FREDHOLM THEORY

Motivation study: $(\mathbb{I} - T) f = g$

In case T is compact, we can prove

Thm (FREDHOLM ALTERNATIVES) $T \in K(X)$,

then (a) $\ker(\mathbb{I} - T)$ is fin dim

(b) $\mathrm{Im}(\mathbb{I} - T)$ is closed and $\mathrm{Im}(\mathbb{I}-T) = {}^{\perp}\underbrace{\ker(\mathbb{I}-T^*)}_{\subset X^*}$

(c) $\ker(\mathbb{I}-T) = \{0\} \iff \mathrm{Im}(\mathbb{I}-T) = X$

(d) $\dim \ker(\mathbb{I}-T) = \dim \ker(\mathbb{I}-T^*)$

=

Solve $u - Tu = g$ $\rightsquigarrow$ study $u - Tu = 0$

two cases

only the trivial solution $u=0$

$\Downarrow$ c)

$\forall g, \exists!\, u \in X : u - Tu = g$

$\exists$ non trivial solution of $u - Tu = 0$

$\Downarrow$ (b)

$\exists\, u : u - Tu = g$

$\Updownarrow$

$(J_X g)(v) = v(g) = 0 \ \forall\, v \in \ker(\mathbb{I}-T^*)$

$\rightsquigarrow$ finitely many constraints

Recall: $A \subseteq X$ lin sub: $A^{\perp} = \{f \in X^* : f(x) = 0 \ \forall x \in A\}$

$A \subseteq X^*$ lin sub: ${}^{\perp}A = \{x \in X : (J_X x)(f) = 0 \ \forall f \in A\}$

PRELIMINARY RESULTS TOWARDS FREDHOLM

Prop 1 $T \in \mathcal{L}(X,Y)$, then

$\exists c>0 : \quad \|Tx\| \geq c \operatorname{dist}(x, \ker T) \iff \operatorname{Im} T$ closed

$$\operatorname{dist}(x,\ker T) = \inf_{\xi : T\xi = 0} \|x - \xi\|$$

proof is based on the analysis of $T|_{X/\ker T}$

$\rightsquigarrow$ few words about quotient spaces.

Lemma If $X_0 \subset X$ closed subsp, X normed

$\Rightarrow$ X/X_0 can be equipped with the norm

$x \sim y \iff x - y \in X_0$

$$\|[x]\| := \inf_{y \in X_0} \|x - y\|$$

Notice that X_0 closed is needed only to prove $\|[x]\| = 0 \Rightarrow [x] = 0$

EXERCISE: prove it!

Q: What about completeness? X Banach $\overset{?}{\Rightarrow}$ X/X_0 is Banach

We need

Lemma $(Z, \|\ \|_Z)$ normed space

$(Z, \|\ \|_Z)$ Banach $\iff$ $\forall \{x_n\} : \sum_{k=1}^{\infty} \|x_k\| < \infty$

$\Downarrow$

$\left\{ \sum_{k \leq n} x_k \right\}$ converges as $n \to \infty$

EXERCISE prove it!

Lemma Let $X_0 \subseteq X$ closed subspace and X Banach
Then $X/_{X_0}$ is Banach

proof Take $\{[x_k]\}_{k\geq 1} \subseteq X/_{X_0}$ s.t. $\sum_{k=1}^{\infty} \|[x_k]\|_{X/X_0} < +\infty$

by previous lemma, if $\sum_{k\leq n} [x_k]$ converges in $X/_{X_0} \Rightarrow X/_{X_0}$ Banach

By definition of $\|[x_k]\|$, $\forall k \ \exists\ e_k \in X_0$ st.

$$\|x_k - e_k\|_X \leq \|[x_k]\|_{X/X_0} + 2^{-k}$$

$$\Rightarrow \sum_k \|x_k - e_k\|_X \leq \sum \|[x_k]\|_{X/X_0} + \sum 2^{-k} < \infty \quad (*)$$

X Banach

$$\Longrightarrow \exists\ \tilde{x} = \lim_{n\to\infty} \sum_{k\leq n} (x_k - e_k)$$

Consider $[\tilde{x}]$, we have

$$\Big\| [\tilde{x}] - \sum_{k=1}^{N} [x_k] \Big\|_{X/X_0} = \Big\| \big[\tilde{x} - \sum_{k=1}^{N} x_k\big] \Big\|_{X/X_0}$$

$\sum e_k \in X_0$

$$\leq \Big\| \tilde{x} - \sum_{k=1}^{N} (x_k - e_k) \Big\|_X$$

$$\leq \sum_{k=N+1}^{\infty} \|x_k - e_k\|_X \xrightarrow[(*)]{n\to\infty} 0$$

$\rightsquigarrow \sum_{k=1}^{N} [x_k]$ converges $\Rightarrow X/_{X_0}$ Banach

$\square$

proof of proposition

(i) $\|Tx\| \geq c \ \mathrm{dist}(x, \ker T) \Rightarrow \operatorname{Im} T$ closed proof $X/\ker T$ Banach space as $\ker T$ is closed

Moreover $\|[x]\|_{X/\ker T} = \inf\limits_{\xi\in\ker T}\|x-\xi\| = \mathrm{dist}(x,\ker T)$ (**)

Define the op $\tilde{T}$ from the following diagram

$$
\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "\pi_0"'] & Y \\
X/\ker T \arrow[ur, "\tilde{T}"'] &
\end{tikzcd}
$$

$\tilde{T}[x] = Tx$ well defined

as $x\sim y$ ($\Leftrightarrow x-y\in\ker T$)

$\rightsquigarrow \tilde{T}[y] = Ty = Tx + \underset{\substack{\| \\ 0}}{T(y-x)}$

$= Tx = \tilde{T}[x]$

and $\|\tilde{T}[x]\|_Y = \|Tx\|_Y \overset{\text{assumption}}{\geq} c\ \mathrm{dist}(x,\ker T)$

$\overset{(**)}{\geq} c\,\|[x]\|_{X/\ker T}$

$\Rightarrow \operatorname{Im}\tilde{T}$ is closed, $\Rightarrow \operatorname{Im}\tilde{T} = \operatorname{Im}T$ closed

general fact: A continuous $\|Ax\| \geq c\|x\| \quad \forall x\in X \Rightarrow \operatorname{Im}A$ closed

Indeed take $(y_m)_{m\geq 1} \subseteq \operatorname{Im}A$, $y_m \to y$

Then $y_m = Ax_m$ for some $x_m\in X$

$$\|y_m - y_n\| = \|A(x_m-x_n)\| \geq c\,\|x_m - x_n\|$$

$\downarrow m,n\to\infty$

0

$\rightsquigarrow \{x_m\}$ is Cauchy $\Rightarrow x_m \to x \Rightarrow \underset{\substack{\| \\ y_m}}{Ax_m} \to \underset{\substack{\| \\ y}}{Ax}$

(i) $\operatorname{Im} T$ closed $\Rightarrow$ $\|Tx\| \geq c \operatorname{dist}(x, \ker T)$

Consider again $\tilde{T} : \underbrace{X/_{\ker T}}_{\text{Banach}} \to \underbrace{\operatorname{Im} T}_{\text{Banach}}$ (closed sub of Banach)

$\tilde{T}$ is bd op.

injective $\left(\tilde{T}[x] = 0 \Rightarrow Tx = 0 \Rightarrow x \in \ker T \right)$ $\Rightarrow [x] = 0$

surjective (trivial)

open mapping thm $\rightsquigarrow$ $\tilde{T}$ is invertible with a bd inverse!

$\Rightarrow \quad \|\tilde{T}^{-1}(y)\|_{X/_{\ker T}} \leq c \|y\| \quad \forall y \in \operatorname{Im} T$

$y = Tx$ $\Rightarrow$ $\|\tilde{T}^{-1}(Tx)\|_{X/_{\ker T}} \leq c \|Tx\| \quad \forall x \in X$

$\|\tilde{T}^{-1}(Tx)\|_{X/_{\ker T}} = \|[x]\|_{X/_{\ker T}}$

$\square$

ANNIHILATORS AND PRE-ANNIHILATORS

Def $A \subseteq X$ lin sub $\quad$ $A^{\perp} = \{ f \in X^* : f(x) = 0 \ \forall x \in A \}$ ANNIHILATOR

$A \subseteq X^*$ lin sub $\quad$ ${}^{\perp}A = \{ x \in X : \underbrace{(J_X x)(f) = 0}_{f(x) = 0} \ \forall f \in A \}$

Lemma (i) $A \subseteq X$ lin sub $\Rightarrow$ ${}^{\perp}(\underbrace{A^{\perp}}_{\subseteq X^*}) = \overline{A}$

(ii) $A \subseteq X^*$ " " $\Rightarrow$ $(\underbrace{{}^{\perp}A}_{\subseteq X})^{\perp} = \overline{A}^{\sigma(X^*, X)}$ (closure in weak $*$ topology)

proof only (i): $x \in \overline{A} \iff f(x) = 0 \quad \forall f \in A^\perp$

$\Rightarrow)$ $\exists (x_n) \subseteq A$ with $x_n \to x$. then $\forall f \in A^\perp$

$$\underbrace{f(x_n)}_{\substack{\shortparallel \\ 0 \ \forall n}} \to f(x) \quad \Rightarrow \quad f(x) = 0 \quad \forall f \in A^\perp$$

$\Leftarrow)$ BC $x \notin \overline{A}$. By Hahn-Banach $\exists f \in X^*$:

$$\begin{cases} f(y) = 0 & \forall y \in A \quad (\Rightarrow f \in A^\perp) \\ f(x) > 0 \end{cases}$$

$\lightning$

$\square$

Lemma (Identities) $T \in \mathcal{L}(X, Y)$, X, Y Banach. Then

(I) $(\operatorname{Im} T)^\perp = \ker T^* \quad (= T')$ (in Y^*)

(II) $\ker T = {}^\perp(\operatorname{Im} T^*)$ (in X)

(III) ${}^\perp(\ker T^*) = \overline{\operatorname{Im} T}$ (in Y)

(IV) $(\ker T)^\perp = \operatorname{Im} T^*$ if $\operatorname{Im} T$ closed (in X^*)

Rem trivial in Hilbert spaces, where $A^\perp \equiv$ usual orthogonal set

example if $y \in (\operatorname{Im} T)^\perp \iff \langle y, Tx \rangle = 0 \quad \forall x$

$$\langle y, Tx \rangle = \langle T^* y, x \rangle \quad \Rightarrow \quad y \in \ker T^*$$

Ex prove the lemma in Hilbert space using scalar product

proof (I) $f \in (\operatorname{Im} T)^\perp \iff f(Tx) = 0 \quad \forall x \in X$

$\iff (T^* f)(x) = 0 \quad \forall x \in X$

$\iff T^* f = 0$ in $X^* \iff f \in \ker T^*$

(II) $x \in \ker T \iff Tx = 0$ in Y

$\iff f(Tx) = 0 \quad \forall f \in Y^*$

$\iff (T^* f)(x) = 0 \quad \forall f \in Y^*$

$\iff (J_X x)(T^* f) = 0 \quad \forall f \in Y^*$

$\iff x \in {}^\perp(\operatorname{Im} T^*)$

(III) we know by (I) $(\operatorname{Im} T)^\perp = \ker T^*$

$${}^\perp\big((\operatorname{Im} T)^\perp\big) = {}^\perp(\ker T^*)$$

$$\underset{\text{lemme}}{\parallel}$$

$$\overline{\operatorname{Im} T}$$

(IV) ($\subseteq$) $x^* \in \operatorname{Im} T^* \Rightarrow x^* = T^* y^*$ for some $y^* \in Y^*$

$\Rightarrow x^*(x) = (T^* y^*)(x) = y^*(Tx)$

$\Rightarrow \forall x \in \ker T$ we have $x^*(x) = 0$

$\Rightarrow x^* \in (\ker T)^\perp$

($\supseteq$) Let $x^* \in (\ker T)^\perp \Rightarrow x^*(\xi) = 0 \quad \forall \xi \in \ker T$

•) $\tilde{x}^* : \dfrac{X}{\ker T} \to \mathbb{C}$,

$[x] \longrightarrow \tilde{x}^*[x] := x^*(x)$

is well defined and continuous

Indeed

$|\tilde{x}^*[x]| = |x^*(x)| = |x^*(x - \xi)| \quad \forall \xi \in \ker T$

$\leq \|x^*\| \, \|x - \xi\| \quad \forall \xi \in \ker T$

$\rightsquigarrow |\tilde{x}^*[x]| \leq \|x^*\| \inf_{\xi \in \ker T} \|x - \xi\| \leq \|x^*\| \, \|[x]\|_{X/\ker T}$

Lemma

$\cdot)$ $\operatorname{Im} T$ closed $\Rightarrow$ $\tilde{T} : \frac{X}{\ker T} \longrightarrow \operatorname{Im} T$

is invertible with bd inverse $\tilde{T}^{-1}$

Put $\quad \psi : \operatorname{Im} T \longrightarrow \mathbb{C}$

$$y \longmapsto \psi(y) := \tilde{x}^*(\tilde{T}^{-1} y)$$

ψ is linear and continuous (composition of continuous maps)
and $\operatorname{Im} T$ closed

By Hahn-Banach, $\exists\, y^* : Y \to \mathbb{C}$ st $y^*|_{\operatorname{Im} T} = \psi$

Claim $\quad T^* y^* = x^*$ in X^*

indeed

$$(T^* y^*)(x) = y^*(Tx) \underset{y^*|_{\operatorname{Im} T} = \psi}{=} \psi(Tx)$$

$$= \tilde{x}^*(\tilde{T}^{-1} T x) = \tilde{x}^*[x]$$

$$= x^*(x) \qquad \forall x \in X$$

$\square$

Lemma (Riesz) X normed vector space and $M \subset X$, $M \neq X$ closed linear space.

Then $\forall \varepsilon > 0$, $\exists\, u \in X$: $\begin{cases} \|u\| = 1 \\ \operatorname{dist}(u, M) \geq 1 - \varepsilon \end{cases}$

proof Take $v \in X \setminus M$. Since M is closed

$$d := \operatorname{dist}(v, M) > 0$$

$$= \inf_{m \in M} \|v - m\|$$

Choose $m_0 \in M$ with

$$d \leq \|v - m_0\| \overset{(*)}{\leq} \frac{d}{1-\varepsilon}$$

Now take $u = \dfrac{v - m_0}{\|v - m_0\|}$; it fulfills the properties

Indeed $\|u\| = 1$ and moreover $\forall\, m \in M$

$$\|u - m\| = \left\| \frac{v - m_0}{\|v - m_0\|} - m \right\| = \left\| \frac{v - \overbrace{(m_0 - m\|v - m_0\|)}^{\in M}}{\|v - m_0\|} \right\|$$

$$\overset{(*)}{\geq} d \, \frac{(1-\varepsilon)}{d} = 1 - \varepsilon$$

$\square$

Cor X Banach with $B_1^X(0)$ is compact then $\dim X < +\infty$

proof B.C. $\dim X = +\infty$. Take $E_1 \subsetneq E_2 \subsetneq E_3 \subsetneq \dots$ seq of nested fin dim sub. Apply Riesz, $\exists\, (u_n)_{n \geq 1}$ with $u_n \in E_n$ and $\operatorname{dist}(u_n, E_{n-1}) \geq 1/2$.

Then $\|u_n - u_m\| \geq 1/2$ $\forall\, m < n$, so $(u_n)_{n \geq 1} \subseteq B_1^X(0)$ but no convergent sub seq. $\lightning$ $\square$

PROOF OF FREDHOLM THEOREM

(a) $\ker(\mathbb{1}-T)$ fin dim

Clearly $Tv = v \quad \forall\, v \in \ker(\mathbb{1}-T)$

Putting

$$X_1 := \ker(\mathbb{1}-T) \subseteq X \quad \text{then} \quad T|_{X_1} = \mathbb{1}_{X_1}$$

but then $\overline{B_1^{X_1}(0)} = \overline{T\, B_1^{X_1}(0)} \subseteq \overline{T\, B_1^{X}(0)}$ $\rightsquigarrow$ compact

$\rightsquigarrow$ $\overline{B_1^{X_1}(0)}$ is compact $\Rightarrow \dim X_1 < +\infty$

(b) (b₁) $\mathrm{Im}(\mathbb{1}-T)$ closed

By previous prop, it is enough to show that $\exists c > 0$:

$$\|(\mathbb{1}-T)v\| \geq c \,\mathrm{dist}(v, \ker(\mathbb{1}-T))$$

B.C. assume $\exists\, (u_j)_{j\geq 1}$ s.t. $(*)$

$$\begin{cases} \mathrm{dist}(u_j, \ker(\mathbb{1}-T)) = 1 \quad \forall j \\ \|(\mathbb{1}-T)u_j\| \leq \dfrac{1}{j} \xrightarrow{j\to\infty} 0 \quad (1) \end{cases}$$

CASE I $(u_j)_{j\geq 1}$ is bd, $\|u_j\| \leq M \quad \forall j$.

$(Tu_j)_{j\geq 1}$ has conv. subseq (T is compact)

$$(2) \qquad Tu_{j_n} \to u_\infty \qquad n\to\infty$$

Then also $u_{j_n} \to u_\infty$, indeed

$$\|u_{j_n} - u_\infty\| \leq \underbrace{\|u_{j_n} - Tu_{j_n}\|}_{\substack{(\mathbb{1}-T)u_{j_n} \\ \downarrow \\ 0 \text{ by } (1)}} + \underbrace{\|Tu_{j_n} - u_\infty\|}_{\substack{\downarrow \text{ by } (2) \\ 0}} \xrightarrow{n\to\infty} 0$$

T bd: $\quad Tu_{n_j} \to Tu_\infty$
$\downarrow \qquad //$
u_∞

$\Rightarrow$ $u_\infty - Tu_\infty = 0$ $\Downarrow$ $u_\infty \in \ker(\mathbb{1} - T)$ ↯
contradicts (✱)

CASE II $(u_j)_{j\geq 1}$ not bd. By (✱),

$\forall j, \exists z_j \in \ker(\mathbb{1}-T): \quad \|u_j - z_j\| \leq 1 + \frac{1}{j}$

Put $w_j := u_j - z_j$.
Want to show that $(w_j)_j$ bd and fulfills (✱) & (1)
$\rightsquigarrow$ back to case I $\rightsquigarrow$ contradiction ↯

(•) $\|w_j\| \leq 2 \quad \forall j$

(1) $\|(\mathbb{1}-T)w_j\| = \|(\mathbb{1}-T)u_j - \underbrace{(\mathbb{1}-T)z_j}_{=0}\| = \|(\mathbb{1}-T)u_j\| \xrightarrow{j\to\infty} 0$

(✱) $\mathrm{dist}(w_j, \ker(\mathbb{1}-T)) = \inf\limits_{z \in \ker(\mathbb{1}-T)} \|\overset{u_j - z_j}{w_j} - z\|$

$= \inf\limits_{z \in \ker(\mathbb{1}-T)} \|u_j - z\| = 1 \quad \forall j$

(b₂) $\underline{\mathrm{Im}(\mathbb{1}-T) = {}^\perp(\ker(\mathbb{1}-T^*))}$

Lemma Identities II

$\overline{\mathrm{Im}(\mathbb{1}-T)} = {}^\perp \ker(\mathbb{1}-T^*)$
$\| \text{(b1)}$
$\mathrm{Im}(\mathbb{1}-T)$

(c) $\underline{\ker(\mathbb{1}-T)=0 \iff \operatorname{Im}(\mathbb{1}-T)=X}$

$\Leftarrow$) BC $\ker(\mathbb{1}-T) = N_1 \neq 0$

Construct $N_k := \ker(\mathbb{1}-T)^k$

Clearly $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots\ldots$

$x \in N_1 \Rightarrow (\mathbb{1}-T)^2 x = (\mathbb{1}-T)\underbrace{(\mathbb{1}-T)x}_{=0} = 0$

We show now that the inclusions are strict

$N_1 \neq 0 \Rightarrow \exists\, x_1 \neq 0 : (\mathbb{1}-T)x_1 = 0$

$\operatorname{Im}(\mathbb{1}-T) = X \Rightarrow \exists\, x_2 \neq 0 : (\mathbb{1}-T)x_2 = x_1$

$\rightsquigarrow (\mathbb{1}-T)^2 x_2 = (\mathbb{1}-T)(\mathbb{1}-T)x_2$

$= (\mathbb{1}-T)x_1 = 0$

$\rightsquigarrow x_2 \in N_2 \setminus N_1$

$\operatorname{Im}(\mathbb{1}-T) = X \Rightarrow \exists\, x_3 \neq 0 ! (\mathbb{1}-T)x_3 = x_2$

$\rightsquigarrow x_3 \in N_3 \setminus N_2$

Thus iterating we have

$N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq N_4 \subsetneq \ldots$ closed

We apply Riesz lemma with $X = N_{k+1}$, $M = N_k$

$\rightsquigarrow$ given N_k, $\exists\, y_{k+1} \in N_{k+1}$: $\left| \begin{array}{l} \|y_{k+1}\| = 1 \\ \operatorname{dist}(y_{k+1}, N_k) \geq 1/2 \end{array} \right.$

We shall show that $\{Ty_n\}$ does not contain a convergent subseq, contradicts compactness of T

Indeed take $m < n \in \mathbb{N}$

$$Ty_n - Ty_m = y_n - y_n + Ty_n - y_m + y_m - Ty_m$$

$$= y_n - \underbrace{(\mathbb{1} - T)y_n}_{} - y_m + (\mathbb{1} - T)y_m$$

$y_n \in N_n \Rightarrow (\mathbb{1}-T)^{n-1}(\mathbb{1}-T)y_n = 0$

$\Rightarrow (\mathbb{1}-T)y_n \in N_{n-1}$

$\in N_m \subset N_{n-1}$

$\in N_{m-1} \subset N_{n-1}$

$$= y_n + \overbrace{N_{n-1}}^{\text{an element of } N_{n-1}}$$

Since $y_n \in N_n \Rightarrow \operatorname{dist}(y_n, N_{n-1}) \geq 1/2$

$\Rightarrow \|Ty_n - Ty_m\| \geq \operatorname{dist}(y_n, N_{n-1}) \geq 1/2$

$\Rightarrow (Ty_n)_{n\geq 1}$ cannot have conv. subseq $\lightning$

<u>Rem</u> Actually we proved that if T compact then

$\exists\, i \in \mathbb{N}$ st $N_k = N_i \quad \forall\, k > i$

(assume it is not true and argue as before)

$\Rightarrow)$ Assume $\ker(\mathbb{1}-T) = 0$

we know by (b) that $\text{Im}(\mathbb{1}-T)$ closed.

By Lemma Identities IV : $\left(\ker(\mathbb{1}-T)\right)^{\perp} = \text{Im}(\mathbb{1}-T^*)$

$$\begin{array}{c} \parallel \\ 0^{\perp} = X^* \end{array}$$

T compact $\Rightarrow$ T^* compact as well

By the previous implication $\Rightarrow$ $\ker(\mathbb{1}-T^*) = \{0\}$

By lemma identities III : ${}^{\perp}\left(\ker(\mathbb{1}-T^*)\right) = \overline{\text{Im}(\mathbb{1}-T)}$

$$X = \text{Im}(\mathbb{1}-T) \quad \text{(by (b))}$$

(d) $\underline{\dim\left(\ker(\mathbb{1}-T)\right) = \dim\left(\ker(\mathbb{1}-T^*)\right)}$

By (a) $\dim \ker(\mathbb{1}-T) = n < \infty$

$\dim \ker(\mathbb{1}-T^*) = p < \infty$

$\ker(\mathbb{1}-T) = \text{span}\langle x_1, \dots, x_n\rangle$

$\ker(\mathbb{1}-T^*) = \text{span}\langle f_1, \dots, f_p\rangle$

Consider functionals $\varphi_1, \dots, \varphi_n \in X^*$: $\varphi_j(x_e) = \delta_{ej}$

elements $u_1, \dots, u_p \in X$: $f_j(u_e) = \delta_{ej}$

(exercise: they exist)

<u>CASE 1</u> $\underline{n < p}$ Define $B \in \mathcal{L}(X)$ by

$$Bx = Tx + \sum_{k=1}^{n} \varphi_k(x)\, u_k$$

B compact ($B =$ compact $+$ fin range op)

Step 1 we show $\ker(\mathbb{1} - B) = 0$

Assume it is not true: $\exists\, x_0 \in X$: $Bx_0 = x_0$, $x_0 \neq 0$

Then $\forall\, f_k \in \ker(\mathbb{1} - T^*)$ we have

$$0 = f_k(\underbrace{Bx_0 - x_0}_{=0}) = \underbrace{f_k(Tx_0 - x_0)}_{f_k \in \ker(\mathbb{1} - T^*)} + \sum_{e=1}^{n} \varphi_e(x_0)\, f_k(u_e)$$

$$(**) \quad = 0 + \varphi_k(x_0) \qquad \forall k$$

$$\Rightarrow \quad x_0 = Bx_0 = Tx_0 + \sum_{k=1}^{n} \underbrace{\varphi_k(x_0)}_{=0\ \forall k} u_k = Tx_0$$

$\rightsquigarrow \quad x_0 \in \ker(\mathbb{1} - T)$

$$x_0 = \sum_{e=1}^{n} a_e x_e$$

Apply φ_m to x_0: $\quad \varphi_m(x_0) = \sum_{e=1}^{n} a_e \underbrace{\varphi_m(x_e)}_{=\delta_{me}} = a_m$

$\forall m \quad \varphi_m(x_0) \overset{\text{by } (**)}{=} 0$

So $a_m = 0 \ \forall m \Rightarrow x_0 = 0$ ↯

Step 2 we show that $\ker(\mathbb{1} - B^*) = 0$

$$\ker(\mathbb{1} - B) = 0 \overset{(c)}{\Rightarrow} \operatorname{Im}(\mathbb{1} - B) = X$$

Lemma Identities I $\qquad \operatorname{Im}(\mathbb{1} - B)^\perp = \ker(\mathbb{1} - B^*)$

$$\operatorname{Im}(\mathbb{1} - B)^\perp = X^\perp = 0$$

Let us prove that $\ker(\mathbb{1} - B^*)$ is actually $\neq 0$

step 3 Actually $\ker(\mathbb{1} - B^*) \neq 0$

We exhibit a non trivial element of $\ker(\mathbb{1} - B^*)$.

$$B^* h = T^* h + \sum_{k=1}^{n} h(a_k)\, \varphi_k \qquad \forall\, h \in X^*$$

(exercise : prove it !)

Since $p > n$, the $f_{n+1} \in \ker(\mathbb{1} - T^*)$. Then

$$(\mathbb{1} - B^*) f_{n+1} = \underbrace{(\mathbb{1} - T^*) f_{n+1}}_{= 0} - \sum_{k=1}^{n} \underbrace{f_{n+1}(u_k)}_{= \delta_{n+1,k} = 0} \varphi_k \Rightarrow$$

$$\Rightarrow \quad f_{n+1} \in \ker(\mathbb{1} - B^*) \neq 0 \quad \lightning$$

CASE II $p < n$ Similar as above (exercise)

$\leadsto \quad p = n$

EXAMPLE Try to solve the integral eq $\boxed{\lightning}$

$$f(x) - \int_0^1 e^{x-y} f(y)\, dy = 1$$

Q: $\exists\, f \in L^p[0,1]$, $p \in (1, \infty)$ which solves eq ?

Apply Fredholm alternative, as

$$(Tf)(x) = \int_0^1 e^{x-y} f(y)\, dy$$

$$= e^x \int_0^1 e^{-y} f(y)\, dy$$

T is bd finite range op, bd $\Rightarrow T \in K(L^p)$, $p \in (1,\infty)$

Check the eq: $g - Tg = 0$

Actually $\dim \ker(\mathbb{1} - T) = \dim \ker(\mathbb{1} - T^*)$

$\rightsquigarrow$ check directly $g - T^* g = 0$

$$(T^* g)(x) = e^{-x} \int_0^1 e^y g(y)\, dy \qquad \text{(exercise)}$$

$$, \quad g - T^* g = g - e^{-x} \underbrace{\int_0^1 e^y g(y)\, dy}_{a \in \mathbb{R}} = 0$$

$$\rightsquigarrow \quad g(x) = a\, e^{-x}$$

You can check that $\forall a \in \mathbb{R}$, this is indeed a solution

$$\Rightarrow \quad \ker(\mathbb{1} - T^*) = \{ a e^{-x} \, ; \; a \in \mathbb{R} \} \quad \text{1-dim}$$

Fredholm alt: $\quad \operatorname{Im}(\mathbb{1} - T) = {}^\perp \ker(\mathbb{1} - T^*)$

So $\quad 1 \in \operatorname{Im}(\mathbb{1} - T) \iff 1 \in {}^\perp \ker(\mathbb{1} - T^*)$

$$\iff \underbrace{\int_0^1 1 \cdot (a e^{-x})}_{\neq 0 \text{ (}= a\text{)}} = 0 \quad \forall a \in \mathbb{R}$$

$$\rightsquigarrow \quad 1 \notin {}^\perp \ker(\mathbb{1} - T^*)$$

$\rightsquigarrow$ the eq does not have a solution!