DIFFERENTIABILITY IN BANACH SPACES

Reference: Ambrosetti – Prodi : "A primer of nonlinear Analysis"

Setting X, Y Banach, $\mathcal{U} \subseteq X$ open

$$F : \mathcal{U} \subseteq X \longrightarrow Y$$

Frechet and Gateaux derivatives

Def F is (Frechet) differentiable at $u \in \mathcal{U} \subseteq X$ if $\exists\, A \in \mathcal{L}(X, Y)$ s.t

$$\lim_{\substack{\|h\|_X \to 0 \\ h \in X}} \frac{\| F(u+h) - F(u) - Ah \|_Y}{\|h\|_X} = 0$$

with this meaning

We write also $F(u+h) = F(u) + Ah + o(\|h\|)$

Idea: A is the best linear approximation of F

Rem •) If A exists, it is unique. Assume the contrary: $\exists\, A, B \in \mathcal{L}(X,Y)$

$$F(u+h) = F(u) + Ah + o(\|h\|)$$
$$F(u+h) = F(u) + Bh + o(\|h\|)$$

$$\Rightarrow \frac{\| Ah - Bh \|}{\|h\|} = \frac{\| -F(u+h) + F(u) + Ah + F(u+h) - F(u) - Bh \|}{\|h\|}$$

$$\Rightarrow \frac{\|Ah - Bh\|}{\|h\|} \xrightarrow{\|h\|\to 0} 0$$

If $A \neq B$, $\exists\, h^* \in X$ st. $(A-B)h^* \neq 0$

Take $h = \varepsilon h^*$, then

$$\lim_{\varepsilon \to 0} \frac{\|(A-B)\varepsilon h^*\|}{\|\varepsilon h^*\|} = \frac{\|(A-B)h^*\|}{\|h^*\|} \neq 0 \quad ⚡$$

We write $A = dF(u) \in \mathcal{L}(X,Y)$

Frechet differential of F at u

•) F Frech diff at $u \Rightarrow F$ continuous at u

Indeed $\|F(u+h) - F(u)\| \leq \|Ah\| + o(\|h\|) \xrightarrow{\|h\|\to 0} 0$

•) Def of differential does not depend on the norm, but only on the topology

$(X, \|\ \|)$, $(X, |||\ |||) \rightsquigarrow$ same diff.

BASIC EXAMPLES

1) $F(u) = y_0 + Tu$, $T \in \mathcal{L}(X,Y)$, $y_0 \in Y$

$\Rightarrow F$ is Frech diff and $\forall u$: $dF(u)[h] = Th$

indeed: $F(u+h) = y_0 + T(u+h) = \underbrace{y_0 + Tu}_{F(u)} + Th$

2) H real Hilbert space

$F : H \to \mathbb{R}$, $F(u) = \|u\|^2$

F is Fréch. diff. at every $u \in H$ and

$$dF(u)[v] = 2 \langle u, v \rangle$$

Indeed: $F(u+h) = \|u+h\|^2 = \langle u+h, u+h \rangle$

$$= \|u\|^2 + \langle h, u \rangle + \langle u, h \rangle + \|h\|^2$$

$$= F(u) + \underbrace{2 \langle u, h \rangle}_{\text{lin. op on } H} + \|h\|^2$$

3) $B : X \times Y \to Z$ bilinear and continuous

B is Fréch diff. at any $(u,v) \in X \times Y$
and

$$dB(u,v)\left[(h,k)\right] = B(h,v) + B(u,k)$$

Indeed: $B(u+h, v+k) = B(u,v) + B(h,v) + B(u,k) +$
$+ B(h,k)$

4) $F : L^p[0,1] \to \mathbb{R}$, $F(u) = \int_0^1 (u(x))^p \, dx$

$p \in \mathbb{N}$

$$F(u+h) = \int (u+h)^p \, dx = \int u^p + p u^{p-1} h + o(h^2)$$

$$= F(u) + \underbrace{\int p u^{p-1} h}_{\text{linear operator, candidate to be the differential}} + o(h^2)$$

$$dF(u)[h] = p \int u^{p-1} h \quad \in \mathcal{L}(L^p, \mathbb{R})$$

$$\left| dF(u)[h] \right| \leq p \int \underbrace{|u^{p-1}|}_{L^{\frac{p}{p-1}}} \underbrace{|h|}_{L^p} \leq C \|u\|_{L^p}^{\frac{p}{p-1}} \|h\|_{L^p}$$

then $\|F(u+h) - F(u) - dF(u)[h]\| \leq \sum_{k=2}^{p} \binom{p}{k} \int \underbrace{|u^{p-k}}_{\in L^{p/p-k}} \underbrace{h^k|}_{\in L^{p/k}}$ $\quad 1 = \frac{k}{p} + \frac{p-k}{p}$

$$\leq \sum_{k=2}^{p} \binom{p}{k} \|u\|_{L^p}^{p-k} \|h\|_{L^p}^{k/p} \leq C_1 \|h\|_{L^p}^2 \quad \text{for } \|h\|_p \ll 1$$

(5) $X = \mathcal{L}(E)$, E Banach space

$\mathcal{U} = \{ T \in X \; ; \; T \text{ is invertible} \}$ (open set)

Consider $F : \mathcal{U} \to \mathcal{U}$, $F(T) = T^{-1}$

Then F is Fréch. diff. and

$$dF(T)[H] = -T^{-1} H T^{-1}$$

Indeed take $\|H\| < \dfrac{1}{2\|T^{-1}\|}$, then

$$F(T+H) = (T+H)^{-1} = \left(T (\mathbb{I} + T^{-1} H) \right)^{-1}$$

$$= (\mathbb{I} + T^{-1} H)^{-1} T^{-1}$$

$$= \sum_{k=0}^{\infty} (-1)^k (T^{-1} H)^k T^{-1}$$

$$= \left(\mathbb{1} - T^{-1}H + o(\|H\|)\right) T^{-1}$$

$$= T^{-1} - \underbrace{T^{-1} H T^{-1}}_{\text{linear part}} + o(\|H\|)$$

Check: $\dfrac{\|(T+H)^{-1} - T^{-1} + T^{-1} H T^{-1}\|}{\|H\|} \xrightarrow{\|H\| \to 0} 0$

6) $G : \mathbb{R} \to X$, X Banach

$\Rightarrow \; dG(\lambda) \in \mathcal{L}(\mathbb{R}, X)$ but of course

$$\mathcal{L}(\mathbb{R}, X) \simeq X$$
$$A \longrightarrow A[1] \in X$$

So we identify $dG(\lambda) \approx dG(\lambda)[1]$

Prop (1) $F, G : \mathcal{U} \subseteq X \to Y$ diff at u_0

$$\Rightarrow \; d(F+G)(u_0) = dF(u_0) + dG(u_0)$$

(2) $F : \mathcal{U} \to Y$, $G : V \to Z$, $\mathcal{U} \subseteq X$, $V \subseteq Y$ open

and $F(\mathcal{U}) \subseteq V$, F diff at u_0,
G diff at $F(u_0)$

$\Rightarrow \; G \circ F$ is diff at u_0 and

$$d(G \circ F)(u_0)[h] = dG(F(u_0))\left[dF(u_0)[h]\right]$$

proof as in $\mathbb{R}^n$.

Def Let $F : \mathcal{U} \subset X \to Y$ diff in $\mathcal{U}$ (diff $\forall u \in \mathcal{U}$)

Consider $F' : \mathcal{U} \longrightarrow \mathcal{L}(X, Y)$ FRECHET DERIVATIVE

$u \longmapsto dF(u)$

IF $F' \in C(\mathcal{U}, \mathcal{L}(X,Y))$, we say F is $C^1(\mathcal{U}, Y)$

Def $F : \mathcal{U} \subseteq X \to Y$ is gateaux diff at $u \in \mathcal{U}$ if $\exists A \in \mathcal{L}(X,Y)$ s.t.

$$\forall h \in X : \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathbb{R}}} \left\| \frac{F(u+\varepsilon h) - F(u)}{\varepsilon} - Ah \right\|_Y = 0$$

(directional derivative: $\forall$ fixed direction, $\exists$ limit)

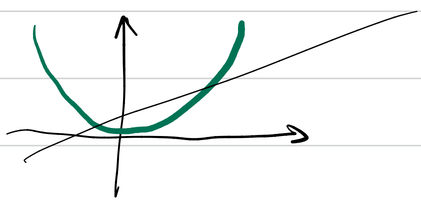
Rem

1) F gat-diff $\Rightarrow$ $A = d^G F(u)$ is unique

2) Frech diff $\Rightarrow$ Gateaux diff and $dF(u) = d^G F(u)$

3) F gat-diff at u $\not\Rightarrow$ F continuous at u

$F : \mathbb{R}^2 \to \mathbb{R}$, $F(x,y) = \begin{cases} 0 & \{y \neq x^2\} \cup \{(0,0)\} \\ 1 & \{y = x^2\} \setminus \{(0,0)\} \end{cases}$



$$\frac{F(\varepsilon \vec{h}) - F(0)}{\varepsilon} = 0 \quad \forall \varepsilon \,!$$

4) F gat diff at u & F cont at u

$\not\Downarrow$

F Frech diff at u

see: $F: L^2[0,1] \to L^2[0,1]$, $F(u) = \sin(u(x))$

○) F continuous at 0: $u_n \to 0$ in L^2
then $\|\sin(u_n(x))\|_{L^2} \to 0$ (exercise)

○) F is G-diff at 0 and $d^G F(0) = \mathbb{1}$

Indeed, $\lim_{\varepsilon\to 0} \left\| \frac{F(\varepsilon h)}{\varepsilon} - h \right\|^2_{L^2} = \lim_{\varepsilon \to 0} \int \left| \frac{\sin(\varepsilon h) - \varepsilon h}{\varepsilon} \right|^2 \xrightarrow{\text{by Lebesgue}} 0$

$\left|\frac{\sin(\varepsilon h) - \varepsilon h}{\varepsilon}\right| \le |h(x)|$

○) F not F-diff: consider $u_\varepsilon = \begin{cases} \frac{\pi}{2} & [0,\varepsilon] \\ 0 & \text{otherwise} \end{cases}$

Then $u_\varepsilon \to 0$ in L^2 but

$$\frac{\|F(u_\varepsilon) - F(0) - u_\varepsilon\|}{\|u_\varepsilon\|} = \frac{\|\sin(u_\varepsilon) - u_\varepsilon\|}{\|u_\varepsilon\|} = \frac{\left(\int_0^\varepsilon (1-\frac{\pi}{2})^2 + \int_\varepsilon^1 0^2\right)^{1/2}}{\frac{\pi}{2}\,\varepsilon^{1/2}}$$

$$\longrightarrow \frac{|1-\frac{\pi}{2}|}{\pi/2} \frac{\cancel{\varepsilon^{1/2}}}{\cancel{\varepsilon^{1/2}}} = c \neq 0$$

Q: How to go from G-diff to F-diff.

Thm $F: \mathcal{U} \subseteq X \to Y$, G-diff at every $u \in \mathcal{U}$. Assume $u \mapsto d^G F(u)$ is continuous at u^* (as a map $\mathcal{U} \to \mathcal{L}(X,Y)$)
$\Rightarrow$ F is Frech-diff at u^* and

$$d^G F(u^*) = dF(u^*)$$

EXERCISE: check that for $F(u) = \sin(u(x))$, $u \mapsto d F^G(u)$ is not continuous at $u^* = 0$

Prop (Mean value thm) $F: \mathcal{U} \to Y$ be G-diff $\forall u \in \mathcal{U}$. Take $u, v \in \mathcal{U}$ st.

$$[u,v] := \{ tu + (1-t)v : t \in [0,1] \} \subset \mathcal{U}$$

$$\Rightarrow \quad \| F(u) - F(v) \| \leq \left(\sup_{w \in [u,v]} \| d^G F(w) \| \right) \| u - v \|$$

<u>proof</u> Assume $F(u) \neq F(v)$. By Hahn–Banach,

$\exists \ \psi \in Y^*$, $\|\psi\|_{Y^*} = 1$ st.

$$\psi(F(u) - F(v)) = \| F(u) - F(v) \|$$

Let $\gamma(t) = tu + (1-t)v$ and consider

$h : [0,1] \to \mathbb{R}$, $\quad h(t) = \psi(F(\gamma(t)))$

$$= \psi(F(tu + (1-t)v))$$

We show h is diff (as map $\mathbb{R} \to \mathbb{R}$)

$$\frac{h(t+\tau) - h(t)}{\tau} = \psi\left(\frac{F(\gamma(t+\tau)) - F(\gamma(t))}{\tau} \right)$$

$\gamma(t+\tau) =$
$= \gamma(t) + \tau(u-v)$

$$= \psi\left(\frac{F(\gamma(t) + \tau(u-v)) - F(\gamma(t))}{\tau} \right)$$

$$\Big\downarrow \ \tau \to 0$$

(F is G-diff $\forall u \in \mathcal{U}$)

$$\psi(d^G F(\gamma(t)) (u-v)) \equiv h'(t)$$

Apply mean-value thm to h:

$$h(1) - h(0) = h'(\theta) \quad \text{for } \theta \in [0,1]$$

$$h'(\theta) = \psi(d^G F(\gamma(\theta))(u-v))$$

$$h(1) - h(0) = \psi(F(u)) - \psi(F(v)) = \| F(u) - F(v) \|$$

$$\|F(u)-F(v)\| \le \underbrace{\|\psi\|_{Y^*}}_{=1} \, \|d^G F(\underbrace{\theta u + (1-\theta)v}_{\in [u,v]})\| \, \|u-v\|$$

$\square$

proof (theorem)

Set $\quad R(h) = F(u^* + h) - F(u^*) - d^G F(u^*)[h]$

claim is equivalent to show $\quad \dfrac{\|R(h)\|_Y}{\|h\|_X} \xrightarrow{\|h\| \to 0} 0$

$u^* \in \mathcal{U}$, $\mathcal{U}$ open
F is G-diff $\quad \Longrightarrow \quad R$ is G-diff. in $B_\varepsilon(0)$ (provided $\varepsilon \ll 1$)

$$d^G R(h)[k] = d^G F(u^*+h)[k] - d^G F(u^*)[k]$$

So we use mean-value thm to estimate

$$\|R(h)\| = \|R(h) - R(0)\| \le \|h\| \sup_{w \in [0,h]} \|d^G R(w)\|$$

$$= \|h\| \sup_{t \in [0,1]} \|d^G R(th)\|_{\mathcal{L}(X,Y)}$$

$$= \|h\| \sup_{t \in [0,1]} \|d^G F(u^* + th) - d^G F(u^*)\|_{\mathcal{L}(X,Y)}$$

$$\Longrightarrow \frac{\|R(h)\|}{\|h\|} \le \sup_{t \in [0,1]} \|d^G F(u^* + th) - d^G F(u^*)\|_{\mathcal{L}(X,Y)} \xrightarrow{\|h\| \to 0} 0$$

by continuity of $d^G F$

$\square$

Another conseq of mean-value property

Prop $F : \mathcal{U} \subseteq X \to Y$, Gateaux diff with $d^G F \equiv 0$ on $\mathcal{U}$, $\mathcal{U}$ connected $\Rightarrow F \equiv \text{const}$

proof $d^G F$ is continuous $\Rightarrow$ F Frech diff $\Rightarrow$ F continuous. Fix $u_0 \in \mathcal{U}$

$$V = \{ u \in \mathcal{U} : F(u) = F(u_0) \}$$

F cont $\Rightarrow$ V is closed: $F(\bar{v}) = F(u_0)$?

We show it is also open: take $\bar{v} \in V$, $\delta > 0$ so that $B_\delta(\bar{v}) \subset \mathcal{U}$. We want to show $B_\delta(\bar{v}) \subseteq V$.

So take $z \in B_\delta(\bar{v})$:

$$\| F(z) - F(\bar{v}) \| \leq \| z - \bar{v} \| \sup_{w \in [z, \bar{v}]} \| d^G F(w) \| = 0$$

$\Rightarrow$ $B_\delta(\bar{v}) \subseteq V$ (V is open)

$\Rightarrow$ $V \equiv \mathcal{U}$ (V open and closed in connected set). ☒

This gives a strategy to prove F is Frechet diff: show that (a) F is G-diff. (b) $d^G F$ continuous $\Rightarrow$ F is diff.

Short remark about integration

Consider $f : [a, b] \to Y$ continuous function

Define $\int_a^b f(t)\,dt$ = strong limit of finite sums $\sum_i f(\xi_i)(t_i - t_{i-1})$, $\xi_i \in [t_{i-1}, t_i]$

The limit exists from the inequality

$$\Big\| \sum_i f(\xi_i)\,(t_i - t_{i-1}) \Big\| \le \sum_i \| f(\xi_i) \| \, |t_i - t_{i-1}|$$

and the fact that $t \to \|f(t)\|$ is continuous and therefore integrable (then the r.h.s. are just Riemann sums)

$$\leadsto \quad \Big\| \int_a^b f(t)\,dt \Big\| \le \int_a^b \|f(t)\|\,dt$$

Take now $F \in C([a,b];Y)$ and set

$$\phi(t) := \int_a^t F(s)\,ds$$

ϕ PRIMITIVE of F

Check that ϕ is diff. and $\phi'(t) = F(t)$ (exercise)

From mean-value thm

$$\| \phi(t_1) - \phi(t_2) \| \le |t_1 - t_2| \sup_{\xi \in [t_1, t_2]} \|F(\xi)\|$$

So if $F(\xi) \equiv 0$ in $[a,b] \Rightarrow \phi(t)$ is constant

Hence if $\phi_1, \phi_2 : [a,b] \to X$ and such that

$$\phi_1', \phi_2' = F \Rightarrow (\phi_1 - \phi_2)' = 0$$

$$\Rightarrow \phi_1 - \phi_2 = \text{const}$$

So the primitive is unique up to a constant

In particular take $F \in C^1([a,b]; X)$

then $\phi(t) = \int_0^t F'(s)\,ds$ and $F(t)$

fulfills $(\phi - F)' = 0$

$\rightsquigarrow$ $\phi(t) - F(t) = \phi(0) - F(0) = -F(0)$

$\rightsquigarrow$ $F(t) - F(0) = \int_0^t F'(s)\,ds$ $(*)$

Finally we have the following:

suppose $[u,v] \subset \mathcal{U}$, $F \in C^1(\mathcal{U}, Y)$,

$\gamma(t) = tu + (1-t)v$, then $F \circ \gamma \in C^1$

and $(F \circ \gamma)'(t) = F'(\gamma(t))(u-v)$

Applying formula $(*)$ we get

$$F(\gamma(1)) - F(\gamma(0)) = \int_0^1 F'(\gamma(t))(u-v)\,dt$$

$[a,b] \longrightarrow \mathcal{L}(X,Y)$

$$\Rightarrow \quad F(u) - F(v) = \left[\int_0^1 F'(tu + (1-t)v)\,dt\right][u-v]$$

very useful to make estimates

Higher order derivatives

Take $F \in C^1(\mathcal{U}, Y)$ and consider
$$F' : \overset{\subset X}{\mathcal{U}} \longrightarrow \mathcal{L}(X,Y)$$
$$u \longmapsto F'(u)$$

Def F is twice F-diff. at $u^* \in \mathcal{U}$ if F' is diff. at u^*
$$d^2F(u^*) := dF'(u^*)$$
F is twice diff in $\mathcal{U}$ if twice diff at u $\forall u \in \mathcal{U}$

Rem $d^2F(u^*) \in \mathcal{L}(X, \mathcal{L}(X,Y))$

$\cong$ isomorphic to

$\mathcal{L}_2(X,Y) = \{$ bilinear cont. maps $X \times X \to Y \}$

$$\mathcal{L}(X, \mathcal{L}(X,Y)) \longrightarrow \mathcal{L}_2(X \times X, Y) \qquad (*)$$
$$A \longrightarrow \phi_A, \quad \phi_A[h,k] = \underbrace{A[h]}_{\in \mathcal{L}(X,Y)}[k]$$
$$h \mapsto \phi(h,\cdot) \longleftarrow \phi$$

Notice that
$$\|T\|_{\mathcal{L}(X;\mathcal{L}(X,Y))} = \sup_{\|h\| \le 1} \|Th\|_{\mathcal{L}(X,Y)}$$
$$\|S\|_{\mathcal{L}_2(X,Y)} = \sup_{\|h\|, \|k\| \le 1} \|S[h,k]\|_Y$$

The isometry $(*)$ is isometric:
$$\|T\|_{\mathcal{L}(X,\mathcal{L}(X,Y))} = \sup_{\|h\|\le 1} \sup_{\|k\|\le 1} \| \underbrace{T[h][k]}_{\phi_T[h,k]} \| = \|\phi_T\|_{\mathcal{L}_2(X,Y)}$$

So from now on $d^2F(u^*) \in \mathcal{L}_2(X,Y)$

Def $F \in \mathcal{C}^2(\mathcal{U}, Y)$ if $F'' : \mathcal{U} \longrightarrow \mathcal{L}_2(X,Y)$
$u \longmapsto F''(u) = d^2F(u)$

is continuous

Prop $F : \mathcal{U} \to Y$ twice diff at $u_0 \in \mathcal{U}$
$\Rightarrow F''(u_0) = d^2F(u_0) \in \mathcal{L}_2(X,Y)$ is symmetric

i.e. $d^2F(u_0)[h, \tilde{h}] = d^2F(u_0)[\tilde{h}, h] \quad \forall h, \tilde{h} \in X$

proof [AP, Thm 3.4]

Analogously define higher order derivatives

Given $F : \mathcal{U} \to Y$ n-times diff in $\mathcal{U}$

$$d^nF(u^*) \in \mathcal{L}_n(X,Y) = \left\{ \begin{array}{l} n\text{-linear bounded maps} \\ \underbrace{X \times \dots \times X}_{n\text{-times}} \to Y \end{array} \right\}$$

Put $F^{(n)} : \mathcal{U} \longrightarrow \mathcal{L}_n(X,Y)$
$u \longmapsto F^{(n)}(u) := d^nF(u)$

the $(n+1)$th diff of F at u^* is the diff at u^* of $F^{(n)}$, i.e.

$$d^{n+1}F(u^*) = dF^{(n)}(u^*) \in \mathcal{L}(X, \mathcal{L}_n(X,Y)) \cong \mathcal{L}_{n+1}(X,Y)$$

Def $F \in C^n(\mathcal{U}, Y)$ if F is n-times diff. in $\mathcal{U}$ and n-th differential

$$F^{(n)} : \mathcal{U} \to \mathcal{L}_n(X, Y)$$
$$u \longrightarrow F^{(n)}(u) := d^n F(u)$$

is continuous

Write $d^n F(u^*) [h_1, \dots, h_n]$

Prop $F : \mathcal{U} \to Y$ is n-times diff in $\mathcal{U}$, then

$$(h_1, \dots, h_n) \to d^n F(u^*) [h_1, \dots, h_n] \quad \text{symmetric}$$

proof [AP, thm 3.5]

Def $F \in C^\infty(\mathcal{U}, Y)$ if $F \in C^n(\mathcal{U}, Y)$ $\forall n$

Taylor formula $F \in C^n(\mathcal{U}, Y)$. Let $u, u+v \in \mathcal{U}$ with $[u, u+v] = \{ tu + (1-t)(u+v) \} \subset \mathcal{U}$

Then

$$F(u+v) = F(u) + dF(u)[v] + \frac{1}{2} d^2F(u)[v,v] + \dots$$
$$+ \frac{1}{(n-1)!} d^{n-1} F(u)[v, \dots, v]$$
$$+ \underbrace{\frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} d^n F(u+tv) [v, \dots, v] \, dt}_{R^n(u,v)}$$

and $\quad \| R^n(u,v) \| \leq \frac{\|v\|^n}{n!} \sup_{t\in[0,1]} \| d^n F(u+tv) \|_{\mathcal{L}_n(X,Y)}$

proof $\quad \gamma : [0,1] \to X \quad , \quad \gamma(t) = u+tv$

put $\quad \phi : [0,1] \to Y \quad , \quad \phi(t) = F(\gamma(t))$

Then $\quad \phi \in C^n$ and $\quad \phi'(t) = (dF)(u+tv)[v]$

$$\vdots$$

$$\phi^{(n)}(t) = (d^n F)(u+tv)[v, \dots, v]$$

Check that

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2}\phi''(0) + \dots + \frac{1}{(n-1)!}\phi^{(n-1)}(0)$$

$$+ \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} \phi^{(n)}(t)\, dt$$

To prove this ↑, e.g. iterate

$$\phi(1) - \phi(0) = \int_0^1 \phi'(t)\, dt = \int_0^1 \left(\phi'(0) + \int_0^t \phi''(t_1)\, dt_1\right) dt$$

$$= \phi'(0) + \int_0^1 \int_0^t \phi''(t_1)\, dt_1\, dt$$

$$= \phi'(0) + \frac{1}{2}\phi''(0) + \dots + \frac{1}{(n-1)!}\phi^{(n-1)}(0) + \underbrace{\int_0^1 \int_0^t \int_0^{t_{n-1}} \phi^{(n)}(t_n)\, dt_n \dots dt_1\, dt}_{\text{change order of integ.}}$$

If $F \in C^\infty(\mathcal{U}, X)$ we might expand formally

$$F(u+v) \text{ " = " } \sum_{n=0}^{\infty} \frac{d^n F(u)[v, \dots, v]}{n!}$$

If $\exists\, r > 0$ s.t. $\quad \sup_{\|v\| \leq r} \left\| F(u+v) - \sum_{\ell=0}^{n} \frac{d^\ell F(u)[v, \dots, v]}{\ell!} \right\| \xrightarrow{n\to\infty} 0$

Then $F(u)$ converges uniformly in $B_r(u)$ to its ∞-Taylor series

$\rightsquigarrow$ F is real analytic

Ex $F \in C^\infty(\mathcal{U}, Y)$, $\exists\ r, C, R > 0$ st.

$$\sup_{\|v\| \leq r} \| d^n F(u+v) \|_{\mathcal{L}_n(X, Y)} \leq \frac{C\, n!}{R^n}$$

$\Rightarrow$ F is analytic at u

Indeed $\displaystyle \| R^n(u,v) \| \leq \frac{\|v\|^n}{n!} \sup_{t \in [0,1]} \| d^n F(u+tv) \|$

hence for $\rho < r$

$$\sup_{\|v\| \leq \rho} \| R^n(u;v) \| \leq \frac{\rho^n}{n!} C \frac{n!}{R^n} \leq C \left(\frac{\rho}{R} \right)^n \xrightarrow{n \to \infty} 0$$

provided $\rho < R$

Partial derivatives

Def $F : \mathcal{U} \subseteq X \times Y \to Z$. F is diff w.r.t. x at (x_0, y_0) if $\exists\ A_{x_0} \in \mathcal{L}(X, Z)$ st.

$$\frac{\| F(x_0 + h, y_0) - F(x_0, y_0) - A_{x_0} h \|}{\| h \|} \xrightarrow{\|h\| \to 0} 0$$

If this is the case, put $d_x F(x_0, y_0) := A_{x_0} \in \mathcal{L}(X, Y)$

Same def for $d_y F(x_0, y_0)$

Rem if F is diff $\Rightarrow$ F has partial derivatives

Indeed let $\sigma_{y_0} : X \to \mathcal{U}$

$x \longmapsto (x, y_0)$

and consider $F(\sigma_{y_0}(x)) : X \to Z$,

Clearly $F(\sigma_{y_0}(x)) = F(x, y_0)$, by chain rule

$$d(F \circ \sigma_{y_0})(x_0)[h] = dF(\sigma_{y_0}(x_0))\, d\sigma_{y_0}(x_0)[h]$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$d_x F(x_0, y_0)[h] \qquad\qquad dF(x_0, y_0)\big[(h, 0)\big]$$

As in the classical case, also converse is true

Prop $F : \mathcal{U} \subseteq X \times V \to Z$, $(u_0, v_0) \in \mathcal{U}$

Assume $d_u F(u,v)$ and $d_v F(u,v)$ exist $\forall (u,v) \in \mathcal{U}$ and $d_u F$, $d_v F$ are continuous (just in open set $\mathcal{U}' \ni (u_0, v_0)$, $\mathcal{U}' \subseteq \mathcal{U}$)

$\Rightarrow$ F is diff at (u_0, v_0) and

$$(dF)(u_0, v_0)[h, k] = d_u F(u_0, v_0)[h] + d_v F(u_0, v_0)[k]$$

proof [AP]

Nemitski operators (composition operators)

Consider $\Omega \subseteq \mathbb{R}^n$ open set ; $f : \Omega \times \mathbb{R} \to \mathbb{R}$

Nemitski operators: $u(x) \longrightarrow f(x, u(x))$

es $f(t) = t^2$ $\quad f(u) = u(x)^2$

We want to study their continuity and differentiability properties

⚠ they depend on the topology!

Thm $f \in C^n(\mathbb{R}, \mathbb{R})$. Consider $F : C^0([0,1]) \to C^0([0,1])$, $u \longmapsto f(u)$

then $F \in C^n(C^0, C^0)$ and $\forall m \leq n$

$$(d^m F)(u)[h_1, \dots, h_m] = f^{(m)}(u) \cdot h_1 \cdot \ldots \cdot h_m$$

In particular $dF(u)[h] = f'(u) \cdot h$ (multiplication operator)

proof by induction: Case $m=1$. Show that

(a) F is G-diff. and $d^G F(u)[h] = f'(u) \cdot h$

(b) $d^G F$ continuous

$\Rightarrow$ F Frech. diff and $dF = d^G F$

Check (a):

$$0 \overset{?}{=} \lim_{t \to 0} \left\| \frac{F(u+th) - F(u)}{t} - f'(u) \cdot h \right\|_{C^0}$$

$$= \lim_{t \to 0} \sup_{x \in [0,1]} \left| \frac{f(u(x) + th(x)) - f(u(x))}{t} - f'(u(x))\, h(x) \right|$$

$$= \lim_{t \to 0} \sup_{x \in [0,1]} \left| \frac{1}{\cancel{t}} \int_0^1 f'(u(x) + sth(x))\, \cancel{t} h(x)\, ds - \underbrace{f'(u(x))\, h(x)}_{\int_0^1 f'(u(x)) \cdot h(x)\, ds} \right|$$

$$= \lim_{t \to 0} \sup_{x \in [0,1]} \left| \int_0^1 \left(f'(u(x) + sth(x)) - f'(u(x)) \right) \cdot h(x)\, ds \right|$$

$$\leq \lim_{t \to 0} \sup_{x} \int_0^1 \left| f'(u(x) + sth(x)) - f'(u(x)) \right| ds \; \|h\|_{C^0}$$

As $u, h \in C^0 \rightsquigarrow \|u\|_\infty + \|h\|_\infty \leq C$

$f \in C^1 \Rightarrow f' \in C^0$ is uniformly continuous over $[-C, C]$

$\rightsquigarrow \forall \varepsilon > 0, \exists \delta : \; |f'(z) - f'(w)| \leq \varepsilon \quad \forall \; |z - w| \leq \delta$

Choose ε, take the corresponding δ and use that, for

$$|u(x) + st\, h(x) - u(x)| \leq t \|h\|_{C^0} \leq \delta$$

(which is true provided $t \leq \frac{\delta}{\|h\|_{C^0}}$)

$$\int_0^1 | f'(u(x) + st\, h(x)) - f'(u(x)) | \, ds \, \|h\|_{C^0} \leq \varepsilon \|h\|_{C^0}$$

and this is true uniformly in x

In particular, $\overline{\lim_{t \to 0}} \sup_x \int_0^1 \cdots \|h\|_{C^0} \leq \varepsilon \|h\|_{C^0}$

Since ε is arbitrary, the $\lim\limits_{t \to 0} = 0$

(b) $$\| d^G F(u) - d^G F(v) \|_{\mathcal{L}(C^0, C^0)} = \sup_{\|h\|_{C^0} \leq 1} \| (f'(u) - f'(v)) \cdot h \|_{C^0}$$

$$\leq \| f'(u) - f'(v) \|_{C^0} \leq \sup_x \| f'(u(x)) - f'(v(x)) \|$$

f' uniformly cont on compact set $\|u\|_{C^0} + \|v\|_{C^0} \leq C$ $\leq \varepsilon$ provided $|u(x) - v(x)| \leq \|u - v\|_{C^0} \leq \delta$

As ε is arbitrary, we get the claim

For higher derivatives one argues analogously $\square$

Other interesting case: $F : L^p \to L^2$

start with continuity

Prop Let $p, r \geq 1$ and $f \in C^0(\mathbb{R}, \mathbb{R})$ st.

$$|f(t)| \leq C (1 + |t|^{p/r})$$

Then $F: L^p([0,1]) \longrightarrow L^2([0,1])$, $u \longmapsto f(u)$

is well defined and continuous.

<u>proof</u> take $u \in L^p$

•) well defined: $\int_0^1 |f(u(x))|^2 dx \leq \int_0^1 c\left(1+|u(x)|^{p/2}\right)^2$

$\overset{2\geq 1}{\leq} C_2 \int_0^1 (1+|u|^p)$ ✓

•) continuity: $u_n \xrightarrow{L^p} u \overset{?}{\Longrightarrow} f(u_n) \xrightarrow{L^2} f(u)$

up to subseq: $u_n \to u$ a.e. $\Rightarrow f(u_n) \to f(u)$ a.e.

and $\exists C_2 > 0$: $|f(u_k)|^2 \leq C_2(1+|u_k|^p)$ $\forall k$

Apply the following:

Thm (Brezis, Thm 4.9) $\{u_n\} \subset L^p([0,1])$, $u \in L^p$ st $u_n \to u$ in L^p. Then $\exists \{u_{n_k}\}$ subseq and $h \in L^p$ st

•) $u_{n_k} \to u$ a.e.

•) $|u_{n_k}| \leq h$ a.e.

We get

$\Rightarrow$ $|f(u_{n_k})|^2 \leq c(1+|h|^p)$

Now use dominated convergence: $\int \underbrace{|f(u_{n_k})}_{\lesssim (1+|h|^p)} - \underbrace{f(u)}_{\lesssim 1+|u|^p}|^2 \longrightarrow 0$

Then upgrade to whole seq in the usual way ☒

(assume $\exists (\tilde{u}_n)_n$ s.t. $f(\tilde{u}_n) \to v_* \neq f(u)$
extract a subseq from $(\tilde{u}_n)$, but then for this sub $f(\tilde{u}_{n_k}) \to f(u)$)

Prop $f \in C^1(\mathbb{R},\mathbb{R})$ s.t.

1) $|f(t)| \le C(1+|t|^{p/r})$

2) $|f'(t)| \le C(1+|t|^{\frac{p}{r}-1})$

with $p > r \ge 1$

$\Rightarrow F: L^p \to L^r$ differ. and

$$dF(u)[h] = f'(u)\cdot h$$

proof Check (a) F is G-diff, $d^G F(u)[h] = f'(u)\cdot h$

(b) $u \to d^G F(u)$ is continuous

Safety check: $d^G F \in \mathcal{L}(L^p, L^r)$ if bounded, ✓

$$\| d^G F(u)[h] \|_{L^r} = \| f'(u)\cdot h \|_{L^r} \le \overbrace{\| f'(u) \|_{L^q}}^{?} \|h\|_{L^p}$$

with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} \rightsquigarrow q = \frac{rp}{p-r}$

$$\rightsquigarrow \int |f'(u)|^{\frac{rp}{p-r}} dx \le C \int \left(1+|u(x)|^{\frac{p-r}{r}}\right)^{\frac{rp}{p-r}}$$

$$\le C_{r,p} \int 1 + |u(x)|^p < +\infty$$

$|f'| \le C(1+|t|^{\frac{p-r}{rp}p})$

$f': L^p \to L^{\frac{rp}{p-r}}$

Moreover, from previous lemma

(*) $u \mapsto f'(u)$ continuous from $L^p \to L^q$, $q = \frac{pr}{p-r}$

(a) $$0 \overset{?}{=} \lim_{t\to 0} \left\| \frac{f(u+th)-f(u)}{t} - f'(u)\cdot h \right\|_{L^r}$$

Call $g_t(x) = \dfrac{f(u(x)+t\,h(x)) - f(u(x))}{t} - f'(u)\cdot h$

Since $f \in C^1 \Rightarrow g_t(x) \xrightarrow{t\to 0} 0$ a.e. (check it)

to confirm a) need to prove $\exists\, z \in L^2$: $|g_t(x)| \le z(x)\ \forall t$
then use dominated convergence

Since $f'(u)\cdot h \in L^2$, it is enough to control only first term

$$\left| \frac{f(u(x)+t\,h(x)) - f(u(x))}{t} \right| \le \int_0^1 |f'(u(x)+ts\,h(x))|\,|h(x)|\,ds$$

$$\le C \int_0^1 |h(x)| \left(1 + |u(x) + ts\,h(x)|^{\frac{p-2}{2}}\right) ds$$

$(x+y)^s \le x^s + y^s$ for $x,y \ge 0$ and $s \in [0,1]$

$$\le C \int_0^1 |h(x)| \left(1 + |u(x)|^{\frac{p-2}{2}} + |s\,h(x)|^{\frac{p-2}{2}}\right) ds$$

$$\le C \left(|h(x)| + |h|\,|u|^{\frac{p-2}{2}} + |h|^{p/2} \right)$$

Since $h \in L^p$, $p > 2 \Rightarrow$ $|h| \in L^2$

$|h|^{p/2} \in L^2$

by Hölder: $|h|\,|u|^{\frac{p-2}{2}} \in L^2$

$\Rightarrow g_t(x) \to 0$ in L^2 $\Rightarrow d^G F(u)[h] = f'(u)h$

b) $\| d^G F(u) - d^G F(v) \|_{\mathcal{L}(L^p, L^2)} \xrightarrow[?]{u \to v \text{ in } L^p} 0$

$$\sup_{\|h\|_{L^p} \le 1} \| f'(u)h - f'(v)h \|_{L^2} \le \| f'(u) - f'(v) \|_{L^q} \sup_{\|h\|_{L^p} \le 1} \|h\|_{L^p}$$

$$\le \| f'(u) - f'(v) \|_{L^q} \xrightarrow{u \to v \text{ in } L^p} 0 \quad \text{by } (*)$$

$\boxtimes$