IMPURITIES, FILAMENTS, AND MEMBRANES PERTURBING LS AND NLS

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TQT

The general framework :

$u \equiv u(t,x)$
$(t,x) \in \mathbb{R} \times \mathbb{R}^d$

$$i\partial_t u = -\Delta u + Vu + \mathcal{N}(u) + \delta_\Gamma u$$

(SCHRÖDINGER)
or also:
$-\partial_t u$ (HEAT)
$-\partial_t^2 u$ (WAVE)

(LAPLACIAN IN $\mathbb{R}^d$)
or also:
$-i\vec{\alpha}\cdot\vec{\nabla} + mc^2\alpha_0$ (DIRAC)
or $\sqrt{1-\Delta}$, etc.

SEMILINEARITY

a LOCALISED PERTURBATION supported on a region $\Gamma \subset \mathbb{R}^d$

A variant relevant in COLD ATOM PHYSICS :

instead of $u \equiv u(t,x)$ (one-body problem)

take $u \equiv u(t, x_1, x_2, \dots, x_K)$ (K-body problem)

$$x_1, \dots, x_K \in \mathbb{R}^d$$

$$-\Delta_x \longrightarrow -\Delta_{x_1} - \Delta_{x_2} - \dots - \Delta_{x_K}$$

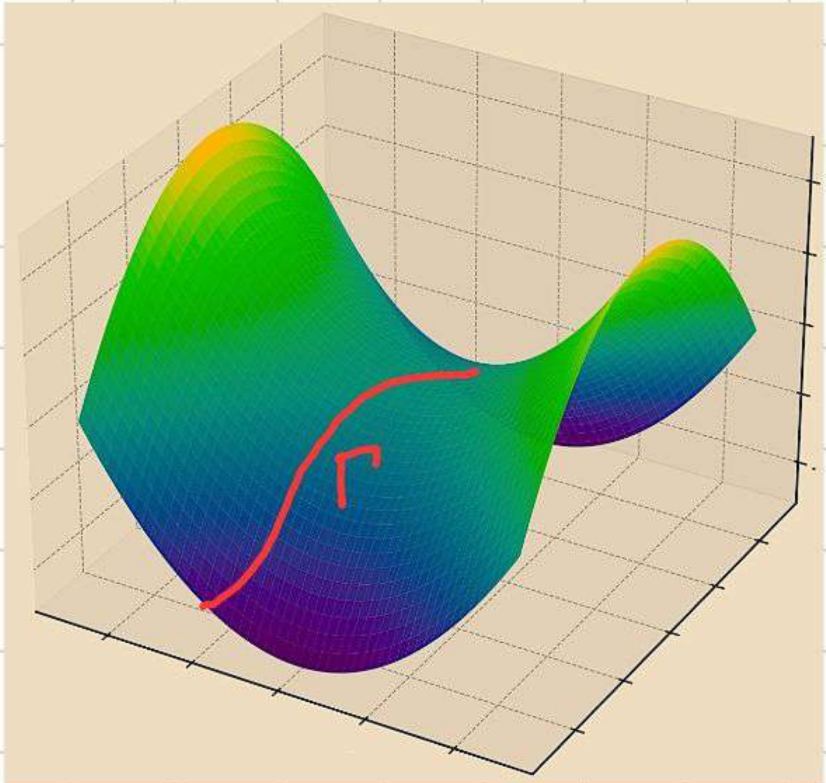
$$\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d \times \dots \times \mathbb{R}^d$$

A variant relevant in CONTROL THEORY :

$$i\partial_t u = -\Delta_{LB} u + \delta_\Gamma u$$

the LAPLACE-BELTRAMI operator on a DEGENERATE RIEMANNIAN MANIFOLD $(\mathcal{M}, g)$

where the metric g blows-up along Γ



« Contact-type interactions » $(\operatorname{codim}\Gamma > 0)$

arise naturally

$\longrightarrow$ as idealised/realistic models
of very localised, singular interactions
in cold atom physics,
condense matter physics, chemistry

$\longrightarrow$ in the Fock space setting of quantum field theory

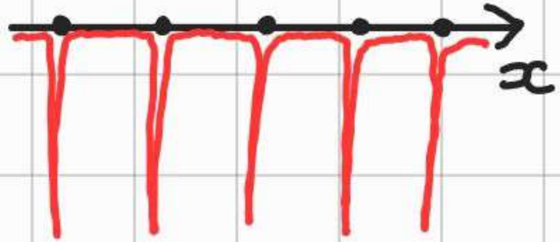
$\longrightarrow$ in stochastic processes for homopolymers

POINT-LIKE PERTURBATIONS

very old idea in Quantum Mechanics:

⟶ Kronig-Penney model of a 1-dim crystal (1931)

$$\boxed{-\frac{d^2}{dx^2} + \alpha \sum_{n\in\mathbb{Z}} \delta(x-n)}$$



⟶ Fermi's δ-pseudopotential for β-decay (1936)

$$\boxed{-\frac{\partial^2}{\partial r^2} - \frac{1}{\alpha}\,\delta(r)\,\frac{\partial}{\partial r}\, r\Big|_{r=0^+}}$$

and neutrons in Hydrogenous substances

$\longrightarrow$ Bethe and Peierls (1935)
Thomas (1935)
Skornyakov and Ter-Martirosyan (1956)

for the TWO-BODY proton–neutron scattering within the 3-body problem

$$\boxed{\begin{aligned} -\Delta\Psi &= E\Psi \\ \frac{\partial}{\partial r}(\ln r\Psi)\Big|_{r=0^+} &= 4\pi\alpha \end{aligned}}$$

the BP contact condition

$$\Rightarrow \Psi \sim \frac{1}{|x_i - x_j|} - \frac{1}{a} \quad \text{as } |x_i - x_j| \to 0 \qquad a = (-4\pi\alpha)^{-1}$$

POINT-LIKE PERTURBATIONS in 1-dim :

$$-\Delta_{\alpha,x_0} = -\frac{d^2}{dx^2} + \alpha \cdot \delta(x-x_0)$$

in the sense of quadratic forms in $L^2(\mathbb{R})$: $\int |\nabla f(x)|^2 + \alpha |f(x_0)|^2$

$$\mathrm{dom}(-\Delta_{\alpha,x_0}) = \left\{ \begin{array}{l} u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{x_0\}) \\ \text{with } u'(x_0^+) - u'(x_0^-) = \alpha\, u(x_0) \end{array} \right\}$$

$$-\Delta_{\alpha,x_0} u = -u''$$

explicit $(-\Delta_{\alpha,x_0} - \lambda \mathbb{1})^{-1}$, $e^{t\Delta_{\alpha,x_0}}$, $e^{-it\Delta_{\alpha,x_0}}$, ...

POINT-LIKE PERTURBATION in 3-dim (at $x_0 = 0$)

$-\Delta_\alpha$ on $L^2(\mathbb{R}^3)$

constructed by
- quadratic forms,
- operator extension,
- resolvent limits,
- etc. etc.

$-\Delta_\alpha$ as the self-adjoint extension of $-\Delta\big|_{C_c^\infty(\mathbb{R}^3 \setminus \{0\})}$ with B.P. contact condition with α

$-\Delta_\alpha$ as resolvent limit of $-\Delta + \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right)$ as $\varepsilon \downarrow 0$

A CLOSE-UP AT $-\Delta_\alpha$ in $\mathbb{R}^3$:

$$\underset{(\lambda>0)}{\mathrm{dom}(-\Delta_\alpha)} = \left\{ g = f_\lambda + \frac{f_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}} G_\lambda \quad \text{with } f_\lambda \in H^2(\mathbb{R}^3) \right\}$$

"regular" H^2-part

"singular" $H^{\frac{1}{2}^-}$-part

$$G_\lambda(x) := \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}$$

hence, $g \sim \left(\frac{1}{|x|} - \frac{1}{a}\right)$ as $|x| \to 0$, $a = -(4\pi\alpha)^{-1}$

$$(-\Delta_\alpha + \lambda) g = (-\Delta + \lambda) f_\lambda$$

A CLOSE-UP AT $-\Delta_\alpha$ in $\mathbb{R}^3$ – THE RESOLVENT

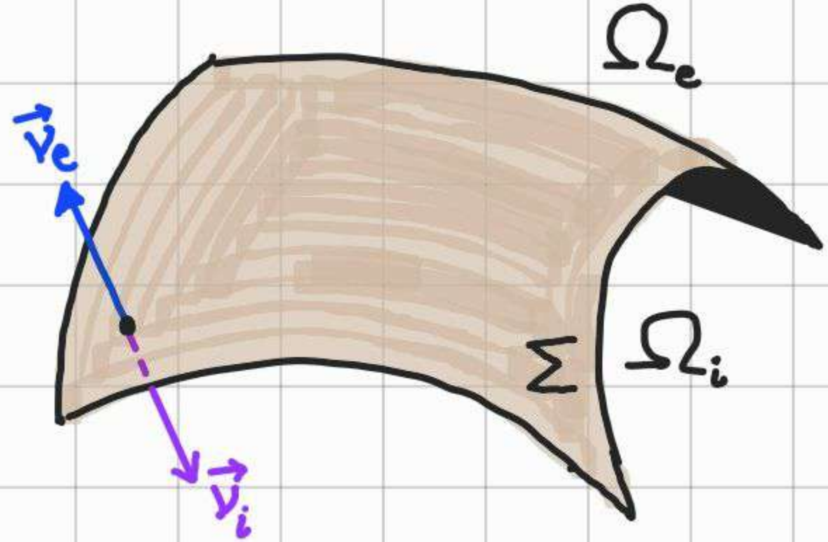
$$(-\Delta_\alpha + \lambda \mathbb{1})^{-1} = (-\Delta + \lambda \mathbb{1})^{-1} + \frac{1}{\alpha + \frac{\sqrt{\lambda}}{4\pi}} P_{G_\lambda} \qquad (\lambda > 0)$$

a rank-one perturbation of the free resolvent

- $e^{t\Delta_\alpha}$, $e^{it\Delta_\alpha}$, $W(e^{it\Delta}, e^{it\Delta_\alpha})$ explicitly known
- dispersive properties, Strichartz, etc. for $-\Delta_\alpha$ explicitly known

Berezin, Faddeev (1961) — renormalisation
Albeverio, Høegh-Krohn - Streit (1977-1980) — Dirichlet forms
Nelson (1977)
Albeverio, Fenstad, Høegh-Krohn } — non-standard analysis
Grossman, Høegh-Krohn, Mebkhout (1980) — resolvents
Albeverio, Høegh-Krohn (1981) — resolvent limits
Albeverio, Gesztesy, Høegh-Krohn (1982) — scattering
Scarlatti, Teta (1990)
Albeverio, Brzeźniak, Dabrowski (1995) } — semi-group and group
D'Ancona, Pierfelice, Teta (2006)
Iandoli, Scandone (2017) } — dispersive estimates
Duchêne, Marzuola, Weinstein (2011) — 1d wave operators
Michelangeli, Ottolini (2016) — self-adjoint realisations
Dell'Antonio, Michelangeli, Scandone, Yajima (2017) — 3d wave ops.
Cornean, Michelangeli, Yajima (2018) — 2d wave operators
Georgiev, Michelangeli, Scandone (2018, 2022, 2024)
Michelangeli, Olgiati, Scandone (2018)
Cacciapuoti, Finco, Noia (2021) } — adapted fractional Sobolev spaces

CONTACT INTERACTIONS ON SURFACES :



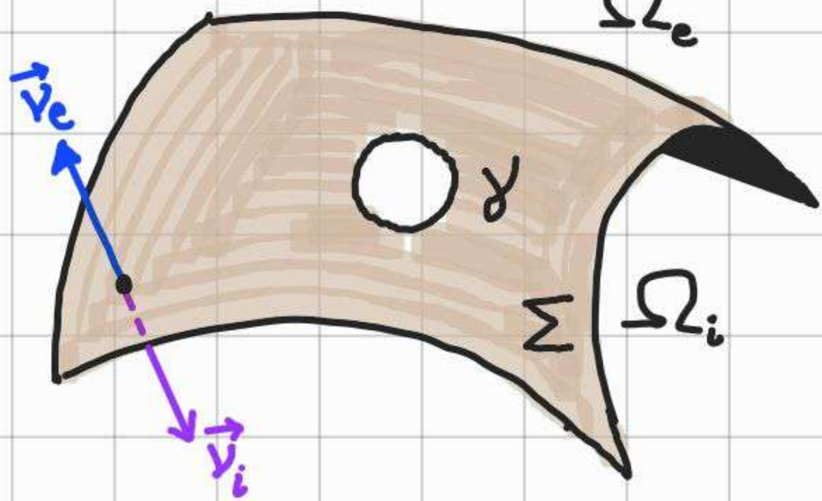
$\alpha \in L^\infty(\Sigma, \mathbb{R})$

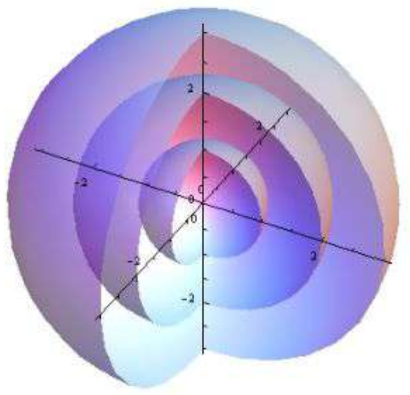
$-\Delta_\alpha f = -\Delta f$

$$\mathrm{dom}(-\Delta_\alpha) = \left\{ \begin{array}{l} f = f_i \oplus f_e \\ f_i \in H^{3/2}(\Omega_i),\ \Delta f_i \in L^2(\Omega_i) \\ (\text{same for } f_e) \\ f_i\big|_\Sigma = f_e\big|_\Sigma \\ \partial_{\nu_e} f_e\big|_\Sigma + \partial_{\nu_i} f_i\big|_\Sigma = \alpha \cdot f\big|_\Sigma \end{array} \right\}$$

VARIANTS :

- membranes with holes :





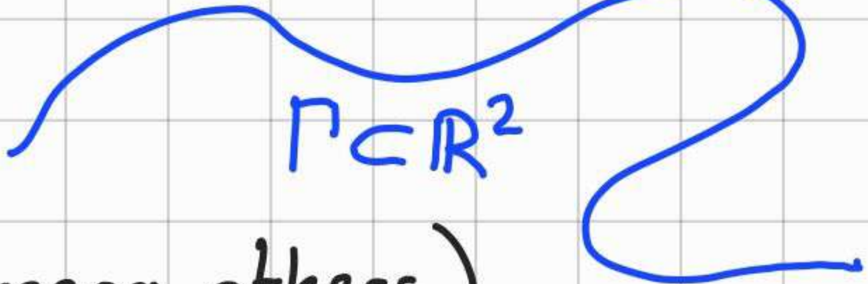
- closed/open surfaces

- concentric delta-shells

- δ' boundary conditions (or variants) instead of δ b.c.

- Dirac instead of Schrödinger

CONTACT INTERACTIONS ON CURVES:



quadratic form construction (among others)

$$Q_\alpha[f,g] := \langle \nabla f, \nabla g \rangle_{L^2(\mathbb{R}^2, \mathbb{C})} + \alpha \langle T_D f, T_D g \rangle_{L^2(\Gamma, \mathbb{C})}$$

$$\mathrm{dom}(Q_\alpha) := H^1(\mathbb{R}^2, \mathbb{C})$$

$T_D : H^1(\mathbb{R}^2, \mathbb{C}) \longrightarrow H^{1/2}(\Gamma, \mathbb{C})$ the Dirichlet trace operator

$Q_\alpha[f,g] = \langle f, -\Delta_{\Gamma,\alpha}\, g \rangle$ for a unique self-adjoint $-\Delta_{\Gamma,\alpha}$

closed and lower semibdd

Bilbao group (Vega, Arrizabalaga, ...)

Prague group (Exner, Lotoreichick, Šeba, Tušek, ...)

Graz group (Behrndt, Holtzmann, Frymark, ...)

Oldenburg group (Pankrashkin, Benhellal, ...)

Lviv group (Holovatyy, ...)

etc. etc.

Typical achievements:

- self-adjointness (via boundary triplets) ✓
- resolvent (via Krĕin formula) ✓
- spectral theory (including new ess spec) ✓
- propagators ✗
- dispersion ✗
- Sobolev spaces ✗

CONTACT INTERACTIONS ON MANIFOLDS (M,g) BY LOCALISED SINGULARITY OF THE METRIC g

PROTOTYPE MODEL: THE GRUSHIN PLANE/CYLINDER

($\longrightarrow$ Quantum Control Theory)

$$M = (\mathbb{R}^-_x \times \mathbb{R}_y) \cup (\mathbb{R}^+_x \times \mathbb{R}_y)$$

$$g_\alpha := dx \otimes dx + |x|^{-2\alpha}\, dy \otimes dy \qquad (\alpha \in \mathbb{R})$$

Riemannian volume form $\mu_\alpha = \sqrt{\det g_\alpha}\, dx \wedge dy$

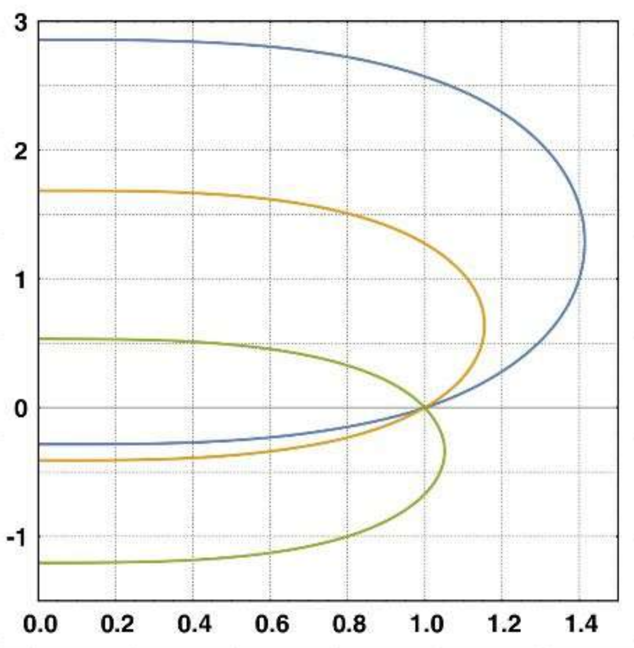
Laplace-Beltrami on $L^2(M, d\mu_\alpha)$ $\quad \Delta_\alpha = \dfrac{\partial^2}{\partial x^2} + |x|^{2\alpha} \dfrac{\partial^2}{\partial y^2} - \dfrac{\alpha}{|x|} \dfrac{\partial}{\partial x}$

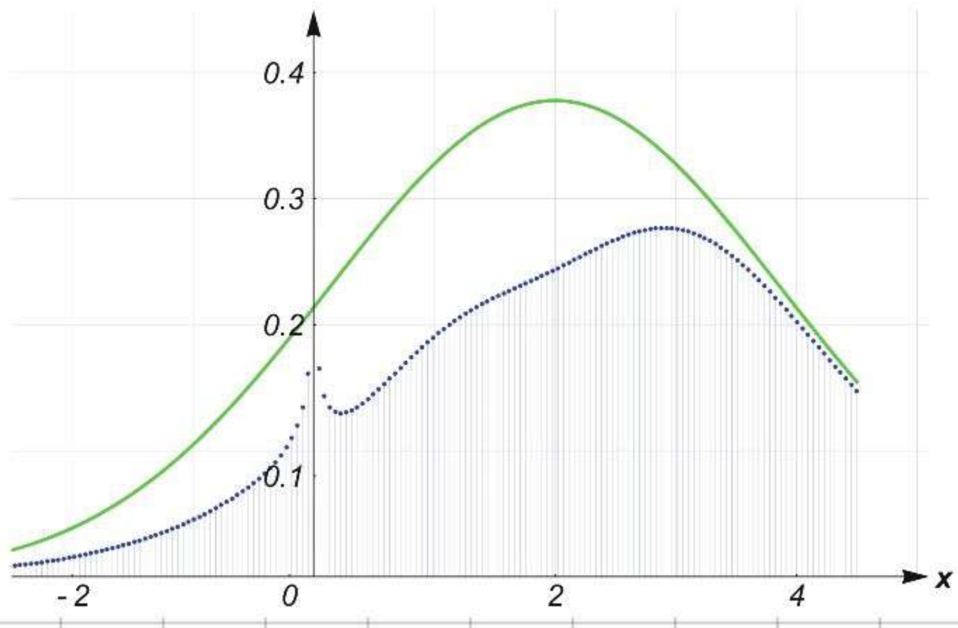
only one self-adjoint realisation of Δ_α for

$\alpha \in (-\infty,-1) \cup [1,+\infty)$ $\longrightarrow$ GEOMETRIC QUANTUM CONFINEMENT

an infinite multiplicity of self-adj. realisations of Δ_α

for $\alpha \in [-1,1)$ $\longrightarrow$ TRANSMISSION





Boscain, Laurent (2013)
Boscain, Prandi (2016)
Boscain, Prandi, Seri (2016)
Prandi, Rizzi, Seri (2018)
Gallone, Michelangeli, Pozzoli (2019)
Gallone, Michelangeli (2020)
Gallone, Michelangeli (2021)
Beshastny (2021)
Beshastny, Boscain, Pozzoli (2021)
Pozzoli (2021)
Gallone, Michelangeli, Pozzoli (2022)

From the LINEAR to the NON-LINEAR PROBLEM

$$i\partial_t u = -\Delta_\alpha u + Vu + \mathcal{N}(u)$$

To develop the full theory (well-posedness, grand states, blow-up, scattering, ...) :

need to characterise the

adapted Sobolev spaces $H^s_\alpha(\mathbb{R}^d) := \mathrm{dom}\,(-\Delta_\alpha)^{s/2}$

$$H^s_\alpha(\mathbb{R}^d) := \mathrm{dom}(-\Delta_\alpha)^{s/2} = \{ g \in L^2(\mathbb{R}^d) \mid -\Delta_\alpha g \in L^2(\mathbb{R}^d) \}$$

$d=3$ (Georgiev, Michelangeli, Scandone 2017)

$$H^s_\alpha(\mathbb{R}^3) = \begin{cases} H^s(\mathbb{R}^3) & s \in (0, \tfrac{1}{2}) \\ H^s(\mathbb{R}^3) \dotplus \mathrm{span}\left\{ \dfrac{e^{-|x|}}{4\pi |x|} \right\} & s \in (\tfrac{1}{2}, \tfrac{3}{2}) \\ \left\{ g = F_\lambda + \dfrac{F_\lambda(0)}{\alpha + \sqrt{\lambda}/4\pi} G_\lambda \;\middle|\; F_\lambda \in H^s(\mathbb{R}^3) \right\} \quad (\lambda > 0) & s \in (3/2, 2) \end{cases}$$

$$H^s_\alpha(\mathbb{R}^d) := \operatorname{dom}(-\Delta_\alpha)^{s/2} = \{ g \in L^2(\mathbb{R}^d) \mid -\Delta_\alpha g \in L^2(\mathbb{R}^d)$$

$d = 2$ (Cacciapuoti, Finco, Noja 2021
Georgiev, Michelangeli, Scandone 2022)

$$H^1(\mathbb{R}^2) \hookrightarrow H^1_\alpha(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2) \qquad p \in [2,\infty)$$

$$H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^2) \subset\subset L^q(\mathbb{R}^2) \qquad q \in (2,\infty)$$

$$H^s_\alpha(\mathbb{R}^2) \cong H^s(\mathbb{R}^2) \qquad s \in [0,1)$$

To give meaning also to

$$H_\alpha^{s,p} := \text{“}\{ g \in L^p \mid (-\Delta_\alpha)^{s/2} g \in L^p \}\text{”}$$

As a matter of fact (Georgiev, Michelangeli, Scandone 2024):

$$H_\alpha^{s,p}(\mathbb{R}^d) \cong H^{s,p}(\mathbb{R}^d)$$

- $d=3$, $s \in (0,1)$, $p \in \left(\frac{3}{2}, \frac{3}{1+s}\right)$
- $d=2$, $s \in (0,2)$, $p \in \left(1, \frac{2}{s}\right)$

regular + singular part

Georgiev, Rastrelli (2023): characterisation of $H_\alpha^{1,p}(\mathbb{R}^2)$

$$i\partial_t u = (-\Delta_\alpha)u + (w * |u|^2)u$$

Michelangeli, Olgiati, Scandone (2018)

LWP in $H^s_\alpha(\mathbb{R}^3)$, $s \in [0,2]$ + global $L^2/H^1_{\alpha,rad}$-solution

Georgiev, Michelangeli, Scandone (2022)

LWP / GWP in $H^1_\alpha(\mathbb{R}^2)$, radial w

$$i\partial_t u = -\Delta_d u + \beta \cdot |u|^{p-1} u$$

Cacciapuoti, Finco, Noja (2021)

LWP in $H^2(\mathbb{R}^d)$ ($p \in [1,\infty), d=2$; $p \in [1, \frac{3}{2}), d=3$)

and global existence in $H^2(\mathbb{R}^d)$

Fukaya, Georgiev, Ikeda (2022) , Finco, Noja (2023)

stability / instability of standing waves ($d=2$)

Cacciapuoti, Finco, Noja (2024)

failure of scattering ($p \in (1,2), d=2$; $p \in (1, \frac{4}{3}), d=3$)

In fact, the non-linear theory is at its early stage:

- a corpus of still miscellaneous results for point perturbations
- virtually unexplored territory for perturbations on curves and surfaces