Non-linear Schrödinger equations with rough, non-Strichartz-controllable magnetic potentials

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Based on

- ❒ standard techniques from non-linear dispersive PDE theory;
- ❒ several seminal contributions from other experts (cited in due time);
- ❒ a few projects of mine and with co-workers, including:

A. Michelangeli, *Global well-posedness of the magnetic Hartree equation with non-Strichartz external fields*, Nonlinearity 28 (2015) 2743-2765;

P. Antonelli (GSSI), A. Michelangeli, R. Scandone, *Global, finite energy, weak solutions for the NLS with rough, time-dependent magnetic potentials*, Z. Angew. Math Phys. (2018) 69:46;

❒ ongoing activity with V. Georgiev (Pisa, and Waseda, and Bulgarian Academy of Science), R. Scandone (GSSI and Naples), and K. Yajima (Tokyo Gakushuin) on point-like perturbations of (magnetic) NLS.

NLS (semi-linear Schrödinger equation) with external magnetic potentials

Class of evolution PDEs of interest:

$$\mathrm{i}\partial_t u = -(\nabla - \mathrm{i}\mathbf{A})^2 u + Vu + \mathcal{N}(u)$$

$$\mathcal{N}(u) = \lambda_1 |u|^{\gamma-1} u + \lambda_2 (|\cdot|^{-\alpha} * |u|^2) u + (W * |u|^2) u \qquad \begin{cases} \gamma \in (1, 1 + \frac{4}{d-2}] \\ \alpha \in (0, d) \\ \lambda_1, \lambda_2 \in \mathbb{R} \end{cases}$$

$$\begin{aligned} V &\equiv V(t,x) : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R} \\ W &\equiv W(x) : \mathbb{R}^d_x \to \mathbb{R} \\ \mathbf{A} &\equiv \mathbf{A}(t,x) \equiv (A_1, \dots, A_d) : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^d \\ &\qquad \text{(mesurable functions)} \end{aligned}$$

in the unknown $u \equiv u(t,x) : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$.

NLS is a basic dispersive model that appears, among others,

- in nonlinear optics,
- in water wave theory,
- in quantum many-body dynamics;

'ordinary' version is without magnetic potential ($\mathbf{A} \equiv \mathbf{0}$):

$$\mathrm{i}\partial_t u = -\Delta_x u + Vu + \lambda_1 |u|^{\gamma-1} u + \lambda_2 (|\cdot|^{-\alpha} * |u|^2) u + (W * |u|^2) u$$

Hartree equation has cubic *non-local* semi-linearity:

$$\mathrm{i}\partial_t u = -\Delta_x u + Vu + (W * |u|^2) u$$

Gross-Pitaevskii equation has cubic *local* semi-linearity:

$$\mathrm{i}\partial_t u = -\Delta_x u + Vu + \lambda |u|^2 u$$

magnetic Laplacian: $-(\nabla - \mathrm{i}\mathbf{A})^2$

stemming from classical / quantum mechanics:

- external electric field: $\mathbf{E} = -\nabla V$,
- external magnetic field: $\mathbf{B} = \nabla \times \mathbf{A}$,
- single particle Hamiltonian: $H = \dfrac{1}{2m}\Big(\mathbf{p} - \dfrac{e\mathbf{A}}{c}\Big)^2 + eV$
  $(\mathbf{p} = -\mathrm{i}\hbar\nabla_x)$

magnetic Sobolev space $H^1_{\mathbf{A}}(\mathbb{R}^d) := \Big\{ f \in L^2(\mathbb{R}^d) \,\Big|\, (\nabla - \mathrm{i}\mathbf{A})f \in L^2(\mathbb{R}^d) \Big\}$
(for $\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$) and $C_c^\infty(\mathbb{R}^d)$ is <u>dense</u> in $H^1_{\mathbf{A}}(\mathbb{R}^d)$

diamagnetism: $\Big|\nabla|f|(x)\Big| \leqslant \Big|(\nabla - \mathrm{i}\mathbf{A})f(x)\Big|$ x-a.e. $\forall f \in H^1_{\mathbf{A}}(\mathbb{R}^d)$
removing $\mathbf{A}$, kinetic energy decreases by replacing f by $|f|$
(at the same time leaving $|f|^2$ unaltered) [Kato, 1972]

observe:

- $f \in H^1_{\mathbf{A}} \Rightarrow (\nabla - \mathrm{i}\mathbf{A})f \in L^2$
  $\Rightarrow -(\nabla - \mathrm{i}\mathbf{A})^2 f$ makes sense as a *distribution*
- $f \mapsto (\nabla - \mathrm{i}\mathbf{A})f$ is a connection on a $U(1)$ bundle over $\mathbb{R}^d$
  $(\nabla - \mathrm{i}\mathbf{A})f =$ 'covariant derivative' (w.r.t. $\mathbf{A}$) of f

notice two opposite rules of thumb:

➠ expand the square
$(\nabla - \mathrm{i}\mathbf{A})^2 f = \Delta f - 2\,\mathrm{i}\,\mathbf{A}\cdot\nabla f - \mathrm{i}(\mathrm{div}\mathbf{A})f - \mathbf{A}^2 f$
and treat it as a 'perturbation' of Δf

➠ NEVER expand the square
exploit operator properties of $(\nabla - \mathrm{i}\mathbf{A})^2$ as a whole

gauge invariance: $\mathbf{A} \mapsto \mathbf{A} + \nabla\xi \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$ stays invariant

Coulomb gauge: $\mathrm{div}\mathbf{A} \equiv 0$
includes the constant-$\mathbf{B}$ case: $\mathbf{A}(x) = (-x_2, 0, 0) \Rightarrow \mathbf{B} = (0, 0, 1)$

For ordinary NLS $(\mathbf{A} \equiv \mathbf{0})$ as well as for magnetic NLS $(\mathbf{A} \not\equiv \mathbf{0})$

$\downarrow \qquad \downarrow \qquad\qquad\qquad \downarrow \qquad \downarrow$

one standard scheme [Kato 1987, 1995; Cazenave and Weissler 1988, ...] to establish the existence and uniqueness of solution in an appropriate sense, e.g., in the H^1- or $H^1_{\mathbf{A}}$-energy space, and then also local well-posedness, is a fixed-point argument based on estimates on space-time size of the free (magnetic) propagator $e^{\mathrm{i}t\Delta}f$, or also $e^{\mathrm{i}t(\nabla-\mathrm{i}\mathbf{A})^2}f$ ('Strichartz estimates').

NEXT:

❐ first a quick glance at the method (for pedagogical purposes);

❐ goal of this talk: to go **beyond** Strichartz-controllable external magnetic fields.

STRICHARTZ ESTIMATES for free Schrödinger propagator $e^{\mathrm{i}t\Delta}$ [Strichartz 1977, Ginibre and Velo 1985, Yajima 1987, Cazenave and Weissler 1988, Keel and Tao 1998]

A pair $(q,r) \in [1,+\infty] \times [1,+\infty]$ is admissible if:

$$\frac{2}{q} = d\Big(\frac{1}{2} - \frac{1}{r}\Big) \quad \text{and} \quad r \in \begin{cases} [2,+\infty] & \text{if } d = 1\,, \\ [2,+\infty) & \text{if } d = 2\,, \\ \left[2, \dfrac{2d}{d-1}\right] & \text{if } d \geqslant 3\,. \end{cases}$$

For any admissible pairs (q,r), $(\widetilde{q},\widetilde{r})$ and interval I with $0 \in \overline{I}$,

- (homogeneous) $\quad \left\| e^{\mathrm{i}t\Delta} u_0 \right\|_{L^q_t(\mathbb{R},L^r_x)} \lesssim \|u_0\|_{L^2_x}$
- (non-homogeneous) $\quad \left\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta} F(s,\cdot)\,\mathrm{d}\tau \right\|_{L^q_t(I,L^r_x)} \lesssim \|F\|_{L^{\widetilde{q}'}_t(I,L^{\widetilde{r}'}_x)}$

where $\frac{1}{\widetilde{q}} + \frac{1}{\widetilde{q}'} = 1 = \frac{1}{\widetilde{r}} + \frac{1}{\widetilde{r}'}$.

FIXED POINT ARGUMENT

for concreteness for $(\bigstar)\begin{cases} \mathrm{i}\partial_t u = -\Delta_x u + \lambda |u|^{\gamma-1} u\,, & (t,x) \in \mathbb{R}\times\mathbb{R}^3 \\ u(0,\cdot) = u_0 \in H^1(\mathbb{R}^3) & \gamma \in (1,5) \end{cases}$

For $M, T > 0$, $r := \gamma + 1$, and (q,r) admissible, set

$$X_{M,T} := \left\{ \begin{array}{c} u \in L^\infty_t([0,T], H^1_x) \cap L^q([0,T], W^{1,r}_x) \\ \|u\|_{L^\infty_t H^1_x} \leqslant M\,, \quad \|u\|_{L^q_t W^{1,r}_x} \leqslant M \end{array} \right\}$$

$$\mathrm{d}(u,v) := \|u-v\|_{L^\infty_t H^1_x} + \|u-v\|_{L^q_t W^{1,r}_x}$$

Can see: $(X_{M,T}, \mathrm{d})$ is a complete metric space.

A H^1-solution u to $(\bigstar)$ is a fixed point $(\Phi(u) = u)$ for the solution map

$$\Phi(u)(t) := e^{\mathrm{i}t\Delta} u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-\tau)\Delta} |u|^{\gamma-1} u(\tau)\,\mathrm{d}\tau\,.$$

Goal: Φ is a contraction in $X_{M,T}$ for suitable M, and T small enough
$\Rightarrow$ **local existence and uniqueness** of solution $u \in L^\infty_t([0,T], H^1_x(\mathbb{R}^3))$
which is then proved to also be in $C([0,T], H^1_x) \cap C^1([0,T], H^{-1}_x)$
('strong solution'), as well as in $L^{\widetilde{q}}([0,T], W^{1,\widetilde{r}}_x(\mathbb{R}^3))$ $\forall$ admissible $(\widetilde{q}, \widetilde{r})$

Φ is a <u>contraction</u> in $X_{M,T}$:

Strichartz on $\quad \Phi(u)(t) := e^{\mathrm{i}t\Delta}u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-\tau)\Delta}|u|^{\gamma-1}u(\tau)\,\mathrm{d}\tau$

$$\begin{aligned}\|\Phi(u)\|_{L_t^q W_x^{1,r}} &\lesssim \|e^{\mathrm{i}t\Delta}u_0\|_{L_t^q W_x^{1,r}} + \Big\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta}|u|^{\gamma-1}u(\tau)\,\mathrm{d}\tau \Big\|_{L_t^q W_x^{1,r}} \\ &\lesssim \|u_0\|_{H_x^1} + \|\,|u|^{\gamma-1}u\,\|_{L_t^{q'} W_x^{1,r'}}\end{aligned}$$

since $r = \gamma + 1$, then $\|\,|u|^{\gamma-1}u\,\|_{L_x^{r'}} \lesssim \|u\|_{L_x^r}^{\gamma}$

Hölder $\;\Rightarrow\;$ $\|\,|u|^{\gamma-1}u\,\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty L_x^r}^{\gamma-1} \|u\|_{L_t^q L_x^r}$

Sobolev embedding $(H_x^1 \hookrightarrow L_x^r)$ $\;\Rightarrow\;$ $\|\,|u|^{\gamma-1}u\,\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{\gamma-1} \|u\|_{L_t^q L_x^r}$

analogously for $\nabla\Big(|u|^{\gamma-1}u\Big)$, whence $\|\,|u|^{\gamma-1}u\,\|_{L_t^q W_x^{1,r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{\gamma-1} \|u\|_{L_t^q W_x^{1,r}}$

this and Hölder in time $\quad\Rightarrow$

$$\|\,|u|^{\gamma-1}u\,\|_{L_t^{q'} W_x^{1,r'}} \lesssim T^{\frac{q-q'}{qq'}} \|\,|u|^{\gamma-1}u\,\|_{L_t^q W_x^{1,r'}} \lesssim T^{\frac{q-q'}{qq'}} \|u\|_{L_t^\infty H_x^1}^{\gamma-1} \|u\|_{L_t^q W_x^{1,r}}$$

Φ is a <u>contraction</u> in $X_{M,T}$:

Strichartz on $\quad \Phi(u)(t) := e^{\mathrm{i}t\Delta}u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-\tau)\Delta}|u|^{\gamma-1}u(\tau)\,\mathrm{d}\tau$

$$\begin{aligned}\|\Phi(u)\|_{L^q_t W^{1,r}_x} &\lesssim \|e^{\mathrm{i}t\Delta}u_0\|_{L^q_t W^{1,r}_x} + \Big\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta}|u|^{\gamma-1}u(\tau)\,\mathrm{d}\tau \Big\|_{L^q_t W^{1,r}_x} \\ &\lesssim \|u_0\|_{H^1_x} + \boxed{\|\,|u|^{\gamma-1}u\,\|_{L^{q'}_t W^{1,r'}_x}}\end{aligned}$$

since $r = \gamma + 1$, then $\|\,|u|^{\gamma-1}u\,\|_{L^{r'}_x} \lesssim \|u\|^{\gamma}_{L^r_x}$

Hölder $\Rightarrow$ $\|\,|u|^{\gamma-1}u\,\|_{L^q_t L^{r'}_x} \lesssim \|u\|^{\gamma-1}_{L^\infty_t L^r_x} \|u\|_{L^q_t L^r_x}$

Sobolev embedding $(H^1_x \hookrightarrow L^r_x)$ $\Rightarrow$ $\|\,|u|^{\gamma-1}u\,\|_{L^q_t L^{r'}_x} \lesssim \|u\|^{\gamma-1}_{L^\infty_t H^1_x} \|u\|_{L^q_t L^r_x}$

analogously for $\nabla\big(|u|^{\gamma-1}u\big)$, whence $\|\,|u|^{\gamma-1}u\,\|_{L^q_t W^{1,r'}_x} \lesssim \|u\|^{\gamma-1}_{L^\infty_t H^1_x} \|u\|_{L^q_t W^{1,r}_x}$

this and Hölder in time $\Rightarrow$

$$\boxed{\|\,|u|^{\gamma-1}u\,\|_{L^{q'}_t W^{1,r'}_x}} \lesssim T^{\frac{q-q'}{qq'}} \|\,|u|^{\gamma-1}u\,\|_{L^q_t W^{1,r'}_x} \lesssim T^{\frac{q-q'}{qq'}} \|u\|^{\gamma-1}_{L^\infty_t H^1_x} \|u\|_{L^q_t W^{1,r}_x}$$

Thus, $\|\Phi(u)\|_{L^q_t W^{1,r}_x} \lesssim \|u_0\|_{H^1_x} + T^{\frac{q-q'}{qq'}} \|u\|^{\gamma-1}_{L^\infty_t H^1_x} \|u\|_{L^q_t W^{1,r}_x}$

and analogously, $\|\Phi(u)\|_{L^\infty_t H^1_x} \lesssim \|u_0\|_{H^1_x} + T^{\frac{q-q'}{qq'}} \|u\|^{\gamma-1}_{L^\infty_t H^1_x} \|u\|_{L^q_t W^{1,r}_x}$

whence

$$\begin{aligned} \|\Phi(u)\|_{L^q_t W^{1,r}_x} + \|\Phi(u)\|_{L^\infty_t H^1_x} &\leqslant C\,\|u_0\|_{H^1_x} + C\,T^{\frac{q-q'}{qq'}}\, M^{\gamma-1}\,\|u\|_{L^q_t W^{1,r}_x} \\ &\leqslant \tfrac{1}{2}M + \tfrac{1}{4}M < M \end{aligned}$$

with the choice $M = 2\,C\,\|u_0\|_{H^1_x}$ and T s.t. $C\,T^{\frac{q-q'}{qq'}}\, M^{\gamma-1} \leqslant \frac{1}{4}$

(doable, because $\gamma \in (1,5) \Rightarrow r \in (2,6) \Rightarrow q > 2 \Rightarrow q > q'$)

$\Rightarrow$ $\Phi(u) \in X_{M,T}$

and the very same reasoning yields also

$\mathrm{d}(\Phi(u),\Phi(v)) \leqslant \kappa\,\mathrm{d}(u,v)$, $\quad \kappa < 1$

a contraction.

The above scheme for $e^{\mathrm{i}t\Delta}$ in $\mathrm{i}\partial_t u = -\Delta_x u + \mathcal{N}(u)$ is well-established and also for $e^{-\mathrm{i}t(-\Delta+V)}$ in $\mathrm{i}\partial_t u = -\Delta_x u + Vu + \mathcal{N}(u)$, see e.g.:

- Sulem, Sulem, *Nonlinear Schrödinger equations*, Springer 1999
- Cazenave, *Semilinear Schrödinger equations*, AMS 2003
- Tao, *Nonlinear dispersive equations*, CBMS 2006
- Lineares, Ponce, *Introduction to nonlinear dispersive equations*, Springer 2015
- ....

Meanwhile, aiming at repeating the same scheme for

$$\mathrm{i}\partial_t u = -(\nabla_x - \mathrm{i}\mathbf{A})^2 u + \mathcal{N}(u),$$

an industry has developed to produce magnetic Strichartz estimates for the magnetic propagator $e^{\mathrm{i}t(\nabla_x - \mathrm{i}\mathbf{A})^2}$

MAGNETIC STRICHARTZ ESTIMATES

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Analogous bounds on space-time size of " $e^{\mathrm{i}t(\nabla_x - \mathrm{i}\mathbf{A})^2} f$ "

two types of conditions needed on $\mathbf{A}$:

① must realise $-(\nabla_x - \mathrm{i}\mathbf{A})^2$ self-adjointly on $L^2(\mathbb{R}^d_x)$
so as to exponentiate it via functional calculus (spectral theorem)
($e^{\mathrm{i}t\Delta}$ is directly given by $e^{-\mathrm{i}tp^2}$ in Fourier transform)

$\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^d)$ $\Rightarrow$ $-(\nabla_x - \mathrm{i}\mathbf{A})^2$ is essentially self-adjoint on $C^\infty_c(\mathbb{R}^d)$
[Kato 1981, Leinfelder and Simader 1981]

② must avoid confinement features of $-(\nabla_x - \mathrm{i}\mathbf{A})^2$
(e.g., eigenvalues/resonances) which would allow, as $t \to \infty$,
non-dispersive components in $e^{\mathrm{i}t(\nabla_x - \mathrm{i}\mathbf{A})^2} f$
(typically: impose some decay on $\mathbf{A}$ and $\mathbf{B}_{\text{tang}} = \frac{x}{|x|} \times \mathbf{B}$
and bounds on local singularities)

(GLOBAL IN TIME) MAGNETIC STRICHARTZ ESTIMATES

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$$\left\| e^{\mathrm{i}t(\nabla-\mathrm{i}\mathbf{A})^2} u_0 \right\|_{L^q_t(\mathbb{R},L^r_x)} \lesssim \|u_0\|_{L^2_x} \qquad \text{(+ non-homogeneous)}$$

for the same admissible pairs (q,r) as for $-\Delta$
(established indeed perturbatively w.r.t. $-\Delta$)

➠ By [Erdoğan, Goldberg, Schlag, 2009] for $d \geqslant 3$, requiring

$$\begin{aligned} \mathbf{A} &\in C^0(\mathbb{R}^d,\mathbb{R}^d)\,, \\ |\mathbf{A}(x)| &\lesssim \langle x\rangle^{-(1+\delta_A)}\,, \qquad (\delta_A > \delta_{A'} > 0) \\ \langle x\rangle^{1+\delta'_A}|\mathbf{A}(x)| &\in \dot{W}^{\frac{1}{2},2d}(\mathbb{R}^d,\mathbb{R}^d)\,, \\ -(\nabla-\mathrm{i}\mathbf{A})^2 &\text{ has no zero-energy resonance and only cont. spectrum} \end{aligned}$$

➠ By [D'Ancona, Fanelli, Vega, Visciglia, 2010] for $d \geqslant 3$
(covering also end-point case $(q,r) = (2,\frac{2d}{d-2})$ for $d \geqslant 4$)
under condition that, practically speaking, correspond to

$$|\mathbf{A}(x)| \lesssim \begin{cases} |x|^{-(1+\delta_A)} \\ |x|^{-(1-\delta_A)} \end{cases} \qquad |\mathbf{B}_{\mathrm{tang}}(x)| \lesssim \begin{cases} |x|^{-(2+\delta_B)} & \text{as } |x| \to \infty \\ |x|^{-(2-\delta_B)} & \text{as } |x| \to 0 \end{cases}$$

for some $\delta_A, \delta_B > 0$ when $d = 3$, $\delta_A = \delta_B = 0$ when $d \geqslant 4$

(GLOBAL IN TIME) MAGNETIC STRICHARTZ ESTIMATES for (mildly) time-dependent $\mathbf{A}$'s, under smallness assumption

➠ By [Georgiev, Stefanov, Tarulli, 2007]:

For $d \geqslant 3$, there is $\varepsilon > 0$ so that for any $\mathbf{A} \equiv \mathbf{A}(t,x)$ with

$$\left\|\nabla_x \mathbf{A}\right\|_{L_t^\infty L_x^{d/2}} + \sup_k \sum_{m\in\mathbb{Z}} 2^m \left\|\mathbf{A}_{<k}\right\|_{L_t^\infty(L^\infty(|x|\sim 2^m))} \leqslant \varepsilon$$

the solution $\boldsymbol{u}$ to $\begin{cases} \mathrm{i}\partial_t u = -(\nabla_x - \mathrm{i}\mathbf{A})^2 u \\ u(0,\cdot) \equiv u_0 \end{cases}$ satisfies

$$\|u\|_{X'} \lesssim \|u_0\|_{L_x^2}$$

where $\|u\|_{X'}^2 := \sum_k \|u\|_{X_k'}^2$ and

$$\|u\|_{X_k'} := \sup_{(q,r)-\text{Strichartz}} \|P_k u\|_{L_t^q L_x^r} + 2^{\frac{k}{2}} \sup_m 2^{-\frac{m}{2}} \|P_k u\|_{L_t^2(L^2(|x|\sim 2^m))}$$

($P_k u :=$ the k^{th} Littlewood-Paley piece of u)

scaling critical case $\mathbf{A}(x) = |x|^{-1}$ *not* covered by the above results

same as criticality for $-\Delta + a|x|^{-2}$ ($\mathbf{A}^2 \leftrightarrow V(x) = |x|^{-2}$), for which Strichartz estimates are proved by [Burq, Planchon, and Stalker, 2003] when $d \geqslant 2$ up to the Hardy threshold $a > -\frac{1}{4}(d-2)^2$

at scaling criticality, magnetic dispersive estimates are available (from which non-endpoint Strichartz estimates follow):

$$\left\| e^{\mathrm{i}t(\nabla - \mathrm{i}\mathbf{A})^2} f \right\|_{L^\infty(\mathbb{R}^d_x)} \leqslant |t|^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d_x)} \qquad d = 2, 3$$

[Fanelli, Felli, Pontelos, Primo, 2013 and 2015]

beyond critical scaling: **counterexamples!**

e.g., for homogeneous potentials $\mathbf{A}(x) = |x|^{-\sigma}\phi(\frac{x}{|x|})$, $\sigma \in (0,1)$ Strichartz estimates fail in $d \geqslant 3$ (apart from trivial case $(q,r) = (\infty, 2)$)

(LOCAL IN TIME) MAGNETIC STRICHARTZ ESTIMATES
(in fact, all what is needed for the contraction argument):

$$\Big\| e^{\mathrm{i}t(\nabla-\mathrm{i}\mathbf{A})^2} u_0 \Big\|_{L_t^q(I,L_x^r)} \lesssim \|u_0\|_{L_x^2} \qquad \text{(for any interval } I \text{ with } 0 \in \overline{I}\,\text{)}$$

now the factors preventing *global* dispersion play no obstruction

at the price of requiring smoothness of $\mathbf{A}$ (needed for constructing the propagator $e^{\mathrm{i}t(\nabla-\mathrm{i}\mathbf{A})^2}$ directly in the form of integral operator via semi-classical parametrix techniques), linear growth of $\mathbf{A}$ at spatial infinity can be covered (thus, including constant $\mathbf{B}$-field) [Yajima, 1991], [Mizutani, 2014]

What about the *vast* regime in between, from $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^d)$, the condition of self-adjointness for $-(\nabla - \mathrm{i}\mathbf{A})^2$, to the requirements for the validity of the known (global-in-time, or also only) local-in-time Strichartz estimates?

This includes physically relevant magnetic fields!

How to establish (local) well-posedness of magnetic NLS **beyond Strichartz-controllable magnetic potentials** ?

One approach: ENERGY METHODS

Yield local well-posedness under very mild assumptions on $\mathbf{A}$ when non-linearity $\mathcal{N}(u)$ is locally Lipschitz in the energy space (then by conservation rules one extends local to global in time).

Theorem. [Michelangeli, 2015]

If $\begin{cases} \mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d) \\ W \in L^{q_0}(\mathbb{R}^d, \mathbb{R}) + L^\infty(\mathbb{R}^d, \mathbb{R})\,, \quad q_0 \geqslant \frac{d}{2}\,, \quad \text{and } W \text{ even} \\ \nabla W \in L^{q_1}(\mathbb{R}^d, \mathbb{R}^d) + L^\infty(\mathbb{R}^d, \mathbb{R}^d)\,, \quad q_1 \geqslant \frac{d}{3} \end{cases}$

then there exists a unique solution $u \in C(\mathbb{R}, H^1_{\mathbf{A}}(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^1_{\mathbf{A}}(\mathbb{R}^d)^*)$

to the Hartree equation $\begin{cases} \mathrm{i}\partial_t u \,=\, -(\nabla - \mathrm{i}\mathbf{A})^2 u + (W * |u|^2)\, u \\ u(0,\cdot) \equiv u_0 \in H^1_{\mathbf{A}}(\mathbb{R}^d) \end{cases}$

and such u also satisfies

$$\sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{H^1_{\mathbf{A}}} < +\infty\,, \qquad \mathcal{M}[u(t)] = \mathcal{M}[u_0]\,, \qquad \mathcal{E}[u(t)] = \mathcal{E}[u_0]$$

where $\begin{cases} \mathcal{M}[u] \,:=\, \|u\|^2_{L^2_x} \;\text{(mass)}\,, \\ \mathcal{E}[u] \,:=\, \displaystyle\int_{\mathbb{R}^d} \Big(\frac{1}{2}\,|(\nabla - \mathrm{i}\mathbf{A})u|^2 + \frac{1}{4}\,(W * |u|^2)|u|^2 \Big) \;\text{(energy)} \end{cases}$

as well as continuous dependence on initial data.

Based on the LIPSCHITZ PROPERTY of $\mathcal{N}(u)=(W*|u|^2)\,u$:
$\|\mathcal{N}(u)-\mathcal{N}(v)\|_{H^1_{\mathbf{A}}} \lesssim_W \Big(\|u\|^2_{H^1_{\mathbf{A}}}+\|v\|^2_{H^1_{\mathbf{A}}}\Big)\|u-v\|_{H^1_{\mathbf{A}}}$

a combination of Hölder + Young + Sobolev embedding $H^1 \hookrightarrow L^r$ and diamagnetic inequality $\Big|\nabla|u|\Big| \leqslant |(\nabla-\mathrm{i}\mathbf{A})u|$.

Then FIXED POINT ARGUMENT in M-ball of $L^\infty_T H^1_{\mathbf{A}} \equiv L^\infty([0,T],H^1_{\mathbf{A}})$ on the solution map

$$\Phi(u)(t) := e^{\mathrm{i}t(\nabla-\mathrm{i}\mathbf{A})^2}u_0 - \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)(\nabla-\mathrm{i}\mathbf{A})^2}(W*|u|^2)\,u(\tau)\,\mathrm{d}\tau$$

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{L^\infty_T H^1_{\mathbf{A}}} &\leqslant \left\|\int_0^t \Big\| e^{\mathrm{i}(t-\tau)(\nabla-\mathrm{i}\mathbf{A})^2}\Big(\mathcal{N}(u(\tau))-\mathcal{N}(v(\tau))\Big)\Big\|_{H^1_{\mathbf{A}}}\mathrm{d}\tau\right\|_{L^\infty_T} \\
&\leqslant C_W T\Big(\|u\|^2_{L^\infty_T H^1_{\mathbf{A}}}+\|v\|^2_{H^1_{\mathbf{A}}}\Big)\|u-v\|_{L^\infty_T H^1_{\mathbf{A}}} \\
&\leqslant 2M^2 C_W\,T\,\|u-v\|_{L^\infty_T H^1_{\mathbf{A}}} \\
&= \tfrac{1}{2}\|u-v\|_{L^\infty_T H^1_{\mathbf{A}}}
\end{aligned}$$

choosing $T=(4M^2C_W)^{-1}$.

Same approach works more generally for

$$\mathrm{i}\partial_t u = -(\nabla - \mathrm{i}\mathbf{A})^2 u + (W * |u|^2)\, u + V u$$

.

under the (beyond-Strichartz) assumptions

$$\begin{cases} \mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d) \\ W \in L^{q_0}(\mathbb{R}^d, \mathbb{R}) + L^\infty(\mathbb{R}^d, \mathbb{R})\,, \quad q_0 \geqslant \frac{d}{2}\,, \quad \text{and } W \text{ even} \\ \nabla W \in L^{q_1}(\mathbb{R}^d, \mathbb{R}^d) + L^\infty(\mathbb{R}^d, \mathbb{R}^d)\,, \quad q_1 \geqslant \frac{d}{3} \\ V \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}) \\ V_- \text{ is } \Delta\text{-form-bounded with relative bound} < 1 \end{cases}$$

which are way weaker than the requirements for known Strichartz estimates for $e^{-\mathrm{i}t(-(\nabla - \mathrm{i}\mathbf{A})^2 + V)}$

now in the **energy space**

$$H^1_{\mathbf{A},V} := \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, (\nabla - \mathrm{i}\mathbf{A}) f \in L^2(\mathbb{R}^d)\,, \, V_+^{1/2} f \in L^2(\mathbb{R}^d) \right\}$$

which can be proved to be the form domain of the closed and lower semi-bounded quadratic form

$$(f, g) \mapsto \int_{\mathbb{R}^d} \Big(\overline{(\nabla - \mathrm{i}\mathbf{A}) f} \cdot (\nabla - \mathrm{i}\mathbf{A}) g + \overline{f}\, V\, g \Big)$$

thus inducing a self-adjoint realisation of $-(\nabla - \mathrm{i}\mathbf{A})^2 + V$.

[Michelangeli, 2015]

Energy methods suited, among others, for (magnetic-)NLS where the linear part is induced by a Schrödinger-type operator $\mathfrak{h}$ of physical relevance, e.g.,

$$\mathfrak{h} = \sqrt{-(\nabla - \mathrm{i}\mathbf{A})^2 + \mu^2} - \mu + V \qquad (\mu \geqslant 0)$$

(semi-relativistic Schrödinger Hamiltonian with external fields), or

$$\mathfrak{h} = \text{Dirac operator with external fields}\,,$$

etc.

In such settings, crucial to characterise $\mathfrak{h}$ self-adjointly on $L^2(\mathbb{R}^d)$ e.g., identifying its form domain ($\longrightarrow$ the energy space)

and to check the Lipschitz property for $\mathcal{N}(u)$ (energy sub-critical):

interplay operator theory + functional analysis.

A second approach to go beyond magnetic Strichartz,
also when $\mathcal{N}(u)$ is non-Lipschitz in energy space:
PARABOLIC ('VISCOSITY') REGULARISATION

$$e^{\mathrm{i}t(\nabla-\mathrm{i}\mathbf{A})^2} \longmapsto e^{(\mathrm{i}+\varepsilon)t(\nabla-\mathrm{i}\mathbf{A})^2} \qquad (\varepsilon>0)$$

Requires:

- ❒ suitable smoothing estimates for the dissipative evolution,
- ❒ a priori estimates for mass and energy, uniform in the regularisation,
- ❒ compactness argument to remove the regularisation locally in time (then gluing to go global).

Pros:

- ❒ can accommodate also external fields $\mathbf{A}\equiv\mathbf{A}(t,x)$, $V\equiv V(t,x)$ that are moderately changing in time around a suitable profile,
- ❒ highly non-Strichartz-controllable $\mathbf{A}$'s and V's.

However:

- ❒ compactness argument loses information on strong solutions: yields existence of global finite-energy weak solutions.

Parabolic regularisation procedures of sort are commonly used in PDEs:

- ❒ vanishing viscosity approximation in fluid dynamics,
- ❒ or in systems of conservation laws,
- ❒ exploited in a similar context by [Guo, Nakamitsu, Strauss, 1994] to demonstrate the existence of finite energy weak solutions to the Maxwell-Schrödinger system

$$\begin{cases} \mathrm{i}\partial_t u = -(\nabla - \mathrm{i}\mathbf{A})^2 u + \phi u + |u|^{\gamma-1} u\,, \quad \phi \equiv (-\Delta)^{-1}|u|^2 \\ \Box u = 2\,(\mathbf{1} - \nabla \mathrm{div} \boldsymbol{\Delta}^{-1})\,\mathfrak{Im}(\overline{u}(\nabla - \mathrm{i}\mathbf{A})u) \end{cases}$$

(charged quantum plasma interacting with its self-generated electromagnetic potential $(V, \mathbf{A})$).

A fairly general model: defocusing magnetic NLS ($d = 3$)

$$
\begin{gathered}
\mathrm{i}\partial_t u = -(\nabla_x - \mathrm{i}\mathbf{A})^2 u + \mathcal{N}(u) \\
\mathcal{N}(u) = \lambda_1 |u|^{\gamma-1} u + \lambda_2 (|\cdot|^{-\alpha} * |u|^2) u \\
\begin{cases} \gamma \in (1,5] \\ \alpha \in (0,3) \\ \lambda_1, \lambda_2 \geqslant 0 \end{cases}
\end{gathered}
\qquad (\clubsuit)
$$

in the unknown $u \equiv u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$
for given real-valued measurable $\mathbf{A} \equiv \mathbf{A}(t,x)$

energy sub-critical:	$\gamma \in (1,5)$,	$\alpha \in (0,3)$
energy critical:	$\gamma = 5$,	$\alpha \in (0,3)$
mass sub-critical:	$\gamma = (1, \frac{7}{3})$,	$\alpha \in (0,2)$

And fairly general classes of magnetic potentials $\mathbf{A}$'s:

$$\widetilde{\mathcal{A}}_1 := \left\{ \mathbf{A} \equiv \mathbf{A}(t,x) \,\middle|\, \begin{array}{c} \mathrm{div}_x \mathbf{A} = 0 \text{ for a.e. } t \in \mathbb{R}, \\ \mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 \text{ such that, for } j \in \{1,2\}, \\ \mathbf{A}_j \in L^{a_j}_{\mathrm{loc}}(\mathbb{R}, L^{b_j}(\mathbb{R}^3, \mathbb{R}^3)) \\ a_j \in (4, +\infty], \quad b_j \in (3,6), \quad \frac{2}{a_j} + \frac{3}{b_j} < 1 \end{array} \right\}$$

$$\widetilde{\mathcal{A}}_2 := \left\{ \mathbf{A} \equiv \mathbf{A}(t,x) \,\middle|\, \begin{array}{c} \mathrm{div}_x \mathbf{A} = 0 \text{ for a.e. } t \in \mathbb{R}, \\ \mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 \text{ such that, for } j \in \{1,2\}, \\ \mathbf{A}_j \in L^{a_j}_{\mathrm{loc}}(\mathbb{R}, W^{1,\frac{3b_j}{3+b_j}}(\mathbb{R}^3, \mathbb{R}^3)) \\ a_j \in (2, +\infty], \quad b_j \in (3,+\infty], \quad \frac{2}{a_j} + \frac{3}{b_j} < 1 \end{array} \right\}$$

$$\mathcal{A}_1 := \left\{ \mathbf{A} \in \widetilde{\mathcal{A}}_1 \mid \partial_t \mathbf{A}_j \in L^1_{\mathrm{loc}}(\mathbb{R}, L^{b_j}(\mathbb{R}^3, \mathbb{R}^3)),\ j = 1,2 \right\}$$

$$\mathcal{A}_2 := \left\{ \mathbf{A} \in \widetilde{\mathcal{A}}_2 \mid \partial_t \mathbf{A}_j \in L^1_{\mathrm{loc}}(\mathbb{R}, L^{b_j}(\mathbb{R}^3, \mathbb{R}^3)),\ j = 1,2 \right\}$$

- $\mathrm{div}_x\mathbf{A} \equiv 0$ (Coulomb gauge) assumed merely for convenience

- both classes $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ contain magnetic potentials $\mathbf{A}$'s for which magnetic Strichartz estimates are not known

- local theory in energy space possible in the larger classes $\widetilde{\mathcal{A}}_1$ or $\widetilde{\mathcal{A}}_2$

- mild amount of extra regularity in time (classes $\mathcal{A}_1$ or $\mathcal{A}_2$) only needed to infer suitable a priori bounds on the solution from estimates on the total energy ($\longrightarrow$ to go global in time)

- regularity in time not needed either for mass sub-critical regime $\gamma = (1, \frac{7}{3})$ $\alpha \in (0,2)$, and when $\max\{b_1, b_2\} \in (3,6)$ [Yajima 1987]

- additional integrability of $\nabla_x \mathbf{A}$ in $\mathcal{A}_2$ needed to cover slow decay of $\mathbf{A}$ at spatial infinity, way slower than the critical $|x|^{-1}$ (even L^∞_x)

EXISTENCE OF GLOBAL, FINITE ENERGY, WEAK SOLUTIONS

Theorem. [Antonelli, Michelangeli, Scandone, 2018]
If $\mathbf{A} \in \mathcal{A}_1$ or $\mathbf{A} \in \mathcal{A}_2$, then for any initial datum $u_0 \in H^1(\mathbb{R}^3)$ the initial value problem for (♣) , i.e.,

$$\begin{cases} \mathrm{i}\partial_t u = -(\nabla_x - \mathrm{i}\mathbf{A})^2 u + |u|^{\gamma-1} u + (|\cdot|^{-\alpha} * |u|^2) u \,, \\ u(0,\cdot) \equiv u_0 \,, \qquad (\gamma \in (1,5] \,,\ \alpha \in (0,3)) \end{cases} \tag{♣}$$

admits a global weak H^1-solution

$$u \in L^\infty_{\mathrm{loc}}([0,+\infty), H^1(\mathbb{R}^3)) \cap W^{1,\infty}_{\mathrm{loc}}([0,+\infty), H^{-1}(\mathbb{R}^3)) \,,$$

and moreover the energy

$$\mathcal{E}[u(t)] := \int_{\mathbb{R}^3} \Big(\tfrac{1}{2} |(\nabla - \mathrm{i}\mathbf{A}(t))\, u|^2 + \tfrac{1}{\gamma+1} |u|^{\gamma+1} + \tfrac{1}{4} (|x|^{-\alpha} * |u|^2) |u|^2 \Big)\, \mathrm{d}x$$

is finite and bounded on compact time intervals.

Main ideas of the technique.

① Introduce a small dissipation term in the equation

$$\mathrm{i}\partial_t u_{\boldsymbol{\varepsilon}} = -(1-\mathrm{i}\,\boldsymbol{\varepsilon})(\nabla - \mathrm{i}\,\mathbf{A})^2 u_{\boldsymbol{\varepsilon}} + \mathcal{N}(u_{\boldsymbol{\varepsilon}}) \qquad (\boldsymbol{\varepsilon} > 0)$$

and treat the approximated problem 'perturbatively'

$$\mathrm{i}\partial_t u_{\boldsymbol{\varepsilon}} = -(1-\mathrm{i}\,\boldsymbol{\varepsilon})\Delta_x u_{\boldsymbol{\varepsilon}} + (1-\mathrm{i}\,\boldsymbol{\varepsilon})(2\,\mathrm{i}\,\mathbf{A}\cdot\nabla u_{\boldsymbol{\varepsilon}} + \mathbf{A}^2 u) + \mathcal{N}(u_{\boldsymbol{\varepsilon}})$$

(not doable in the Hamiltonian case $\varepsilon = 0$: $\mathbf{A}\cdot\nabla$ is *not* a Kato-small perturbation of $-\Delta$).

② For $e^{(\mathrm{i}+\varepsilon)t\Delta} = e^{\varepsilon t\Delta} e^{\mathrm{i}t\Delta}$ combine space-time (Strichartz) bounds for heat and for Schrödinger propagator and obtain **spacetime estimates for the heat-Schrödinger flow**.

For $\varepsilon > 0$, $T > 0$, and admissible pair (q, r) $\left(\text{i.e., } \frac{2}{q} = \frac{3}{2} - \frac{3}{r},\ r \in [2, 6]\right)$:

<u>homogeneous Strichartz estimates</u>

$$\|e^{(\mathrm{i}+\varepsilon)t\Delta} f\|_{L^q([0,T],L^r(\mathbb{R}^3))} \lesssim \|f\|_{L^2(\mathbb{R}^3)}$$

<u>inhomogeneous retarded Strichartz estimates</u>

$$\left\| \int_0^t e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta} F(\tau)\, \mathrm{d}\tau \right\|_{L^q([0,T],L^r(\mathbb{R}^3))} \lesssim_\varepsilon T^\theta \|F\|_{L^s([0,T],L^p(\mathbb{R}^3))}$$

with $\frac{2}{s} + \frac{3}{p} < \frac{7}{2}$, $\begin{cases} \frac{1}{2} \leqslant \frac{1}{p} \leqslant 1 & 2 \leqslant r < 3 \\ \frac{1}{2} \leqslant \frac{1}{p} < \frac{1}{r} + \frac{2}{3} & 3 \leqslant r \leqslant 6 \end{cases}$ and $\theta := \frac{7}{4} - \frac{1}{s} - \frac{3}{2p} > 0$

and, if in addition $(q, r) \neq (2, 6)$ (<u>non-endpoint case</u>),

$$\left\| \nabla \int_0^t e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta} F(\tau)\, \mathrm{d}\tau \right\|_{L^q([0,T],L^r(\mathbb{R}^3))} \lesssim_\varepsilon T^\theta \|F\|_{L^s([0,T],L^p(\mathbb{R}^3))}$$

with $\frac{2}{s} + \frac{3}{p} < \frac{5}{2}$, $\frac{1}{2} \leqslant \frac{1}{p} < \frac{1}{r} + \frac{1}{3}$, and $\theta := \frac{5}{4} - \frac{1}{s} - \frac{3}{2p} > 0$

③ Exploit the above bounds to establish the existence of the linear magnetic viscous propagator for

$$\mathrm{i}\partial_t u_\varepsilon = -(1-\mathrm{i}\varepsilon)\Delta_x u_\varepsilon + (1-\mathrm{i}\varepsilon)(2\,\mathrm{i}\,\mathbf{A}\cdot\nabla u_\varepsilon + \mathbf{A}^2 u_\varepsilon) + \mathcal{N}(u_\varepsilon)$$

namely the family $\{\mathcal{U}_{\varepsilon,\mathbf{A}}(t,\tau)\}_{t,\tau}$ of operators on $H^1(\mathbb{R}^3)$ satisfying

- $\mathcal{U}_{\varepsilon,\mathbf{A}}(t,s)\,\mathcal{U}_{\varepsilon,\mathbf{A}}(s,\tau) = \mathcal{U}_{\varepsilon,\mathbf{A}}(t,\tau)$ for any $\tau < s < t$,
- $\mathcal{U}_{\varepsilon,\mathbf{A}}(t,t) = \mathbb{1}$,
- the map $(t,\tau)\mapsto\mathcal{U}_{\varepsilon,\mathbf{A}}(t,\tau)$ is strongly continuous in $H^1(\mathbb{R}^3)$,

such that the regularised IVP is equivalent to the integral problem

$$u_\varepsilon(t) = \mathcal{U}_{\varepsilon,\mathbf{A}}(t,0)u_0 - \mathrm{i}\int_0^t \mathcal{U}_{\varepsilon,\mathbf{A}}(t,\tau)\mathcal{N}(u_\varepsilon)(\tau)\,\mathrm{d}\tau$$

(for time-independent $\mathbf{A}$, $\mathcal{U}_{\varepsilon,\mathbf{A}}(t,0)$ would just be $e^{(\mathrm{i}+\varepsilon)t(\nabla-\mathrm{i}\mathbf{A})^2}$)

and derive Strichartz-type estimates

$$\big\|\mathcal{U}_{\varepsilon,A}(t,\tau)f\big\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \lesssim_{\varepsilon,\mathbf{A},T} \|f\|_{H^1(\mathbb{R}^3)}$$

(+ retarded ones).

④ Perform a standard (Strichartz-based) contraction argument for

$$u_\varepsilon(t) = \mathcal{U}_{\varepsilon,\mathbf{A}}(t,0)\,u_0 - \mathrm{i}\int_0^t \mathcal{U}_{\varepsilon,\mathbf{A}}(t,\tau)\,\mathcal{N}(u_\varepsilon)(\tau)\,\mathrm{d}\tau$$

under the assumption $\mathbf{A} \in \widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2$

$\Rightarrow$ local well-posedness of the regularised magnetic NLS
in $C([0,T_{\mathrm{max}}),H^1(\mathbb{R}^3))$

in the energy sub-critical regime $\gamma \in (1,5)$, $\alpha \in (0,3)$.

For the energy critical case $\gamma = 5$:

$F \equiv |u|^4 u$ not covered by the above heat-Schrödinger Strichartz bounds (would require $p = \frac{6}{5}$, $s = 0$ therein, $T^\theta = T^0 = 1$).

Yet, can treat it a la [Cazenave, Weissler, 1990]

using energy dissipation

$\Rightarrow$ existence and uniqueness in $C([0,T_{\mathrm{max}}),H^1(\mathbb{R}^3))$.

⑤ Establish uniform-in-ε a priori bounds for the solution to

$$u_\varepsilon(t) = \mathcal{U}_{\varepsilon,\mathbf{A}}(t,0)\,u_0 - \mathrm{i}\int_0^t \mathcal{U}_{\varepsilon,\mathbf{A}}(t,\tau)\,\mathcal{N}(u_\varepsilon)(\tau)\,\mathrm{d}\tau\,,$$

provided that $\mathbf{A} \in \mathcal{A}_1 \cup \mathcal{A}_2$ ($\leftarrow$ need a bit of time-regularity for $\mathbf{A}(t,x)$):

$$\begin{aligned} \sup_{t\in[0,T]} \mathcal{M}[u_\varepsilon] \quad &\lesssim \quad 1\,, \\ \sup_{t\in[0,T]} \mathcal{E}[u_\varepsilon] \quad &\lesssim_{A,T} \quad 1\,, \\ \|u_\varepsilon\|_{L^\infty([0,T),H^1(\mathbb{R}^3))} \quad &\lesssim_{A,T} \quad 1\,, \end{aligned}$$

constants depending on

$$\|\partial_t \mathbf{A}_j(t,\cdot)\|_{L^1([0,T],L^{b_j}(\mathbb{R}^3))}, \quad \|\mathbf{A}_j(t,\cdot)\|_{L^1([0,T],L^{b_j}(\mathbb{R}^3))}, \quad j\in\{1,2\}$$

(hence non-uniformity in T of the above bounds only due to the fact that $\mathbf{A}, \partial_t\mathbf{A} \in L^1_{\mathrm{loc}}$ in time, i.e., $\mathbf{A} \in AC_{\mathrm{loc}}$ in time).

Observe: due to the defocusing structure of the regularised problem,

$$\|u_\varepsilon(t)\|^2_{H^1_{\mathbf{A}(t)}} \leqslant \mathcal{M}[u_\varepsilon](t) + \mathcal{E}[u_\varepsilon](t)\,, \qquad t\in[0,T)\,.$$

⑥ (standard:) under the assumptions $\mathbf{A} \in \mathcal{A}_1 \cup \mathcal{A}_2$

$$\left.\begin{array}{c}\text{uniform-in-}\varepsilon\\ \text{a priori bounds}\\ \text{for the solution } u_\varepsilon\end{array}\right\} \Rightarrow \begin{array}{c}\text{GLOBAL EXISTENCE}\\ \text{AND UNIQUENESS}\\ \text{of \textit{strong} solution } u_\varepsilon\\ \text{to the regularised magnetic NLS}\end{array}$$

For the energy sub-critical exponent $\gamma \in (1,5)$, also complete GWP.

For mass sub-critical exponents $\gamma \in (1, \frac{7}{3})$, $\alpha \in (0,2)$
can actually assume just $\mathbf{A} \in \widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2$
an proceed through this simpler, alternative path:

➠ LWP in $L^2(\mathbb{R}^3)$ for ε-NLS by fixed point argument
using space-time estimates for the heat-Schrödinger flow;

➠ extend solution u_ε globally using mass (L^2) conservation;

➠ mass sub-critical nonlinearity $\Rightarrow$ convenient commutator estimate
on $[\nabla_x, (\nabla_x - \mathrm{i}\mathbf{A})^2] \quad \Rightarrow \quad$ *global* persistence of H^1_x-regularity for u_ε.

⑦ Remove the regularisation ($\varepsilon \downarrow 0$) *locally in time*

$$\text{from} \quad \mathrm{i}\partial_t u_\varepsilon = -(1-\mathrm{i}\,\boldsymbol{\varepsilon})(\nabla - \mathrm{i}\,\mathbf{A})^2 u_\varepsilon + \mathcal{N}(u_\varepsilon)\,, \qquad u_\varepsilon(t,\cdot) \equiv u_0$$
$$\text{to} \quad \mathrm{i}\partial_t u = -(\nabla - \mathrm{i}\,\mathbf{A})^2 u + \mathcal{N}(u)\,, \qquad u(t,\cdot) \equiv u_0\,, \qquad (\clubsuit)$$

via compactness argument, extracting a subsequence from $(u_n)_{n\in\mathbb{N}}$ (with $u_n \equiv u_{\varepsilon_n}$, $\varepsilon_n \equiv \frac{1}{n} \to 0$, solutions to regularised problems).

First, by uniform-in-ε a priori bounds, and up to subsequence,

$$u_n \xrightarrow{n\to\infty} u \qquad \text{weakly-}* \text{ in } L^\infty([0,T], H^1(\mathbb{R}^3))\,.$$

for some u *to be identified* as solution to ($\clubsuit$).

Next, exploit uniform-in-ε a priori bounds so as to prove, up to subseq.,

$$\begin{aligned}
\mathbf{A}_i \cdot \nabla u_n &\to X_i && \text{weakly-}* \text{ in } L^\infty([0,T], L^{p_i}(\mathbb{R}^3))\,,\\
\mathbf{A}_i \cdot \mathbf{A}_j u_n &\to Y_{ij} && \text{weakly-}* \text{ in } L^\infty([0,T], L^{p_{ij}}(\mathbb{R}^3))\,,\\
|u_n|^{\gamma-1} u_n &\to \mathcal{N}_1 && \text{weakly-}* \text{ in } L^\infty([0,T], L^{p(\gamma)}(\mathbb{R}^3))\,,\\
(|\cdot|^{-\alpha} * |u_n|^2) u_n &\to \mathcal{N}_2 && \text{weakly-}* \text{ in } L^\infty([0,T], L^{\widetilde{p}(\alpha)}(\mathbb{R}^3))
\end{aligned}$$

for $i,j \in \{1,2\}$ and suitable exponents $p_i, p_{ij}, p(\gamma), \widetilde{p}(\alpha)$.

.....and identify pointwise the above limits as the counterparts for u

.

$$\mathbf{A}_i \cdot \nabla u_n \to \mathbf{A}_i \cdot \nabla u \,,$$
$$\mathbf{A}_i \cdot \mathbf{A}_j u_n \to \mathbf{A}_i \cdot \mathbf{A}_j u \,,$$
$$|u_n|^{\gamma-1} u_n \to |u|^{\gamma-1} u \,,$$
$$(|\cdot|^{-\alpha} * |u_n|^2) u_n \to |\cdot|^{-\alpha} * |u|^2) u \,,$$

proceeding this way:

- $\nabla u_n \to \nabla u$ weakly-$*$ in $L^2_t L^2_x$ by compactness,
  $\mathbf{A}_j \eta \in L^2_t L^2_x$ for any $\eta \in L^2_t L^{p'_j}_x$, with $\frac{1}{p_j} = \frac{1}{2} + \frac{1}{b_j}$,
  because $\mathbf{A}_j \in L^\infty_t L^{b_j}_x$ and $\frac{1}{p'_j} + \frac{1}{b_j} = \frac{1}{2}$, therefore
  $$\int_0^T \int_{\mathbb{R}^3} \mathbf{A}_i \cdot (\nabla u_n - \nabla u) \overline{\eta} \, \mathrm{d}x \, \mathrm{d}t \; = \; \int_0^T \int_{\mathbb{R}^3} (\nabla u_n - \nabla u) \cdot \mathbf{A}_i \overline{\eta} \, \mathrm{d}x \, \mathrm{d}t \; \to \; 0$$
  whence $\mathbf{A}_i \cdot \nabla u_n \to \mathbf{A}_i \cdot \nabla u \qquad$ weakly-$*$ in $L^\infty([0,T], L^{p_i}(\mathbb{R}^3))$;
- Aubin-Lions compactness lemma, whence
  $u_n\big|_\Omega \to u\big|_\Omega$ strongly in $L^M([0,T], L^4(\Omega))$, $\qquad M \in [1, +\infty]$
  for every open bounded $\Omega \subset \mathbb{R}^3$.

⑧ Final refinement of $(u_n)_{n\in\mathbb{N}}$ such that its limit u is a weak H^1-solution to (♣) for all times in $[0, N]$, and iterating over N

$\Rightarrow$ global weak H^1-solution with finite energy for a.e. $t \in \mathbb{R}$.

Had we assumed $\mathbf{A} \in AC$ globally in time, this step not needed; must do it here because $\mathbf{A} \in AC_{\text{loc}}$ in time so the uniform-in-ε a priori bounds are T-dependent.