FUNCTIONAL CALCULUS

We have seen how to define $f(A)$ for any function f when A is a compact selfadjoint operator.
We want to extend this construction when A is a general bd selfadjoint operator:

ULTIMATE GOAL: Define $f(A)$ when $f = 1\!\!1_\Omega$, $\Omega \subseteq \mathbb{R}$ borellian subset.
In fact it turns out that $1\!\!1_\Omega(A)$ is $\perp$ projection which generalize the orthog. projections over the space $\ker(A - \lambda I)$.
First step: f continuous

CONTINUOUS FUNCTIONAL CALCULUS

Thm (cont. func. calc) $A \in \mathcal{L}(H)$, $A = A^*$
then $\exists !\ \Phi : \mathcal{C}(\sigma(A)) \longrightarrow \mathcal{L}(H)$

$$f \longmapsto \Phi(f) \equiv f(A)$$

with these properties:

(1) Φ algebraic $*$-homomorphism

$$\phi(fg) = \phi(f)\,\phi(g)$$
$$\phi(\lambda f + \mu g) = \lambda \phi(f) + \mu \phi(g)$$
$$\phi(1) = 1\!\!1$$
$$\phi(\bar{f}) = \phi(f)^*$$

(2) If $f(\lambda) = \lambda \Rightarrow \phi(f) = A$

(3) Φ is isometric:

$$\forall f \quad \|\Phi(f)\|_{\mathcal{L}(H)} = \sup_{\lambda \in \sigma(A)} |f(\lambda)| = \|f\|_{C(\sigma(A))}$$

(4) Spectral mapping $\quad \sigma(f(A)) = f(\sigma(A))$

(5) If $Ax = \lambda x \Rightarrow f(A)x = f(\lambda)x$

(6) If $f \geq 0 \Rightarrow f(A) \geq 0$ (i.e. $\langle f(A)x, x\rangle \geq 0 \ \forall x$)

Scheme of the proof

1. For any polynomial $p \in \mathbb{C}[t]$, define
$$p = \sum c_k t^k$$
$$p(A) = \sum_{k=1}^{n} c_k A^k$$

2. For $A \in \mathcal{L}(H)$, $A = A^*$ prove that

$$\| p(A) \|_{\mathcal{L}(H)} = \sup_{\lambda \in \sigma(A)} |p(\lambda)| = \| p \|_{C(\sigma(A))}$$

$p \mapsto p(A)$ ISOMETRY!

3. $\sigma(A)$ is a compact set of $\mathbb{R}$
By Stone-Weierstrass polynomials are dense in $C(\sigma(A))$
$\rightsquigarrow$ Define $f(A) = \lim_{n\to\infty} p_n(A)$ where $p_n \to f$
and (p_n) are polynomials

We split the proof in several lemmas

Lemma (SPECTRAL MAPPING FOR POLYNOMIALS)
Let $A \in \mathcal{L}(X)$, then $\forall p \in \mathbb{C}[t]$ polynomial

$$\sigma(p(A)) = p(\sigma(A)) = \{ p(\lambda) : \lambda \in \sigma(A) \}$$

<u>Rem</u> It holds in Banach space and not self adj. op.

<u>proof</u> Fix $\lambda \in \mathbb{C}$, consider the polynomial

$$p(t) - \lambda$$

and denote by $\lambda_1, \dots, \lambda_n$ the roots of it. Then

(a) $p(\lambda_i) = \lambda \quad \forall\, i = 1, \dots, n$

(b) $p(t) - \lambda = c\,(t - \lambda_1) \cdots (t - \lambda_n)$

$\Rightarrow \quad p(A) - \lambda \mathbb{I} = c\,(A - \lambda_1) \cdots (A - \lambda_n) \qquad (\ast)$

Let $c \neq 0$ (otherwise $p(t) \equiv \lambda$ const and claim is trivial)

<u>CLAIM</u> $\lambda \in \rho(p(A)) \iff \lambda_i \in \rho(A) \quad \forall i$

$\Leftarrow$) From $(\ast)$, $\prod_j (A - \lambda_j)$ is inv. with bd inverse

so $p(A) - \lambda$ is inv. $\rightarrow \lambda \in \rho(p(A))$

$\Rightarrow$) BC, assume $\exists\, \lambda_i$ so that $A - \lambda_i$ not invertible.
Assume it is λ_1 (otherwise commute the terms)

so $\begin{cases} \ker(A - \lambda_1) \neq 0 & \Rightarrow \ker(A - \lambda_2) \cdots (A - \lambda_n)(A - \lambda_1) \neq 0 \\ \operatorname{Im}(A - \lambda_1) \neq X & \operatorname{Im}(A - \lambda_1)(A - \lambda_2) \cdots (A - \lambda_n) \neq X \end{cases}$

so $\prod (A - \lambda_i)$ not invertible $\Rightarrow \lambda \in \sigma(p(A))$ ↯

By claim we know that

$$\lambda \in \sigma(p(A)) \iff \exists\, i : \lambda_i \in \sigma(A)$$

$$\overset{(\ast)}{\iff} \lambda \in p(\sigma(A))$$

$\overset{(*)}{\Rightarrow}$) by (a) $\lambda = p(\lambda_i)$ and $\lambda_i \in \sigma(A) \Rightarrow \lambda \in p(\sigma(A))$

$\overset{(*)}{\Leftarrow}$) $\lambda = p(z)$ for some $z \in \sigma(A)$

by (b), $z \in \{\lambda_1, \dots, \lambda_n\}$ so $\exists\, i: \lambda_i \in \sigma(A)$

$\Rightarrow \lambda \in \sigma(p(A))$

$\square$

II important is isometric property but only in Hilb space and for self adj op.

Lemma $A \in \mathcal{L}(H)$, $A = A^*$. Let $p \in \mathbb{C}[t]$. Then

$$\|p(A)\|_{\mathcal{L}(H)} = \|p\|_{C(\sigma(A))}$$

proof Let $p(t) = \sum_{k=0}^{n} c_k t^k$. Since $c_k \in \mathbb{C}$

in general $p(A)$ not self adjoint and

$$p(A)^* = \sum \overline{c_k} A^k = \overline{p}(A)$$

(*) $$\underbrace{p(A)^* p(A)}_{\text{self adjoint}} = \left(\sum \overline{c_k} A^k\right)\left(\sum c_\ell A^\ell\right) = (\overline{p}p)(A)$$

$$\Rightarrow \|p(A)\|^2_{\mathcal{L}(H)} = \|p(A)^* p(A)\|_{\mathcal{L}(H)}$$

$$\overset{(*)}{=} \|(\overline{p}p)(A)\|_{\mathcal{L}(H)}$$

$(\overline{p}p)(A)$ selfadjoint

For T selfadjoint $\|T\| = r(T)$

$$= r((\overline{p}p)(A))$$

$$= \sup_{\lambda \in \sigma((\overline{p}p)(A))} |\lambda|$$

spectral mapping $\sigma((\overline{p}p)(A)) = (\overline{p}p)(\sigma(A))$

$$= \sup_{\lambda \in \sigma(A)} |(\overline{p}p)(\lambda)| = \sup_{\lambda \in \sigma(A)} |p(\lambda)|^2 = \|p\|^2_{C(\sigma(A))}$$

$\square$

proof (cont. func calc) $\forall p$ polynomial we set

$$\phi(p) = p(A) \quad (\text{hence } \phi(1) = 1(A) = 1 = \mathbb{1})$$

Take $f \in C(\sigma(A))$. As $\sigma(A)$ is compact, by Stone-Weierstrass thm (polynomials are dense in $C(\sigma(A))$)
$\exists\ (p_n)_{n\geq 1}$ polynomials with $\| f - p_n \|_{C(\sigma(A))} \to 0$

Consider $(p_n(A))_{n\geq 1}$: want to prove it is Cauchy seq in $\mathcal{L}(H)$

$$\| p_n(A) - p_m(A) \|_{\mathcal{L}(H)} = \| (p_n - p_m)(A) \|_{\mathcal{L}(H)}$$

$$= \| p_n - p_m \|_{C(\sigma(A))} \xrightarrow{n,m\to\infty} 0$$

$\rightsquigarrow$ $(p_n(A))_n$ is Cauchy in $\mathcal{L}(H)$! Put

$$\phi(f) := f(A) := \lim_{n\to\infty} p_n(A) \qquad (\lim \text{ in } \mathcal{L}(H))$$

EXERCISE: verify that limit does not depend on approx seq, i.e. if $q_n \to f$ then $\lim_{n\to\infty} q_n(A) = f(A)$

Proof of the properties: (idea: verify them for polynomials and pass to the limit)

1) $*$-homomorphism: take $f, g \in C(\sigma(A))$ and show that $\phi(fg) = \phi(f)\,\phi(g)$

Approximate with polynomials: $p_n \to f$, $q_n \to g$ in $C(\sigma(A))$

$\rightsquigarrow$ $p_n q_n \to fg$ in $C(\sigma(A))$

$\Rightarrow$ $p_n(A)\, q_n(A) \longrightarrow f(A)\, g(A)$ (limit in $\mathcal{L}(H)$)

$\quad\ \ (p_n q_n)(A) \longrightarrow (fg)(A)$

($p_n(A)\, q_n(A) = (p_n q_n)(A)$)

Other identities proved similarly

[In general: properties true for polynomials, they stay true by taking limits thanks to isometry property]

(2) $f(t) = t \in C[t] \Rightarrow \Phi(f) = A$ by def.

(3) $\| f(A) \|_{\mathcal{L}(H)} = \lim_{n\to\infty} \| p_n(A) \|_{\mathcal{L}(H)} = \lim_{n\to\infty} \|p_n\|_{C(\sigma(A))} = \|f\|_{C(\sigma(A))}$

with p_n polynomials with $p_n \to f$ in $C(\sigma(A))$

(4) $\sigma(f(A)) = f(\sigma(A))$

($\subseteq$) We show $\rho(f(A)) \supseteq (f(\sigma(A)))^c$

So take $\lambda \in \mathbb{C}$ with $\lambda \notin \underbrace{\{ f(t) : t \in \sigma(A) \}}_{\text{compact}}$

Define $g(t) = \dfrac{1}{f(t) - \lambda} \in C(\sigma(A))$

By the prop. of func. calculus $g(A)$ is bdd op.

$$g(A)(f(A) - \lambda) = [g(f-\lambda)](A) = [1](A) = \mathbb{1}$$
$$\| \qquad\qquad$$
$$(f(A) - \lambda) g(A) \quad \sim$$

$\rightsquigarrow$ $g(A)$ is the inverse of $f(A) - \lambda$ $\rightsquigarrow$ $\lambda \in \rho(f(A))$

($\supseteq$) Let $\lambda \in f(\sigma(A))$, so $\lambda = f(\mu)$ with $\mu \in \sigma(A)$

take $p_n \to f$ in $C(\sigma(A))$

Then $p_n(A) - p_n(\mu)\mathbb{1} \longrightarrow f(A) - f(\mu)\mathbb{1}$ in $\mathcal{L}(H)$

CLAIM: $p_n(A) - p_n(\mu)\mathbb{1}$ is not invertible $\forall n$

$\rightsquigarrow$ $f(A) - f(\mu)\mathbb{1}$ is not invertible (inv. ops form an open set) $\rightsquigarrow$ $\underset{\lambda}{\overset{\|}{f(\mu)}} \in \sigma(f(A))$

proof of claim: $\sigma(p_n(A)) = p_n(\sigma(A))$

$\mu \in \sigma(A) \Rightarrow p_n(\mu) \in p_n(\sigma(A)) = \sigma(p_n(A))$

(5) $Ax = \lambda x \Rightarrow f(A)x = f(\lambda)x$

True for polynomials: $p(A)x = \sum c_k A^k x = \sum c_k \lambda^k x$

$= p(\lambda)x$

then take $p_n \rightrightarrows f$ in $C(\sigma(A))$.

$$\begin{array}{ccc} p_n(A)x & = & p_n(\lambda)x \\ \downarrow & & \downarrow \\ f(A)x & & f(\lambda)x \end{array} \qquad \forall n$$

(6) $f \geq 0 \Rightarrow \sqrt{f} \geq 0$ and $\sqrt{f} \in C(\sigma(A))$

Hence put

$B := (\sqrt{f})(A)$ it is selfadjoint and

$B^2 = (\sqrt{f})(A)(\sqrt{f})(A) = (\sqrt{f}\sqrt{f})(A) = f(A)$

$\rightsquigarrow (f(A)x, x) = (B^2 x, x) = (Bx, Bx) = \|Bx\|^2 \geq 0$

uniqueness Assume that γ is an other funct. calculus fulfilling the same properties.

then $\psi(1) = \mathbb{1}$

$\psi(p) = A$ for $p(t) = t$

By the $*$-homeomorphism prop, $\forall p \in \mathbb{C}[t]$

$$\psi(p) = p(A) = \phi(A)$$

↳ by our def of ϕ

Any 2 cont. func. calculus must coincide on polynomials
But polynomials are dense ⇝ they coincide everywhere:
$f \in C(\sigma(A))$, $p_n \to f$

$$\| \psi(f) - \psi(p_n) \|_{\mathcal{L}(H)} = \| \psi(f - p_n) \|_{\mathcal{L}(H)} = \| f - p_n \|_{C(\sigma(A))} \to 0$$

$$\| \psi(f) - \phi(p_n) \|_{\mathcal{L}(H)}$$

Hence $\phi(p_n) \to \psi(f) = \phi(f)$

☒

Cor 1 $A \geq 0$, $A = A^*$, then $\exists\, B = B^*$ s.t.

$$B^2 = A \quad ⇝ \quad B = \sqrt{A}$$

proof $A \geq 0 \Rightarrow \sigma(A) \subseteq [0, +\infty)$

so $f(t) = \sqrt{t} \in C(\sigma(A))$

Define $B = f(A)$, it fulfills $B^2 = f^2(A) = A$ ☒

Cor 2 $A = A^*$, $\lambda \notin \sigma(A) \Rightarrow \| (A - \lambda)^{-1} \| = \dfrac{1}{\operatorname{dist}(\lambda, \sigma(A))}$

proof $\lambda \notin \sigma(A) ⇝ g(t) = \dfrac{1}{t - \lambda} \in C(\sigma(A))$ and $g(t)(t-\lambda) = 1$

⇝ $g(A)(A-\lambda) = (A-\lambda)\, g(A) = \mathbb{1}$ ⇝ $g(A) = (A-\lambda)^{-1}$, then use (3) ☒

Spectral measure

BOREL σ-ALGEBRA: smallest σ-algebra containing open sets

σ-algebra: ∘) closed under complement
∘) " " countable union & intersection

$A \in \mathcal{L}(H)$, $A = A^*$, by cont. funct calculus we have a map

$$C(\sigma(A)) \longrightarrow \mathcal{L}(H)$$
$$f \longmapsto f(A)$$

Now fix $x \in H$ and consider the map

$$\ell_x : C(\sigma(A), \mathbb{R}) \longrightarrow \mathbb{R}$$
$$f \longmapsto \langle f(A)x, x\rangle$$

This map is
- well defined: f real $\Rightarrow f(A) = f(A)^* \Rightarrow \langle f(A)x, x\rangle \in \mathbb{R}$
- linear: $\ell_x(\alpha f_1 + \beta f_2) = \alpha\, \ell_x(f_1) + \beta\, \ell_x(f_2)$
- continuous: $|\ell_x(f)| \le \|f(A)\|_{\mathcal{L}(H)} \|x\|^2 \le \|f\|_{C(\sigma(A))} \|x\|^2$
- positive: $\langle f(A)x, x\rangle \ge 0$ for $f \ge 0$.
  ↳ i.e. $\ell_x(f) \ge 0$ for $f \ge 0$

So $\forall x \in H$, $\ell_x \in C(\sigma(A); \mathbb{R})^*$ and positive.

Can we identify this functional?

Thm (Riesz–Markov thm) Let X be a compact Hausdorff space, then $\forall$ positive linear functional ℓ on $C(X;\mathbb{R})$ there is a unique Radon measure (real-valued, positive) on the Borel σ-algebra of X for which

$$\ell(f) = \int_X f \, d\mu \qquad \forall f \in C(X;\mathbb{R})$$

Proof [RS, Thm IV.14]

RADON: ∘) $\mu(K) < +\infty$ $\forall K$ compact
∘) OUTER REGULAR: $\forall E$ Borel, $\mu(E) = \inf\limits_{U \text{ open},\, U \supseteq E} \mu(U)$
∘) INNER REGULAR: $\forall E$ Borel, $\mu(E) < \infty$: $\mu(E) = \sup\limits_{K \subseteq E,\, K \text{ compact}} \mu(K)$

Application: $\forall x \in H$, $\exists!$ Radon measure μ_x on $(\sigma(A), \mathrm{Bor}(\sigma(A)))$ s.t.

$$\ell_x(f) = \langle f(A)x, x\rangle = \int_{\sigma(A)} f(\lambda)\, d\mu_x(\lambda), \quad \forall f \in C(\sigma(A);\mathbb{R})$$

Def μ_x is the spectral measure of A associated to x

Rem (1) $\mu_x(\sigma(A)) = \int_{\sigma(A)} 1 \cdot d\mu_x(\lambda) = \langle \underset{\overset{\shortparallel}{I}}{1(A)}\, x, x\rangle = \|x\|^2$

$\rightsquigarrow$ μ_x is finite measure with total mass $\|x\|^2$

(2) $\underset{\int f(\lambda)\, d\mu_{\alpha x}}{\underset{\shortparallel}{\langle f(A)\alpha x, \alpha x\rangle}} = |\alpha|^2 \langle f(A)x, x\rangle = |\alpha|^2 \int f(\lambda)\, d\mu_x(\lambda)$ $\quad \forall f \in C(\sigma(A);\mathbb{R})$

By the uniqueness of spectral measure: $|\alpha|^2 \mu_x = \mu_{\alpha x}$

EXAMPLES: (1) $A = A^*$, let x be eigenvector: $Ax = \tilde{\lambda} x$. Then

$$\underset{\int f(\lambda)\, d\mu_x}{\underset{\shortparallel}{\langle f(A)x, x\rangle}} = \langle f(\tilde{\lambda})x, x\rangle = f(\tilde{\lambda})\|x\|^2 \qquad \forall f \in C(\sigma(A))$$

$\Rightarrow$ $\mu_x = \delta_{\tilde{\lambda}} \|x\|^2$ (pure point measure)

(2) $L^2([0,1])$, $(Au)(t) = t\, u(t)$. We know $\sigma(A) = [0,1]$

What is μ_u for $u \in L^2([0,1])$?

$p_n \to f$ in $C(\sigma(A))$

$$f(A)u = \lim_{n\to\infty} p_n(A)u = \lim_{n\to\infty} p_n(t)u = \underbrace{f(t)u}_{\text{multiplication op by } f(t)}$$

$$\underset{\int_{\sigma(A)} f(\lambda)\, d\mu_u(\lambda)}{\underset{\shortparallel}{\langle f(A)u, u\rangle}} = \int f(t)\, |u(t)|^2\, dt$$

$\Rightarrow$ $\mu_u = |u(t)|^2\, dt$ (a.c. with respect to Lebesgue)

EXERCISE: (1) H separable Hilbert space, $T = T^*$ compact
given $x \in H$, compute μ_x

(2) $L^2([0,1])$, $T = M_g$ multiplication op by $g \in C^0([0,1];\mathbb{R})$
Compute μ_u for $u \in L^2$.

Let us come back to the def. of spectral measure

$$\underline{\langle f(A) x, x \rangle} = \int_{\sigma(A)} f(\lambda) \, d\mu_x(\lambda) \qquad \forall f \in C(\sigma(A);\mathbb{R})$$

↳ this is a quadratic form

linear in 1 comp.
conj. lin in the other

From a quadratic form, we can define a sesquilinear form by polarization

$$\langle f(A) x, y \rangle = \frac{1}{4} \Big[\langle f(A)(x+y), x+y \rangle - \langle f(A)(x-y), x-y \rangle - i \langle f(A)(x-iy), x-iy \rangle + i \langle f(A)(x+iy), (x+iy) \rangle \Big]$$

(check it!)
$$= \frac{1}{4} \Big[\int f(\lambda) \big(d\mu_{x+y} - d\mu_{x-y} - i \, d\mu_{x-iy} + i \, d\mu_{x+iy} \big) \Big]$$

$$= \int f(\lambda) \, d\mu_{x,y}$$

where $\mu_{x,y} := \frac{1}{4} \Big[\mu_{x+y} - \mu_{x-y} - i \mu_{x-iy} + i \mu_{x+iy} \Big]$

$\mu_{x,y}$ is a complex-Borel measure

$\left[\begin{array}{l} \mu : \text{Borel } \sigma\text{-algebra} \to \mathbb{C} \\ \sigma\text{-additive} : \mu\left(\bigcup_{n=1}^{\infty} M_n \right) = \sum \mu(M_n) \\ \forall (M_n)_{n \geq 1} \text{ pairwise disjoints} \end{array} \right.$

Rem 1
- $\mu_{x,y}$ is finite
- $\overline{\mu_{x,y}} = \mu_{y,x}$

$\Big\}$ from $\langle f(A) x, y \rangle$

$\odot \quad x \to \mu_{x,y}$ is linear

$y \mapsto \mu_{x,y}$ is conjugate linear

$\odot \quad \mu_{x,x} = \mu_x$

$$\int f(\lambda)\, d\mu_{x,y}$$

Rem 2 splitting $f : \sigma(A) \to \mathbb{C}$ as $\mathrm{Re} f + i \,\mathrm{Im} f$, one has $\langle f(A)x, y\rangle = \int f(\lambda)\, d\mu_{x,y}$ holds $\forall f \in C(\sigma(A); \mathbb{C})$

KEY OBSERVATION! $\int f(\lambda)\, d\mu_{x,y}$ it is defined on

$$\mathcal{B}_b(\sigma(A)) := \Big\{ f : \sigma(A) \to \mathbb{C} : \text{Borel measurable bounded; } \sup_{\lambda \in \sigma(A)} |f(\lambda)| < \infty \Big\}$$

Indeed just note that

$$\left| \int_{\sigma(A)} f(\lambda)\, d\mu_{x,y} \right| \leq \frac{1}{4}\Big[\int |f(\lambda)|\, d\mu_{x+y} + \int |f(\lambda)|\, d\mu_{x-y} + \int |f(\lambda)|\, d\mu_{x-iy} + \int |f(\lambda)|\, d\mu_{x+iy} \Big]$$

$$\leq \sup_{\lambda \in J} |f(\lambda)| \frac{1}{4} \Big[\|x+y\|^2 + \|x-y\|^2 + \|x-iy\|^2 + \|x+iy\|^2 \Big]$$

$$\leq 4 \sup_J |f(\lambda)| \left(\|x\|^2 + \|y\|^2 \right) < +\infty$$

So we define $\forall f \in \mathcal{B}_b(\sigma(A))$

$$B_f(x,y) := \int_{\sigma(A)} f(\lambda)\, d\mu_{x,y}(\lambda)$$

Rem Clearly we can define B_f for a function $f: \mathbb{R} \to \mathbb{C}$ with the same definition.

But B_f depends only on the values of $f|_{\sigma(A)}$

Similarly, given a Borel set $E \subseteq \mathbb{R}$, we can put

$$\mu_{x,y}(E) := \mu_{x,y}(E \cap \sigma(A))$$

Lemma $B_f : H \times H \to \mathbb{C}$ is a bounded sesquilinear form

If f is real valued, B_f fulfills $B_f(x,y) = \overline{B_f(y,x)}$

proof ⊙ $x \mapsto B_f(x,y)$ linear

$y \to B_f(x,y)$ conj. linear.

⊙ $\sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |B_f(x,y)| \leq 4 \sup_{\lambda \in J} |f(\lambda)| \left(\|x\|^2 + \|y\|^2\right) \leq 8 \sup_{\lambda \in J} |f(\lambda)|$

⊙ f real: $B_f(x,y) = \int_{\sigma(A)} f(\lambda)\, d\mu_{x,y}(\lambda) = \int_{\sigma(A)} f(\lambda)\, \overline{d\mu_{y,x}}$

$$= \overline{\int_{\sigma(A)} f(\lambda)\, d\mu_{y,x}} = \overline{B_f(y,x)}$$

$\square$

We can use Riesz-representation theorem to write a sesquilinear form as a scalar product:

Thm $B : H \times H \to \mathbb{C}$ sesquilinear, bounded form

i.e. $\sup_{\|x\|, \|y\| \le 1} |B(x,y)| = M < +\infty$

$\Rightarrow \exists!\ S \in \mathcal{L}(H)$ s.t.

$$B(x,y) = \langle Sx, y \rangle \qquad \forall x, y \in H$$

If B symmetric $\left(B(x,y) = \overline{B(y,x)} \right)$, then $S = S^*$.

proof $\forall\ x \in H$, the map

$$\ell_x : H \to \mathbb{C}$$
$$y \mapsto \ell_x(y) = \overline{B(x,y)}$$

is a linear functional $\left(\ell_x(\alpha y) = \overline{B(x, \alpha y)} = \alpha \underbrace{\overline{B(x,y)}}_{\alpha\, \ell_x(y)} \right)$

bounded: $\left(\|\ell_x\| = \sup_{\|y\| \le 1} |\ell_x(y)| \le \sup_{\|y\| \le 1} |B(x,y)| \le M\|x\| \right)$

So apply Riesz thm $\Rightarrow \exists!\ X_x \in H$ with $\|X_x\| \le M\|x\|$ $(*)$
s.t.

$$\underbrace{\ell_x(y)}_{\overline{B(x,y)}} = \langle y, X_x \rangle \qquad \Rightarrow B(x,y) = \langle X_x, y \rangle$$

Put $S : H \to H$
$$x \mapsto (Sx) := X_x$$

S is linear and bounded (thanks to $(*)$)

$\leadsto$ $B(x,y) = (Sx, y) \qquad \forall x, y \in H$

If B symmetric ; $\quad B(x,y) = \overline{B(y,x)} \quad \forall x,y \in H$

$$\begin{array}{ccc} B(x,y) & = & \overline{B(y,x)} \\ \shortparallel & & \shortparallel \\ (Sx,y) & & \overline{(Sy,x)} = (x,Sy) = (S^*x,y) \end{array}$$

$\rightsquigarrow \; S = S^*$

$\blacksquare$

We apply Riesz thm to B_f

$\rightsquigarrow \; \exists! \; A_f \in \mathcal{L}(H)$ st.

$$\langle A_f x, y\rangle = B_f(x,y) = \int_{\sigma(A)} f(\lambda) \, d\mu_{x,y}(\lambda) \quad \forall x,y \in H$$

Rem (1) $f \mapsto A_f$ is linear as $\left| \begin{array}{l} f \mapsto B_f \text{ is linear} \\ B_f \to A_f \text{ is unique} \end{array}\right.$

(2) $f \in C(\sigma(A))$

$$\langle A_f x, y\rangle = B_f(x,y) = \int_{\sigma(A)} f(\lambda) \, d\mu_{x,y} = \langle f(A)x, y\rangle$$

defined via cont. funct calculus for $f|_{\sigma(A)}$

$\rightsquigarrow \; A_f = f(A)$

(3) Clearly A_f is well defined for any bounded Borel function $f : J \supseteq \sigma(A) \to \mathbb{C}$ for J compact, depending only on $f|_{\sigma(A)}$

Thm (Bounded Borel funct calculus)

$A \in \mathcal{L}(H)$, $A = A^*$, then $\exists!\ \hat{\Phi} : \mathcal{B}_b(J) \to \mathcal{L}(H)$

$f \longmapsto \hat{\Phi}(f)$

where J compact interval, $J \supseteq \sigma(A)$, st

(1) $\hat{\Phi}$ algebraic $*$-homomorphism

$$\hat{\Phi}(fg) = \hat{\Phi}(f)\,\hat{\Phi}(g)$$
$$\hat{\Phi}(\alpha f + \beta g) = \alpha\,\hat{\Phi}(f) + \beta\,\hat{\Phi}(g)$$
$$\hat{\Phi}(1) = \mathbb{1}$$
$$\hat{\Phi}(\bar{f}) = \hat{\Phi}(f)^*$$

(2) for $f(\lambda) = \lambda$, $\quad \hat{\Phi}(f) = A$

(3) $\|\hat{\Phi}(f)\|_{\mathcal{L}(H)} \leq \sup_{\lambda \in J} |f(\lambda)|$

(4) $\sigma(\hat{\Phi}(f)) \subseteq \overline{f(\sigma(A))}$

(5) If $Ax = \lambda x \Rightarrow \hat{\Phi}(f)\,x = f(\lambda)\,x$

(6) $f \geq 0 \Rightarrow \hat{\Phi}(f) \geq 0$

(7) $(f_n)_{n \geq 1}$ seq in $\mathcal{B}_b(J)$ st $\begin{cases} \sup_{\lambda \in J} |f_n(\lambda)| \leq M & \forall n \\ f_n(\lambda) \to f(\lambda) & \forall \lambda \end{cases}$

$\Rightarrow \hat{\Phi}(f_n)\,x \to \hat{\Phi}(f)\,x \quad \forall x \in H$

i.e. $\hat{\Phi}(f_n) \to \hat{\Phi}(f)$ strongly (not uniformly)

proof (1) exercise, but we prove $\hat{\Phi}(fg) = \hat{\Phi}(f)\hat{\Phi}(g)$

preliminary observation: If $(f_n)_{n\geq 1}$ with $\left| \begin{array}{l} \sup_{\lambda\in J} |f(\lambda)| \leq M \\ f_n \to f \text{ pointwise everywhere} \end{array} \right.$

$$\rightsquigarrow \langle \hat{\Phi}(f_n)x,y\rangle = \int_{\sigma(A)} f_n(\lambda)\, d\mu_{x,y} \longrightarrow \int_{\sigma(A)} f(\lambda)\, d\mu_{x,y} = \langle \hat{\Phi}(f)x,y\rangle$$

by Lebesgue's dominated conv. theorem.

$$\rightsquigarrow \hat{\Phi}(f_n) \longrightarrow \hat{\Phi}(f) \qquad (*)$$

Next we need a lemma from measure theory

Lemma $K \subset \mathbb{R}$ compact, Set $\mathcal{B}_b(K)$ bounded Borel-measurable function. Let $U \subseteq \mathcal{B}_b(K)$ st.

(i) $U \supseteq C(K)$

(ii) U closed under "bounded pointwise convergence", i.e.

$$(f_n)_n \subseteq U, \left| \begin{array}{l} f_n(\lambda) \to f(\lambda) \quad \forall \lambda \in K \\ \sup_{\lambda\in K} |f_n(\lambda)| \leq M \quad \forall n \end{array} \right. \Rightarrow f \in U$$

$$\Longrightarrow U = \mathcal{B}_b(K)$$

proof exercise [Werner, Lemma VII.1.5]

Back to prove $\hat{\Phi}(fg) = \hat{\Phi}(f)\hat{\Phi}(g) \qquad (**)$

Tricky limit procedure: If $f, g \in C(J)$, $\hat{\phi}(f) = f(A)$ and we have $(*)$.

If $g \in C(J)$, set $U = \{ f \in \mathcal{B}_b(J) : \hat{\phi}(fg) = \hat{\phi}(f)\hat{\phi}(g) \}$

Then $U \supset C(J)$. We show U is closed under "b p. c.". Take $(f_n)_{n \geq 1} \subset U$, $\begin{cases} \sup_\lambda |f_n(\lambda)| \leq M \ \forall n \\ f_n \to f \text{ pointwise} \end{cases}$

$$\langle \hat{\phi}(f_n)\hat{\phi}(g) x, y \rangle \xrightarrow[(*)]{n \to \infty} \langle \hat{\phi}(f)\hat{\phi}(g) x, y \rangle$$

$\| \ f_n \in U$

$$\langle \hat{\phi}(f_n g) x, y \rangle \xrightarrow[(*)]{} \langle \hat{\phi}(fg) x, y \rangle$$

$$\rightsquigarrow \quad \hat{\phi}(fg) = \hat{\phi}(f)\hat{\phi}(g) \quad \Rightarrow f \in U$$

$$\overset{\text{Lemma}}{\Rightarrow} \quad U = \mathcal{B}_b(J)$$

Finally take $f \in \mathcal{B}_b(J)$, put

$$V = \{ g \in \mathcal{B}_b(J) : \hat{\phi}(fg) = \hat{\phi}(f)\hat{\phi}(g) \}$$

We just proved $V \supset C(J)$. Again one shows that V closed under "b p.c" $\Rightarrow V = \mathcal{B}_b(J)$.

(2) $f(\lambda) = \lambda \in C(J) \rightsquigarrow \hat{\phi}(f) = f(A) = A$ (by continuous f.c. case)

(3) f real $\Rightarrow f(A)$ selfadjoint

$$\|f(A)\| = \sup_{\|x\| \leq 1} |\langle f(A)x, x \rangle| \leq \sup_{\lambda \in \sigma(A)} |f(\lambda)|$$

f complex: $\|\hat{\phi}(f)\|^2 = \|\hat{\phi}(f)^* \hat{\phi}(f)\| = \|\hat{\phi}(\bar{f})\hat{\phi}(f)\| = \|\hat{\phi}(|f|^2)\| \leq \sup_\lambda |f(\lambda)|^2$

(4) ~ (6) exercise

(7) $\begin{cases} f_n \to 0 & \text{pointwise} \\ \|f_n\|_\infty < M & \forall n \end{cases}$

$$\|\hat{\phi}(f_n)x\|^2 = \langle \hat{\phi}(f_n)x, \hat{\phi}(f_n)x\rangle$$

$$= \langle \hat{\phi}(f_n)^* \hat{\phi}(f_n)x, x\rangle$$

$$= \langle \hat{\phi}(|f_n|^2)x, x\rangle \longrightarrow 0$$

by Lebesgue dom. conv. thm.

<u>uniqueness</u> ψ another func. calc.

(1)+(2) $\Rightarrow$ $\psi(p) = \phi(p)$ $\forall p$ polynomial

(3) $\Rightarrow$ $\psi(f) = \phi(f)$ $\forall f \in C(J)$
by Stone-Weierstrass

(7) $\Rightarrow$ $\psi(f) = \phi(f)$ $\forall f \in \mathcal{B}_b(J)$

Indeed put $U := \{ f \in \mathcal{B}_b(J) : \psi(f) = \phi(f) \}$

We have proved $U \supseteq C(J)$
take $(f_n)_n \subset U$ with $\begin{cases} f_n \to f & \text{pointwise} \\ \|f_n\| \leq C \end{cases}$

Then $\langle \psi(f_n)x, y\rangle \longrightarrow \langle \psi(f)x, y\rangle$
$\langle \phi(f_n)x, y\rangle \longrightarrow \langle \phi(f)x, y\rangle$

$\Rightarrow f \in U \Rightarrow U = \mathcal{B}_b(J)$ ∎

ROAD MAP

$A = A^*$, $A \in \mathcal{L}(H)$

polynomial $\quad p(A) = \sum c_k A^k \quad \forall$ polynomial

$$\left\{ \begin{array}{l} \text{SMT: } \sigma(p(A)) = p(\sigma(A)) \\ \|p(A)\| = \|p\|_{C(\sigma(A))} \end{array} \right.$$

CONT. FUNC. CALC: $\quad C(\sigma(A)) \longrightarrow \mathcal{L}(H)$

$$f \longmapsto f(A) = \lim_{n \to \infty} p_n(A)$$

by Stone-Weierstrass $(p_n)_n$ polynomials, $p_n \to f$

Fix $x \in H$: $\quad C(\sigma(A), \mathbb{R}) \longrightarrow \mathbb{R}$

$$f \longmapsto \langle f(A)x, x \rangle$$

This map is linear and continuous: $\in C(\sigma(A))^*$

Riesz: $\exists !\ \mu_x$ SPECTRAL MEASURE;

$$\langle f(A)x, x \rangle = \int_{\sigma(A)} f(\lambda)\, d\mu_x(\lambda) \qquad \forall f \in C(\sigma(A))$$

By polarization

$$\underbrace{\langle f(A)x, y \rangle = \int_{\sigma(A)} f(\lambda)\, d\mu_{x,y}(\lambda)}_{\text{sesquilinear bounded operator}} \qquad \forall f \in C(\sigma(A))$$

$\mu_{x,y}$ by polarization

It also makes sense $\forall f \in \mathcal{B}_b(J)$, $J \supseteq \sigma(A)$
J compact

$$B_g(x,y) := \int_{\sigma(A)} g(\lambda) \, d\mu_{x,y}(\lambda), \quad \forall g \in \mathcal{B}_b(J)$$

Riesz: $\exists!$ $g(A) = \hat{\phi}(g)$ s.t

$$B_g(x,y) = \langle g(A) x, y \rangle \qquad \forall x,y \qquad (*)$$

BOREL FUNC. CALC: $\mathcal{B}_b(J) \longrightarrow \mathcal{L}(H)$

$$g \longmapsto g(A) \quad \text{defined by } (*)$$

indicator function over Ω

EX take $g(x) = \mathbb{1}_\Omega(x)$, $\Omega \subseteq J$

$$E^A(\Omega) = \mathbb{1}_\Omega(A) = \hat{\phi}(\mathbb{1}_\Omega)$$

$E^A(\Omega)$ is an orthogonal projection:

indeed: $E^A(\Omega)^* = \hat{\phi}(\mathbb{1}_\Omega)^* = \hat{\phi}(\overline{\mathbb{1}_\Omega}) = E^A(\Omega)$

$$E^A(\Omega)^2 = \hat{\phi}(\mathbb{1}_\Omega)\hat{\phi}(\mathbb{1}_\Omega) = \hat{\phi}(\underbrace{\mathbb{1}_\Omega^2}_{= \, \mathbb{1}_\Omega}) = E^A(\Omega)$$

Next step: reconstruct A starting from this projections