"The Hidden Math of Ocean Waves Crashes Into View": the corrected story

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Recently, the Quanta article *"The Hidden Math of Ocean Waves Crashes Into View"* [1] discussed recent progress on the linear instability of Stokes wave solutions of the water wave equations under long-wave perturbations. In particular, it presented results obtained by three of us (Massimiliano, Alberto, and Paolo) concerning the spectrum near the origin of the linearized water-wave operator at small-amplitude Stokes waves [4], as well as results obtained jointly by four of us (Massimiliano, Livia, Alberto, Paolo) on the spectrum away from the origin [7], that rigorously justify more than a decade long of numerical computations of such spectra developed and carried on by one of us (Bernard) together with several collaborators (Katie Oliveras, Olga Trichtchenko, Ryan Creedon).

We are honored that Quanta chose to highlight such a beautiful and classical problem, and we appreciate that all five of us were interviewed—a practice that, as we understand, is not common.

However, the Quanta article [1] contains several inaccuracies and does not factually represent our collaboration or our interactions with others mentioned in the piece. Quanta's reconstruction does not reflect our perspective, despite our clear articulation of it.

As explained by the Quanta editors, it is not their practice to share an article with its subjects prior to publication. Consequently, we were unable to raise our concerns before [1] appeared online. On October 31, 2025, Alberto, Paolo, and Livia wrote to the journalist to convey these concerns. After several email exchanges with him and the responsible editor, only some of the proposed changes have been implemented in the online version of the paper. The current version does not yet represent our perspective.

Therefore, in what follows, we have taken the Quanta article [1] and modified and added some phrases, that we put in *italics*. These revisions in no way diminish the efforts of Quanta—and, in this case, of the journalist, whose work we sincerely appreciate—in communicating complex mathematics to a broad audience, a mission that we view as both valuable and essential. We did not alter any of these scientific descriptions. Rather, our revisions are intended solely to ensure that the facts are presented accurately and that credit is assigned appropriately, in keeping with the foundational principles of academic integrity.

We hold Quanta in high esteem and wish for it to continue to stand as a trusted and respected source of scientific communication on which the community can confidently rely. While we fully respect and acknowledge the editorial independence of Quanta's journalists —a foundational principle of free journalism— we believe that the concerns of the scientific community, particularly when raised by the principal contributors to the work being reported, should be carefully taken into consideration. It is precisely for this reason that we must continue to insist on these points.

For our part, we will continue to collaborate on these fascinating problems with the same enthusiasm and dedication.

The Hidden Math of Ocean Waves Crashes Into View

The best perk of Alberto Maspero's job, he says, is the view from his window. Situated on a hill above the ancient port city of Trieste, Italy, his office at the International School for Advanced Studies (SISSA) overlooks a broad bay at the northern tip of the Adriatic Sea. "It's very inspiring," the mathematician said. "For sure the most beautiful view I've ever had."

Italians call Trieste la città della bora, after its famed "bora" wind, which blows erratically down off the Alps and over the city. When the bora is strong enough, it drives the waves into reverse. Instead of breaking against the docks, they stream away from the city, back toward the open sea.

But they never actually get there. Watching from his window on these gusty days, Maspero can see the retreating waves slowly disperse as they exit the port, eventually giving way to a calm, still surface.

The equations that mathematicians use to study the flow of water and other fluids — which Leonhard Euler first wrote down nearly 300 years ago — look simple enough. If you know the location and velocity of each droplet of water, and simplify the math by assuming there's no internal friction, or viscosity, then solving Euler's equations will allow you to predict how the water will evolve over any time period. The rich menagerie of phenomena we see in the world's oceans — tsunamis, whirlpools, riptides — are all solutions to Euler's equations.

But the equations are usually impossible to solve. Even one of the simplest and most common kinds of solutions — one that describes a steady train of gently rolling waves — is a mathematical nightmare to extract from Euler's equations. Until about 30 years ago, the bulk of what we knew about these waves came only from a mix of real-world observations and guesswork. For the most part, proofs seemed like a fantasy.

"Before starting math, I thought water waves were something very understood — not a problem at all," said Paolo Ventura, a postdoctoral fellow at the Swiss Federal Technology Institute of Lausanne and Maspero's former PhD student, *whose primary advisor was Massimiliano Berti, also at SISSA [8].* "But in reality, they are just strange."

One strange phenomenon that has perplexed mathematicians for decades is that, even when friction is minimal, that steady train of gently rolling waves still eventually falls apart and becomes irregular. Mathematicians hadn't expected to see such unstable behavior emerge from such a simple starting point. They wanted to prove it — to show that instabilities arise naturally from the Euler equations. But they couldn't figure out how to do it.

Now Berti, Maspero and Ventura, along with Livia Corsi of Roma Tre University, have finally presented such a proof, showing exactly when these instabilities occur and when they don't [7]. The result is just the latest in a renaissance that's starting to transform our mathematical understanding of Earth's waves. Mathematicians have been using new computational tools to formulate conjectures about how waves behave. And they're now developing sophisticated pen-and-paper techniques to prove those conjectures.

"It's not one particular thing. It's a whole wave of new types of analysis in multiple directions," said Walter Strauss, a mathematician at Brown University. "I'm very impressed."

A Slow Tide

The ancient Greeks often compared the unsteady beat of waves against the shore to laughter. Considering how those waves have eluded human understanding, perhaps they were right: The ocean has been laughing at us all along.

Even at the height of the Enlightenment in the late 17th and early 18th centuries, when waves took up much of the scientific discourse, the ocean always seemed to have the last word. A number of scientists had measured the speed of sound waves, and Newton and his detractors were locked in a conflict over the wavelike nature of light. But the oldest waves known to humans remained a mathematical enigma.

It would take more than a century for this to start to change. In the early 1800s, Sir George Stokes became fascinated with ocean waves when, as a boy, he was swimming near his home in Sligo, Ireland, and an enormous wave almost dragged him out to sea. In 1847, he published a monumental treatise on the topic. He started with Euler's equations for a fluid with no viscosity and added the mathematical condition that its top surface be totally "free" — allowed to take any shape it pleased.

"They don't look bad," Strauss said of the resulting equations. "But just take a look at a lake with a little wind on it. You get all these complicated forms, like whitecaps and rolling waves, some parallel to each other, some not."

Each of these varied forms, when understood as a solution to Euler's equations, is mathematically distinct and terribly unwieldy. Make the tiniest change to the fluid's initial state, and it might evolve in a vastly different way — bumps and eddies can become rogue waves and tsunamis.

These free, moving surfaces were what Stokes wanted to study. But the challenge was immense. Describing the motion of water confined within a box, or flowing through a pipe, is hard enough. But then, at least, you know where the system's edges lie — no water can extend beyond those boundaries. If there's no restriction other than the force of gravity on how high the water can reach and what shape it can take, the math becomes far more difficult.

"If I go to the beach at seven in the morning, it's going to be very calm," Corsi said. "But if you really look at the surface, how it moves, it's a mess."

Still, Stokes was able to conjecture one solution: that it's possible for the surface of the water to form evenly spaced waves that travel in a single direction.

In the 1920s, mathematicians proved Stokes' conjecture. Furthermore, they found that if there are no external disturbances, these solutions to the Euler equations persist forever: Once they form, so-called Stokes waves will continue cruising gaily along the water's surface for all time, their form unchanged.

But what if the wake of a passing boat crosses the waves' path? Will the waves absorb this disturbance and maintain their form, or will they be disrupted permanently, transforming into an entirely different pattern of waves?

For decades, mathematicians assumed that Stokes waves are stable, meaning that any small distortion will have a minimal effect. After all, the real world is full of such complications, yet the seas are rife with Stokes waves. If they fell apart at the tiniest poke, they'd never survive long enough to make it to shore.

Still, in 1967, the mathematician T. Brooke Benjamin decided to verify this basic assumption. He had his student Jim Feir perform a series of experiments in a wave tank — a narrow rectangular pool with an oscillating rudder at one end that could produce Stokes waves. But Feir couldn't get the waves to reach the other end of the pool. At first, he thought there was a problem with the experimental setup. But soon it became apparent that the waves were, surprisingly, unstable.

In 1995, mathematicians finally proved that such "Benjamin-Feir instabilities" are an inevitable consequence of the Euler equations [2]. But the work left researchers wondering about the nature of these instabilities. Which kinds of disturbances can kill waves, and which can't? How rapidly do the instabilities balloon? Could a gust of wind at the center of the Pacific cause a train of waves to strike Malibu Beach weeks later, or would the formation break down before reaching the shore?

Strange Archipelagos

Maspero had never thought to wonder why the waves exiting Trieste's bay were dying. That would change when his colleague, Berti, had a chance encounter at a 2019 workshop on the mathematics of waves.

There, he met Bernard Deconinck, an applied mathematician at the University of Washington who, along with Katie Oliveras of Seattle University, *then his PhD student*, had been mapping all the different instabilities that could destroy Stokes waves. A few years earlier, the pair had noticed an astonishing pattern, and they hadn't been able to stop thinking about it.

When a perfect train of Stokes waves encounters a disturbance that distorts the waves' shape, sometimes the effects of the disturbance grow to destroy the entire train, and sometimes they barely interfere. The outcome depends on the frequency of the disturbance — how much it oscillates compared to the length of the original wave. A kayak, which produces a wake that consists of short, frequent oscillations, will deliver a higher-frequency impact than a massive ocean liner, which produces longer and slower oscillations.

In general, mathematicians expect waves to recover more easily from higher-frequency disruptions like the kayak's, because their impacts are limited to a smaller region of a passing wave at any given moment. The wake of the ocean liner, on the other hand, can affect the entire wave at once, permanently disrupting it. Benjamin-Feir instabilities are caused by low-frequency disruptions.

In 2011, Deconinck and Oliveras simulated different disturbances with higher and higher frequencies and watched what happened to the Stokes waves. As they expected, for disturbances above a certain frequency, the waves persevered.

But as the pair continued to dial up the frequency, they suddenly began to see destruction again. At first, Oliveras worried that there was a bug in the computer program. "Part of me was like, this can't be right," she said. "But the more I dug, the more it persisted."

In fact, as the frequency of the disturbance increased, an alternating pattern emerged. First there was an interval of frequencies where the waves became unstable. This was followed by an interval of stability, which was followed by yet another interval of instability, and so on.

Deconinck and Oliveras published their finding as a counterintuitive conjecture [3]: that this archipelago of instabilities stretches off to infinity. They called all the unstable intervals "isole" — the Italian word for "islands."

It was strange. The pair had no explanation for why instabilities would appear again, let alone infinitely many times. They at least wanted a proof that their startling observation was correct.

For years, no one could make any progress. Then, at a 2019 workshop, Deconinck approached Berti. *He knew that he had a lot of experience studying the math of wavelike phenomena in a variety of contexts. Perhaps he and his collaborators could figure out a way to prove that these striking patterns arise from the Euler equations.*

Still, it took a while for things to get going. In the summer of 2020, Berti attended a talk by Strauss on his own breakthrough with Nguyen on the Benjamin-Feir stability problem [9]. The methods used by Nguyen and Strauss resonated with Berti and soon after, he brought in Maspero and new PhD student Ventura.

They started with the lowest set of frequencies that seemed to cause waves to die, *i.e, the Benjamin-Feir instabilities.* First, they applied techniques from physics to represent each of these low-frequency instabilities as arrays, or matrices, of 16 numbers. These numbers encoded how the instability would grow [4] and distort the Stokes waves over time. The mathematicians realized that if one of the numbers in the matrix was always zero, the instability would not grow, and the waves would live on. If the number was positive, the instability would grow and eventually destroy the waves.

To show that this number was positive for the first batch of instabilities, the mathematicians had to compute a gigantic sum. It took 45 pages and nearly a year of work to solve it. Once they'd done so, they turned their attention to the infinitely many intervals of higher-frequency wave-killing disturbances — the isole. *Berti and Maspero invited Corsi to join them on the project, as her expertise brought in exciting methods that would prove essential.*

First, the team of four figured out a general formula — another complicated sum — that would give them the number they needed for each isola. *Then Ventura used a computer program to solve the formula for the first 21 isole.* (After that, the calculations got too complicated for the computer to handle.) The numbers were all positive, as expected — and they also seemed to follow a simple pattern that implied they would be positive for all the other isole as well.

But a pattern isn't a proof, and the group wasn't sure how to proceed. So they turned to a global community of computer experts for help.

The Levee Breaks

Maspero had been scouring the mathematical literature for anything that could help him. The problem, he decided, was that he needed to somehow simplify the calculations he had to make. He found a book [5] in which Doron Zeilberger, a mathematician at Rutgers University, *together with Marko Petkovsek and Herbert Wilf,* outlined algorithmic approaches to performing difficult algebraic calculations on a computer. Unable to adapt them to his case, Maspero reached out to Zeilberger directly.

"We have recently encountered certain combinatorial problems that we cannot solve," his email to Zeilberger began. "We wonder if you can help us."

Zeilberger was intrigued. "The question was exactly my cup of tea," he said. With some work, he was able to get his computer, which he calls Shalosh B. Ekhad (and which appears as a co-author on all his papers), to compute sums for the first 2,000 isole, verifying that the outputs were all positive and that they conformed to the pattern the Italian team had identified. Then he called on his network of computer-algebra enthusiasts to help, offering to make a 100 donation to the On-Line Encyclopedia of Integer Sequences [6] in the name of whoever could establish that the pattern persisted forever.

In February 2024, Zeilberger paid up. After a lengthy email exchange with two of his frequent collaborators *Mark van Hoeij and Christoph Koutschan*, they came back with a complete proof that the sums would never equal zero.

Deconinck and Oliveras had been right: Their isole were real. The result means that mathematicians now finally know precisely [7] which types of disturbances will kill a Stokes wave and which will not — something they have hoped to understand for two centuries.

"It's just like, holy crap, thank you," Oliveras said.

It also leaves mathematicians with more work to do. Why do waves live and die in this alternating pattern? "OK, those isole were real," she said. "Now we have to pay attention to them."

The result is just the latest in a recent spate of papers that aim to illuminate the mathematics of water waves. Mathematicians are combining advances in computational and theoretical techniques to better understand solutions to the Euler equations, allowing them to prove more and more conjectures about how waves behave. The italian team hope that their methods can now be used to solve other problems in this area.

As for the bora-blown waves outside Maspero's office window, and their eventual decline into flat water — at the moment, he can't say for sure whether his team's math explains this precise phenomenon. "I don't know if there is a connection," he said. "But I love to think it's the same instabilities."

References

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