ belonging to different chains:

$$V = (a_1, a_2, a_1^\dagger, a_2^\dagger, a_3, a_4, a_3^\dagger, a_4^\dagger)^{tr}$$

Entangled mesosocopic modes from different chains

- Chain 1: $a_1\,,\,a_1^\dagger$ Chain 2: $a_3\,,\,a_3^\dagger$
- $\Omega_\beta^{(1,3)}$ restricted to $a_1\,,\,a_1^\dagger$, $a_3\,,\,a_3^\dagger$: Gaussian and separable

$$R_\beta^{(13)} = \left(1-\mathrm{e}^{-\beta\eta}\right)^2 \mathrm{e}^{-\beta\eta a_1^\dagger a_1}\,\mathrm{e}^{-\beta\eta a_3^\dagger a_3}$$

$$\Sigma_\beta^{(13)} = \frac{1}{\mathrm{e}^{\beta\eta}-1}\begin{pmatrix}\mathrm{e}^{\beta\eta} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \mathrm{e}^{\beta\eta} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

- Time-invariant and thus separable at all times: $\Omega_\beta^{(13)}\circ\Phi_t = \Omega_\beta^{(13)}$

Squeezing: time-changing state, separable at $t = 0$

- Squeezed state:

$$\rho_r^\beta = S_1(r)\, S_3(r)\, R_\beta^{(13)}\, S_3^\dagger(r)\, S_1^\dagger(r)$$

$$S_j^\dagger(r)\, a_j^\dagger\, S_j(r) = \cosh(r)\, a_j^\dagger + \sinh(r)\, a_j$$

- Squeezed state: Gaussian, Not Φ_t-invariant

$$\nu_r^\beta(t)\,(D_{13}(z)) = \mathrm{Tr}\left(\rho_r^\beta\, \Phi_t\,[D_{13}(z)]\right) = \mathrm{Tr}\left(\rho_r^\beta(t)\, D_{13}(z)\right)$$

- $\nu_r^\beta(t)$: time-changing, Gaussian 2-mode state
- Check of Simon criterion on its covariance matrix $\Sigma_r^\beta(t)$

Entanglement quantification for continuous variables

A. Isar, Open Sys. Inf. Dynamics **18** (2011)

- Send $a_i^\dagger \mapsto a_i\,, \quad a_i^\dagger \mapsto a_i\,, \qquad a_i^\dagger a_i \mapsto a_i^\dagger a_i$
- Induced transformation of the covariance matrix:

$$\Sigma_r^\beta(t) \mapsto \widetilde{\Sigma}_r^\beta(t)$$

- Simon criterion: $\nu_r^\beta(t)$ separable **IFF**

$$\widetilde{\Sigma}_r^\beta(t) + \frac{\Sigma_3}{2} \geq 0$$

- Smallest symplectic eigenvalue of $\widetilde{\Sigma}_r^\beta(t)$: $g(\widetilde{\Sigma}_r^\beta(t))$
- Logarithmic negativity: entanglement quantifier

$$E(t) = \max\left\{0, -\frac{1}{2}\log_2(4g(\widetilde{\Sigma}_r^\beta(t)))\right\}$$

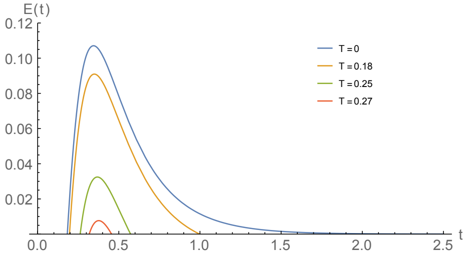

Figure : $E(t)$: $r = 1$, $\gamma = 1$, varying T.

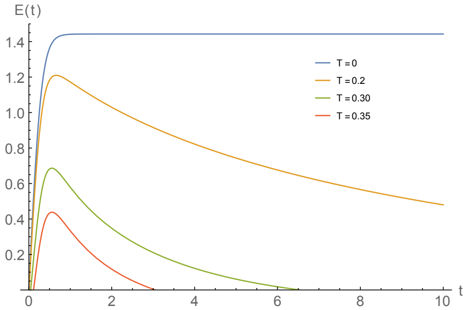

Figure : $E(t)$: $r = 1$, $\gamma = 0$, varying T