Functional analysis

Sheet 4 — SS 21

Bifurcation problems and Shauder fixed point

1. **Traveling wave solutions for the Whitham equation.** The Whitham equation was proposed as a simplified description of uni-directional wave motion at the surface of an inviscid fluid. It reads

$$\eta_t + \mathcal{M}\eta_x + \eta_x \eta = 0, \quad x \in \mathbb{R}$$

where $\eta \colon \mathbb{R} \to \mathbb{R}$ and $\mathcal{M}$ is the Fourier multiplier operator

$$\mathcal{M}[\eta] := \sum_{j \neq 0} \sqrt{\frac{\tanh(\mathtt{h}j)}{j}}\, \eta_j \, e^{\mathrm{i}jx}$$

and $\mathtt{h} > 0$ is a parameter describing the depth of the fluid (you can put $\mathtt{h} = 1$). Discuss the existence of small amplitude, space periodic, traveling wave solutions, namely solutions of the Whitham equation of the form

$$\eta(t, x) = \breve{\eta}(x - ct), \quad \breve{\eta}(x + 2\pi) = \breve{\eta}(x)$$

2. Consider the stationary Gross-Pitaevskii equation

$$-u''(x) + V(x)u(x) + u^3(x) = \lambda u(x)$$

for $u \in H^2(\mathbb{R})$, $\lambda \in \mathbb{R}$ and $V \in C_c^\infty(\mathbb{R})$. Let $\lambda_0 < 0$ be a simple eigenvalue of $-\partial_x^2 + V(x) \colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$. Show that λ_0 is a bifurcation point.
Hint: show that for $\lambda < 0$ the operator $-\partial_x^2 - \lambda$ is invertible from $H^2(\mathbb{R}) \to L^2(\mathbb{R})$ and moreover that the operator $(-\partial_x^2 - \lambda)^{-1} M_V$ is compact from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where M_V is the multiplication operator by $V(x)$. For this last just show that

3. The bending of an elastic rod can be described be the boundary value problem

$$\begin{cases} u'' + \lambda \sin(u) = 0 \\ u'(0) = u'(2\pi) = 0 \end{cases}$$

with $x \in \mathbb{T}$. Find all the bifurcation points.

4. Prove the following version of Schauder theorem: Let K be a convex compact set in a normed space X and let $f \colon K \to K$ be a continuous mapping. Then there exists $x \in K$ with $f(x) = x$.

5. Let X be a compact Hausdorff space, and let $P(X)$ be the set of all Borel probability measures on X. Let $\mu \in P(X)$. A measure μ is said to be invariant with respect to a μ-measurable map $f : X \to X$ if

$$\mu(A) = \mu(f^{-1}(A)) \quad \forall A \;\; \mu-\text{measurable set}$$

   (a) Check that $P(X)$ is convex and closed in the weak* topology, and actually is weakly* compact.

   (b) Given a continuous map $f \colon X \to X$, prove there exists an invariant measure for f.

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