Functional analysis

Sheet 2 — SS 21

Spectrum

1. Let $T \in \mathcal{L}(X)$ be invertible. Prove that $\sigma(T^{-1}) = \{\lambda^{-1} \colon \lambda \in \sigma(T)\}$.

2. Let $\mathcal{H}$ be a Hilbert space and U a unitary operator. Prove that $\sigma(U) \subseteq \{z \colon |z| = 1\}$.

3. Compute the spectrum of the following operators, and describe the puntual, continuous and residual parts:
   (a) $T \colon L^2([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C})$, the multiplication operator by a function $v \in C^\infty([0,1],\mathbb{R})$, i.e. $(Tf)(x) = v(x)f(x)$.
   (b) T as above, but now as an operator on $C^0([0,1],\mathbb{C})$ and with $v \in C^\infty([0,1],\mathbb{R})$.
   (c) $U \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ the shift operator, $(Ux)_j = x_{j+1}$, $\forall j \in \mathbb{Z}$.
   (d) The orthogonal projection $P \colon \mathcal{H} \to \mathcal{H}$ over the span of finitely many linearly independent vectors.
   (e) The operator $K \colon L^2([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C})$ defined by $(Kf)(t) = \int_0^1 k(t,s)f(s)\mathrm{d}s$ where

$$k(t,s) = \begin{cases} 1, & s \le t \\ 0, & s > t. \end{cases}$$

4. Consider the operator T on $\ell^2(\mathbb{N})$ defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots) \ .$$

   Prove that T is compact and compute $\sigma(T)$ and its nature (point spectrum, residual, continuous).

5. Let X be a complex Banach space and $A \in \mathcal{L}(X)$. Prove that if A^2 has an eigenvalue, then A also does.
   Hint: use that $A^2 - \lambda = (A + \lambda^{1/2})(A - \lambda^{1/2})$.

6. Let $\mathcal{H}$ be Hilbert and E_1, E_2 two closed linear subspaces such that $H = E_1 \oplus E_2$. Let $T \in \mathcal{L}(\mathcal{H})$ leaving invariant E_j, i.e. $T \colon E_j \to E_j$, $j = 1, 2$. Show that

$$\sigma(T) = \sigma(T|_{E_1}) \cup \sigma(T|_{E_2})$$

7. Construct a bounded linear operator on the complex space ℓ^2 such that its spectrum consists of the two points 0 and 1 that are not its eigenvalues.
   HINT: consider first a compact operator without eigenvalues.

Alberto Maspero: alberto.maspero@sissa.it