Functional analysis

Sheet 1 — SS 21

Bounded and compact operators

1. True or false? If true then prove it, if false furnish a counterexample!
   (a) Every linear operator with a finite dimensional range is compact.
   (b) Let $K \in \mathcal{L}(X)$ be compact and injective. Then K cannot be surjective.
   (c) Symmetric operators defined everywhere on an Hilbert space $\mathcal{H}$ are continuous.
   (d) Let H be a Hilbert space and let $A_n \in \mathcal{L}(H)$ be such that $A_n x \to 0$ for every $x \in H$. Then $A_n^* x \to 0$ for every x.

2. Shur Test. Define the linear operator

$$[A_K u](x) := \int_{\mathbb{R}^d} K(x,y) u(y)\, dy.$$

   Assume that

$$\|A\|_{L^\infty_x L^1_y} := \sup_x \int |K(x,y)|\, dy < \infty, \qquad \|A\|_{L^\infty_y L^1_x} := \sup_y \int |K(x,y)|\, dx < \infty$$

   Then $A \colon L^p \to L^p$ is bounded $\forall p \in [1, +\infty]$ and

$$\|Au\|_{L^p} \leq \|A\|^{1-\frac{1}{p}}_{L^\infty_x L^1_y} \, \|A\|^{\frac{1}{p}}_{L^\infty_y L^1_x} \, \|u\|_{L^p}.$$

3. Consider the space $\ell^2(\mathbb{N})$.
   (a) Let $T_n(x_1, x_2, \ldots) := (\frac{1}{n}x_1, \frac{1}{n}x_2, \ldots)$. Then $T_n \to 0$ uniformly.
   (b) Let $S_n(x_1, x_2, \ldots) := (\underbrace{0, \ldots, 0}_{n \text{ times}}, x_{n+1}, x_{n+2}, \ldots)$. Then $S_n \to 0$ strongly, not uniformly.
   (c) Let $W_n(x_1, x_2, \ldots) := (\underbrace{0, \ldots, 0}_{n \text{ times}}, x_1, x_2, \ldots)$. Then $W_n \to 0$ weakly, not strongly.

4. Consider the multiplication operator $T \colon L^2([0,1]) \to L^2([0,1])$, $(Tu)(x) = v(x)u(x)$, where $v \in C^\infty([0,1])$, $v \neq 0$. Prove that T is not compact.

5. (a) Find in $L^p[0, \pi/2]$, $1 \leq p < \infty$, the solution of the equation

$$f(x) = \lambda \int_0^{\pi/2} \cos(x-y)\, f(y) \mathrm{d}y$$

   (it will depend on λ).
   (b) Decide if there exists a solution in $L^p[0, \pi/2]$ for $1 < p < \infty$ of the equation

$$f(x) - \lambda \int_0^{\pi/2} \cos(x-y)\, f(y) \mathrm{d}y = 1$$

6. Let E be a closed linear subspace in $C[0,1]$ such that $E \subset C^1[0,1]$.
   Prove that $\dim E < \infty$.
   Hint: observe that $C^1[0,1]$ belongs to the range of a compact operator and hence cannot contain infinite-dimensional closed subspaces.

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