Functional analysis

Sheet — SS 23

Exam of Functional Analysis

1 Linear Analysis

1. Theory: Let H be a Hilbert space, and A be a bounded self-adjoint operator on H.
   (i) Recall the definitions of spectrum $\sigma(A)$, point spectrum $\sigma_p(A)$, continuous spectrum $\sigma_c(A)$
   (ii) Characterize $\sigma(A)$, $\sigma_p(A)$ and $\sigma_c(A)$, in terms of the projection-valued measure associated with A (with proof)
   (iii) Suppose f is continuous and not non-negative on $\sigma(A)$. Prove that there exists $u \in H$ such that $\langle u, f(A)u \rangle < 0$

2. Practice: Let A be the operator on $C([0,1])$, equipped with the sup norm, defined by the formula

$$Au(t) = \frac{1}{t}\int_0^t u(s)ds, \qquad Au(0) = u(0)$$

   (i) Compute the norm and the spectral radius of A
   (ii) Find the eigenvalues of A
   (iii) Is A compact?
   (iv) Discuss whether $\lambda = 0$ is in the spectrum, and determine its nature

3. Bonus: A bounded linear operator $U \in \mathcal{L}(H)$ on a Hilbert space H is *unitary* if $UU^* = U^*U = \mathbb{I}$, and two operators $A, B \in \mathcal{L}(H)$ are unitarily equivalent if there exists a unitary operator such that $UAU^* = B$. Prove whether the following pairs of operators are unitarily equivalent or not:
   (i) The operators multiplication by $f(t) = t$ and $f(t) = t^2$ on $L^2([0,1])$
   (ii) The operators multiplication by $f(t) = t$ and $f(t) = t^2$ on $L^2([0,2])$

2 Nonlinear analysis

1. Theory: Discuss the existence of solutions of the Sturm-Liouville operator

$$\begin{cases} -u'' + V(x)u = g \\ u(0) = u(1) = 0 \end{cases}$$

   for $V, g \in C^0([0,1])$ (do not prove Lax-Milgram, but apply it).

2. PRACTICE: Consider the semilinear Dirichlet problem

$$(D) \qquad \begin{cases} u'' + au + b(u) = h(x), \\ u(0) = u(1) = 0 \end{cases}$$

with $h \in C^0([0,1])$ and $b \in C^1(\mathbb{R})$. Assume that $a \notin \{n^2\pi^2\}_{n\in\mathbb{N}}$ and that

(H1) exists $M > 0$ such that $|b(s)| \le M$ for all $s \in \mathbb{R}$;

(H2) $a + b'(s) < \pi^2$ for all $s \in \mathbb{R}$.

Prove that for all $h \in C^0([0,1])$ there exists a unique classical solution to (D) following the following scheme:

(i) prove the uniqueness of solutions;
(*Hint:* you will need the Poincaré inequality with the best constant

$$\int_0^1 u^2 \mathrm{d}x \le \frac{1}{\pi^2} \int_0^1 (u')^2 \mathrm{d}x \ , \quad \forall u \in H_0^1 \ . \tag{1}$$

You do not need to prove it.)

(ii) prove that $\mathrm{Im}(G(u))$ is open, where $G(u) := u'' + au + b(u)$;

(iii) prove that $\mathrm{Im}(G(u))$ is closed;
(*Hint:* take a sequence $(h_n)_n \subset L^2$ such that $h_n \to h$ in L^2 and consider $(u_n)_{n\in\mathbb{N}} \subseteq H_0^1$ with $G(u_n) = h_n$. Assume that $(u_n)_{n\in\mathbb{N}}$ is unbounded in L^2, and deduce a contradiction for $z_n := u_n / \|u_n\|_{L^2}$.)