Functional analysis

Sheet 3 — SS 21

Functional calculus and spectral theorem

1. Recall that, given a selfadjoint operator $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, the spectral measure μ_x fulfills
$$\langle f(T)x, x\rangle = \int_{\sigma(T)} f(\lambda) \mathrm{d}\mu_x, \qquad \forall x \in \mathcal{H}$$
Compute the spectral measure μ_x in case
   (a) T is a compact operator.
   (b) T is the multiplication operator on $L^2[0,1]$ by a smooth function $g(x)$.

2. Suppose that selfadjoint operators A and B on a separable Hilbert space commute. Then for any bounded Borel functions φ and ψ the operators $\varphi(A)$ and $\psi(B)$ commute.

3. Let A_n and A be selfadjoint operators on a separable Hilbert space H and $f \in C_b(\mathbb{R})$.
   (a) Suppose that $A_n \to A$ in the operator norm. Prove that $f(A_n) \to f(A)$ in the operator norm.
   (b) Suppose that $A_n x \to Ax$ for all $x \in H$. Prove that $f(A_n)x \to f(A)x$ for all vectors $x \in H$.

4. Let $A \in \mathcal{L}(\mathcal{H})$ be selfadjoint. Put $E_\lambda := E^A((-\infty, \lambda])$. Let $\lambda_0 \in \sigma(A)$. Prove that
   (a) $\lambda_0 \in \sigma_p(A) \quad \Longleftrightarrow \quad E_\lambda \not\to E_{\lambda_0}$ strongly when $\lambda \nearrow \lambda_0$.
   (b) $\lambda_0 \in \sigma_c(A) \quad \Longleftrightarrow \quad E_\lambda \to E_{\lambda_0}$ strongly when $\lambda \nearrow \lambda_0$.

5. Let $A \in \mathcal{L}(\mathcal{H})$, $A = A^*$. Consider the Schrödinger equation
$$\mathrm{i}\partial_t \psi = A\psi.$$
For any $t \in \mathbb{R}$ define
$$U(t) := \int_J e^{-\mathrm{i}t\lambda} \, \mathrm{d}E(\lambda)$$
where $E(\cdot)$ is the PVM of the operator A. Prove the following:
   (a) $U(t)$ is a 1-parameter semigroup, unitary, strongly continuous, i.e.
$$U(0) = 1, \qquad U(t+s) = U(t)U(s), \qquad U(t)^* = U(t)^{-1}, \qquad U(s)\psi \overset{s \mapsto t}{\to} U(t)\psi$$
   (b) For any $\psi \in \mathcal{H}$ one has
$$A\psi = \lim_{t \to 0} \mathrm{i} \frac{U(t)\psi - \psi}{t} = \mathrm{i} \left.\frac{d}{dt}\right|_{t=0} U(t)\psi$$
   (c) $\psi(t) := U(t)\psi_0$ is the unique solution of the equation $\mathrm{i}\partial_t \psi = A\psi$ with initial data ψ_0.

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6. **Multivariable Bounded Borel Functional Calculus.** Let $A_1, \ldots, A_n$ be selfadjoints bounded operators on $\mathcal{H}$ pairwise commuting, i.e. $[A_j, A_i] = 0$ for any i, j, where $[A, B] := AB - BA$. Let $\sigma := \sigma(A_1) \times \cdots \times \sigma(A_n)$. There exists a unique map

$$\Phi \colon \mathcal{B}_b(\sigma) \to \mathcal{B}(\mathcal{H})$$
$$f \to \Phi(f) \equiv f(A_1, \ldots, A_n)$$

such that

(i) Φ is a unital-*-algebra homeomorphism;

(ii) $||\Phi(f)|| \leq ||f||_{L^\infty(\sigma)}$;

(iii) if $f(t_1, \ldots, t_n) = t_i$, then $f(A_1, \ldots, A_n) = A_i$.

(iv) if $(f_n)_n$ is a bounded sequence in $\mathcal{B}_b(\sigma)$ converging pointwise to f, then $\Phi(f_n)$ converges strongly to $\Phi(f)$.

Hint: follow the following steps:

- for any Borel subsets $\Omega_1, \ldots, \Omega_n$ of $\sigma(A_1), \ldots, \sigma(A_n)$ consider the function on σ

$$h(x_1, \cdots, x_n) = 1_{\Omega_1}(x_1) \cdots 1_{\Omega_n}(x_n)$$

  and put

$$h(A_1, \ldots, A_n) = 1_{\Omega_1}(A_1) \cdots 1_{\Omega_n}(A_n)$$

  using Borel functional calculus. Verify that $h(A_1, \ldots, A_n)$ is a orthogonal projection.

- If f is a simple function of the form

$$f(x_1, \ldots, x_n) = \sum c_k h_k$$

  with h_k as above and having zero products, verify that

$$\|f(A_1, \ldots, A_n)\| \leq \|f\|_\infty$$

- Use approximation arguments to extend to a genereal bounded borel function $f(x_1, \ldots, x_n)$

7. **Spectral theorem for normal operators.** Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator, i.e. $A^*A = AA^*$. There exists a unique projection valued measure defined on the borelian sets of $\sigma(A) \subset \mathbb{C}$ such that

$$A = \int_{\sigma(A)} z \, \mathrm{d}E(z)$$

Hint: • Write $A = A_1 + \mathrm{i}A_2$ with

$$A_1 = \frac{A + A^*}{2}, \qquad A_2 = \frac{A - A^*}{2\mathrm{i}}$$

and check that A_1, A_2 are selfadjoint commuting operators.

• Put $E(M_1 \times M_2) = E^{A_1}(M_1) \, E^{A_2}(M_2)$ for any M_1, M_2 Borel sets of $\mathbb{R}$ and show that one can extend this to be a PVM on $\mathbb{R}^2$.

• If f is a Borelian function of $\mathbb{R}$, show that

$$\int_{\mathbb{R}} f(\lambda_k) \mathrm{d}E^{A_k}(\lambda_k) = \int_{\mathbb{R}^2} f(\lambda_k) \, dE(\lambda_1, \lambda_2), \qquad k = 1, 2$$

8. **Spectral theorem for unitary operators.** Let $U \in \mathcal{L}(\mathcal{H})$ be a unitary operator, i.e. $U^*U = UU^* = Id$. There exists a unique projection valued measure defined on the borelian sets of $[0, 2\pi]$ so that

$$U = \int_0^{2\pi} e^{\mathrm{i}t} \, \mathrm{d}E(t)$$

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