Functional analysis

Sheet 4 — SS 21

Calculus in Banach spaces

1. (a) Let $f \in C^1(\mathbb{R})$ with $f(0) = 0$ and consider

$$F \colon \ell^p(\mathbb{Z}, \mathbb{R}) \to \ell^p(\mathbb{Z}, \mathbb{R}), \qquad (x_n)_{n\in\mathbb{Z}} \mapsto (f(x_n))_{n\in\mathbb{Z}} \ .$$

Prove it is Frechet differentiable and its differential is $dF(x)[h] = (f'(x_n)h_n)_{j\in\mathbb{Z}}$

(b) Let H be a real Hilbert space, $A \in \mathcal{L}(H)$ and $F(x) := \langle Ax, x\rangle$. Compute $d^n F$ for any $n \geq 1$.

(c) Let H be a *complex* Hilbert space; the function $f(u) := |u|^2 = \langle u, u\rangle_H$ is not differentiable at $u = 0$. However, writing $u = x + \mathrm{i}y$ and identifying H with $H_\mathbb{R} \oplus \mathrm{i}H_\mathbb{R}$ with $H_\mathbb{R}$ a real Hilbert space, then the real and imaginary parts are differentiable. Compute them.

2. Consider $L^p([0, 1])$ and let $G \colon \mathbb{C} \to \mathbb{R}$ be real differentiable with

$$|G(z)| \leq C(1 + |z|^p), \quad \sqrt{|\partial_x G(z)|^2 + |\partial_y G(z)|^2} \leq C(1 + |z|^{p-1}), \quad z = x + \mathrm{i}y$$

Prove that

$$F(u) := \int_0^1 G(u)\, dx$$

is Gateaux differentiable and

$$d^G F(u)[h] = \int_0^1 (\partial_x G(u))\mathrm{Re}(h) + (\partial_y G)(u)\mathrm{Im}(h))\ dx$$

Is it also differentiable?

3. Let $X := \mathcal{L}(E)$, E a Banach space. Prove that the mapping

$$H \mapsto e^H$$

is Frechet differentiable and find its derivative.

4. Consider the nonlinear problem on the torus $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases} -u'' + g(u) = f \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \end{cases}$$

where $g \colon \mathbb{R} \to \mathbb{R}$ is a smooth function with $g'(0) > 0$. Prove that for any $f \in L^2(\mathbb{T})$ sufficiently small, the nonlinear problem has a solution. What about generic f (not necessarily small)?

5. Let S be the unit sphere in $L^2([0, 1])$, i.e.

$$S := \{u \in L^2 \colon \ \ \|u\|_{L^2} = 1\} \ .$$

(a) Let $F(u) := \int_0^1 u(x)\mathrm{d}x$. Use the method of Lagrange multiplier to find all points where F attains its maximum value on S.

(b) Let $\tilde{F}(u) := \int_0^1 |u|^{3/2}(x)\mathrm{d}x$. Find the maximum value of $\tilde{F}$ on S.

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6. **Minimal surfaces.** Consider the ball $B_1(0) \subset \mathbb{R}^2$ and let $g : \partial B_1(0) \to \mathbb{R}$ a smooth function. A smooth function $u\colon B_1(0) \to \mathbb{R}$ is a minimal surface with boundary g if

$$\begin{cases} (1+u_y^2)u_{xx} + (1+u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0 \\ u|_{\partial B_1(0)} = g \end{cases}$$

Prove that for any $g \in C^2(\partial B_1(0))$ sufficiently small, there is a unique solution $u \in C^2(\overline{B_1(0)})$.
Hint: study first the existence and uniqueness of solutions of the linear problem

$$\begin{cases} -\Delta u = f & \text{in } B_1(0) \\ u = g & \text{in } \partial B_1(0) \end{cases}$$

either via classical techniques (Green's function), Lax-Milgram (requires higher dimensional Sobolev spaces) or variational methods (namely proving that the solution of the elliptic problem is the only minimizer of

$$I[w] := \frac{1}{2}\int_{B_1(0)} |\nabla u|^2 - wf\mathrm{d}x$$

with w belonging to the admissible set $\mathcal{A} := \{w \in C^2(B_1(0)) \,|\, w = g \text{ on } \partial U\}$)

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