LAGRANGE MULTIPLIERS

X Banach space, $\mathcal{U} \subseteq X$ open, $f: \mathcal{U} \to \mathbb{R}$
$g: \mathcal{U} \to \mathbb{R}^d$, $d \geq 1$

$f, g \in C^1$

We want to find necessary conditions for existence of

$$(*) \qquad \min_{x \in \mathcal{U}:\ g(x)=0} f(x)$$

CONSTRAINED OPTIMIZATION PROBLEM

If $\dim X < +\infty$: Lagrange multipliers $\qquad g(x) = \begin{pmatrix} g^1(x) \\ \vdots \\ g^d(x) \end{pmatrix}$

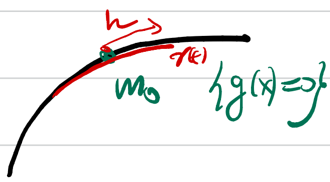
If x solves $(*)$ $\Rightarrow$ $\exists\, \lambda \in \mathbb{R}^d$: $\quad df(x) = \sum_j \lambda_j\, dg^j(x)$

We want to extend this result to ∞-dim spaces:

Thm H Hilbert space. Let $f: \mathcal{U} \to \mathbb{R}$, $g: \mathcal{U} \to \mathbb{R}^d$, $\mathcal{U} \subseteq H$ open, $f, g \in C^1$. Assume that $\exists\ m_0 \in M = \{x \in \mathcal{U}: g(x) = 0\}$ solving $(*)$ and assume also that $dg(m_0)$ is surjective. $\in \mathcal{L}(H, \mathbb{R}^d)$
Then $\exists\ \lambda \in \mathbb{R}^d$:

$$df(m_0) = \sum_{j=1}^{d} \lambda_j\, dg^j(m_0)$$

proof Step 1 Description of tangent space:



$$T_{m_0} M := \left\{ h \in H:\ \exists\ \gamma \in C^1((-\varepsilon, \varepsilon), H) \text{ with } \begin{array}{l} \gamma(t) \subseteq M \quad \forall t \\ \gamma(0) = m_0 \\ \frac{d}{dt}\gamma(t)\Big|_{t=0} = d\gamma(0)[1] = h \end{array} \right\}$$

We want to prove

$$T_{m_0} M = \ker dg(m_0)$$

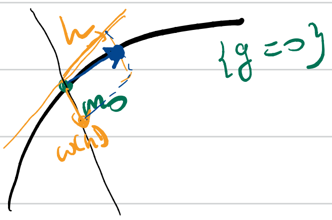
($\subseteq$) Let $h \in T_{m_0} M$, then $\exists$ $\gamma(t)$ with $\gamma(0) = m_0$, $\dot\gamma(0) = h$ and

$$g(\gamma(t)) = 0 \quad \forall t$$

$$0 = \frac{d}{dt} g(\gamma(t)) = dg(\gamma(t)) \frac{d}{dt}\gamma(t) \overset{t=0}{\rightsquigarrow} 0 = dg(m_0)[h]$$

$\rightsquigarrow$ $h \in \ker dg(m_0)$

($\supseteq$) Let $h \in \ker dg(m_0)$, we want to construct a curve $\gamma \in C^1$ with $\gamma(0) = m_0$, $\dot\gamma(0) = h$ and $\gamma(t) \in M$ $\forall t$, i.e. $g(\gamma(t)) = 0$ $\forall t$.



Idea: $\forall \|h\| \le \varepsilon$ construct a vector $w(h) \in H$ so that $\underbrace{m_0 + h + w(h) \in M}$

as in fin dim, we describe the constrained as a graph over a ball

We decompose $H = \underbrace{\ker dg(m_0)}_{H_0} \oplus \underbrace{(\ker dg(m_0))^{\perp}}_{H_1}$

Note that H_0 and H_1 are both closed: H_0 is the kernel of bd lin op, H_1 is the $\perp$ of vector space

Introduce
$$\mathcal{G} : H_0 \times H_1 \longrightarrow \mathbb{R}^d$$
$$(h, w) \longrightarrow \mathcal{G}(h,w) = g(m_0 + h + w)$$

Then:
- $\mathcal{G}(0,0) = g(m_0) = 0$
- $\mathcal{G} \in C^1$
- $d_w \mathcal{G}(0,0)$ is invertible $\in \mathcal{L}(H_1, \mathbb{R}^d)$

$$d_w \mathcal{G}(0,0)[\hat w] = dg(m_0)[\hat w]$$

The map $dg(m_0): H_1 \longrightarrow \mathbb{R}^d$ is

- surjective: $\forall y \in \mathbb{R}^d: \quad \exists x \in H: \quad dg(m_0)x = y$

  then write $x = x_0 + x_1 \in \ker dg(m_0) \oplus \underbrace{(\ker dg(m_0))^{\perp}}_{H_1}$

  $\leadsto dg(m_0)x = dg(m_0)x_1 = y$

- injective: $dg(m_0)w = 0 \overset{\text{for some } w\in H_1}{\Longrightarrow} w \in H_0 \cap H_1 = \{0\}$
- bounded: $dg(m_0) \in \mathcal{L}(H_1, \mathbb{R}^d)$
- H_1 Banach: it is closed in Banach

open mapping:

$$\Longrightarrow \quad [dg(m_0)]^{-1} \in \mathcal{L}(\mathbb{R}^d; H_1)$$

$$\overset{\text{IFT}}{\Longrightarrow} \quad \exists!\ \omega: \begin{array}{ccc} B_\varepsilon^{H_0}(0) & \longrightarrow & B_\delta^{H_1}(0) \\ h & \longrightarrow & \omega(h) \end{array} \text{ of class } C^1, \text{ so that}$$

$$0 \equiv \mathcal{G}(h, \omega(h)) = g(m_0 + h + \omega(h)) \qquad \forall h \in B_\varepsilon^{H_0}(0)$$

So in particular, $\forall h \in B_\varepsilon^{H_0}(0)$, $m_0 + h + \omega(h) \in M$

Moreover: $\omega(0) = 0$,

$$d\omega(0)[\hat{h}] = -\left[d_w \mathcal{G}(0,0)\right]^{-1} d_h \mathcal{G}(0,0)[\hat{h}]$$

$$= -\left[d_w \mathcal{G}(0,0)\right]^{-1} dg(m_0)[\hat{h}] = 0$$

$\hat{h} \in H_0 = \ker dg(m_0)$

Now put $\gamma(t) := m_0 + th + \omega(th) \in C^1((-\varepsilon, \varepsilon), H)$

we have $\gamma(0) = m_0$

$$\frac{d}{dt}\gamma(0) = h + \underbrace{d\omega(0)[h]}_{=0} = h$$

$\gamma(t) \in M \quad \forall t$ by construction of ω:

$$g(m_0 + th + \omega(th)) = 0 \qquad \forall t$$

<u>Step 2</u> Take $h \in T_{m_0} M$ and curve $\gamma(t)$ in M with h as tangent vector at 0.

Consider now $f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$, and we have

$$f(\gamma(t)) \geq f(\gamma(0)) \quad \forall t$$

since m_0 solves the constrained minimization problem

So $f \circ \gamma$ has minimum at $t=0$:

$$0 = \frac{d}{dt} f(\gamma(t)) \Big|_{t=0} = df(\gamma(0)) \left[\tfrac{d}{dt} \gamma(0) \right] = df(m_0)[h]$$

this holds true $\forall\, h \in T_{m_0} M \equiv \ker\, dg(m_0)$

Equivalently, since $df(m_0) \in \mathcal{L}(H, \mathbb{R}) \simeq H^*$

$(\mathbb{R}^d)^* \to H^*$

$$df(m_0) \in \left(\ker\, dg(m_0) \right)^{\perp} = \operatorname{Im} [dg(m_0)]^*$$

provided $\operatorname{Im} dg(m_0)$ closed

$\Uparrow$

$\operatorname{Im} dg(m_0) = \mathbb{R}^d$ so closed

$\Rightarrow \exists\, \lambda \in \mathbb{R}^d : \quad df(m_0) = (dg(m_0))^* [\lambda]$

Finally $\quad dg(m_0)^* : (\mathbb{R}^d)^* \longrightarrow H^*$

$$\lambda \longrightarrow [dg(m_0)]^*(\lambda)$$

$$dg(m_0)^*(\lambda)[h] = \lambda\big(\underbrace{dg(m_0)(h)}_{\in \mathbb{R}^d}\big) = \sum \lambda_j\, dg^j(m_0)[h]$$

$$\Rightarrow \exists\, \lambda \in \mathbb{R}^d : \quad df(m_0) = \sum \lambda_j\, dg^j(m_0)$$ ☒

Next goal: extend to general Banach spaces:

Where did we use Hilbert? $H = H_0 \oplus H_1$, H_0, H_1 closed

So in general X, we would like to have similar decomposition:

$X = \ker\, dg(m_0) + L$, $\quad L \cap \ker\, dg(m_0) = \{0\}$

L closed

Complementary subspaces (Brezis, 2.4)

Def Let $G \subseteq X$ be a <u>closed</u> subspace of X Banach
$L \subseteq X$ is a <u>complement</u> of G if
- L is closed
- $G \cap L = \{0\}$ and $G + L = X$

$z = x_1 + y_1 = x_2 + y_2$
$\Rightarrow x_1 - x_2 = y_2 - y_1 \in G \cap L = \{0\}$
$\Rightarrow$

Rem if G, L are complementary:
$\forall z \in X, \exists!\, x \in G, y \in L : \quad z = x + y$
So we can introduce a projector op
$$P : X \to G$$
$$z \to (Pz) := x \quad \text{so that} \quad z = x + y \in G + L$$

Lemma X Banach, G, L complementary subspaces. Then
$\exists\, C \geq 0$ so that $\forall\, z = x + y \in G + L$
$$\|x\| \leq C \|z\| \quad , \quad \|y\| \leq C \|z\|$$
i.e. $P : X \to G$ is continuous projection

proof Consider $G \times L$ with norm $\|(x,y)\| = \|x\| + \|y\|$
and the map
$$T : G \times L \to G + L = X \quad \rightsquigarrow \quad T \text{ is}$$
$$(x, y) \to x + y$$
- linear
- continuous
- surjective
- injective $\leftarrow$ uniqueness of decomp.

open mapping thm
$\Rightarrow$ T^{-1} exists and bounded:
$$\|(x,y)\| \leq \|T^{-1} z\| \leq C \|z\|$$
$\square$

Rem 1) every fin dim sub G of X admits a complement. Indeed $G = \text{span}\langle \vec{e}_1, \dots, \vec{e}_n \rangle$,
$\forall x \in G, \quad x = \sum x_i \vec{e}_i$
Define lin func. $\varphi_i : G \to \mathbb{R}, \quad \varphi_i(x) = x_i$
Extend φ_i to a cont. lin funct $\tilde{\varphi}_i : X \to \mathbb{R}$ by

Hahn-Banach, with $\tilde{\varphi}_i|_G = \varphi_i$

Then $L = \bigcap^n \tilde{\varphi}_i^{-1}(0)$ is a complement of G

.) closed (finite intersection of closed)

.) $G \cap L = \{0\}$, ($g \in G \cap L$ has $g = \sum g_i \cdot e_i$ and $\tilde{\varphi}_a(g) = \varphi_a(g) = g_a = 0 \ \forall a$)

.) $G + L = X$: ($x \in X$, put $x_i := \tilde{\varphi}_i(x)$ and $g_i = \sum x_i \cdot e_i$. Then $x - g \in L$)

2) In Hilbert, every closed subspace has a complement:
Just take $\perp$

3) In every Banach space not isomorphic to Hilbert, $\exists$ closed subspaces without complement
(Lindenstrauss - Tzafriri, Israel J. Math, 1971)

We need additional conditions ensuring $\ker T$, with T lin op, has a complement.
This cond. is that T has right inverse

Def $T \in \mathcal{L}(X, Y)$ has a right inverse if $\exists\, S \in \mathcal{L}(Y, X)$ so that

$$TS = \mathbb{1}_Y \qquad (TSy = y \quad \forall y \in Y)$$

T has left inverse if $\exists\, S \in \mathcal{L}(Y, X)$ with

$$ST = \mathbb{1}_X$$

Rem T has right inverse $\Rightarrow$ T surjective
T has left inverse $\Rightarrow$ T injective
$\Leftarrow$) implications NOT true, but we have characterization

Prop 1) Assume $T \in \mathcal{L}(X, Y)$ surjective. Then

T has right inverse $\Leftrightarrow$ $\ker T$ has complement

2) $T \in \mathcal{L}(X, Y)$ injective
T has left inverse $\Leftrightarrow$ $\operatorname{Im} T$ closed and has complement

proof only 1):

$\Rightarrow$) Claim: $\operatorname{Im} S$ is complementary for $\ker T$ ← right inverse

•) $\ker T \cap \operatorname{Im} S = \{0\}$:

$x \in \ker T$, $x = Sy$ for some $y \in Y$. Then

$0 = Tx = TSy = y \Rightarrow x = Sy = S0 = 0$

•) $\operatorname{Im} S$ closed:

$(x_n)_n \subset \operatorname{Im} S$ with $x_n \to x$. $x_n = Sy_n$, $y_n \in Y$

$\rightsquigarrow y_n = TSy_n = Tx_n \to Tx$

Thus
$$\begin{array}{ccc} x_n & = & Sy_n \\ \downarrow & & \downarrow \\ x & & S(Tx) \end{array} \rightsquigarrow x \in \operatorname{Im} S$$

•) $\ker T + \operatorname{Im} S = X$

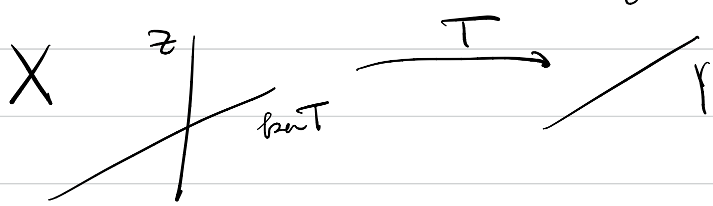
If $x \in \ker T$ ✓ $(x = x + 0)$

If $x \notin \ker T$, put $y := Tx \neq 0$

$$\rightsquigarrow x = \underbrace{Sy}_{\in \operatorname{Im} S} + \underbrace{x - Sy}_{\in \ker T:}$$ $T(x - Sy) = Tx - TSy = y - y = 0$

$\Leftarrow$) By assumption $\ker T$ has complement. Consider $P: X \to \ker T$ projection (it is continuous by previous lemma)

put $Z = (\mathbb{1} - P)X$ so that Z is the complementary of $\ker T$ and Z closed by assumption



claim: $T|_Z$ is bijective

put $A = T|_Z : Z \to Y$. A is
•) linear and bounded
•) injective: $Ax = 0 \Rightarrow T|_Z x = 0$ $\rightsquigarrow x \in \ker T \cap Z = \{0\}$
•) surjective (T surjective and Z complementary $\ker T$)

$\circ$) Z Banach (Z closed)

open mapping
$\Rightarrow$ $\exists\ A^{-1} : Y \to Z \ \in \mathcal{L}(Y, Z)$

If we put $i : Z \to X$ continuous, then
$i \circ A^{-1} \in \mathcal{L}(Y, X)$ is a right inverse.

$(T \circ i \circ A^{-1})(y) = T|_Z A^{-1}(y) = A A^{-1}(y) = y$

$\square$

Back to Lagrange multipliers in Banach space

Thm X Banach, $f : \mathcal{U} \subseteq X \to \mathbb{R}$, $g : \mathcal{U} \to \mathbb{R}^d$, C^1 maps,
Assume $m_0 \in M = \{x \in \mathcal{U} : g(x) = 0\}$ solves $(*)$ and
$dg(m_0)$ surjective. Then $\exists\ \lambda \in \mathbb{R}^d$:

$$df(m_0) = \sum_{j=1}^{d} \lambda_j \, dg^j(m_0)$$

proof As before, but modify step 1 ($\supseteq$)
Let $h \in \ker dg(m_0) = X_0$
We want X_0 to have complement: find X_1 closed with
$X = X_0 + X_1$, $X_0 \cap X_1 = \{0\}$

As $dg(m_0)$ surjective, $\ker dg(m_0)$ complementable $\Leftrightarrow$ $dg(m_0)$ right inverse

construct a right inverse: take $x_1, \dots, x_d \in X$ so that
$dg(m_0)[x_i] = \vec{e_i} \in \mathbb{R}^d$ $\quad \vec{e_i} = (0, \dots, 1, \dots 0)$
(possible as $dg(m_0)$ surjective) $\quad$ i-th component

Put $B : \mathbb{R}^d \to X$
$\vec{c} := (c_1, \dots, c_d) \to \sum_{i=1}^{d} c_i x_i$

Then B is $\circ$) continuous: $\|B\vec{c}\|_X = \|\sum c_i x_i\|_X \leq \sup_i \|x_i\| \sum |c_i|$
$\leq C_f \|\vec{c}\|$

$\circ$) right inverse: $dg(m_0)[B\vec{c}] = dg(m_0)[\sum c_i x_i]$
$= \sum c_i \, dg(m_0)[x_i] = \sum c_i \vec{e_i}$ ✓
$= \vec{c}$

$\rightsquigarrow$ X is complementable, $X = X_0 + X_1$ with X_1 closed

Then follow previous proof and apply IFT to prove $T_{m_0} M = \ker dg(m_0)$ and prove step 2 as before

$\boxtimes$

Application Look for non trivial solutions of

$$(*) \begin{cases} -u'' = u^3 \\ u(0) = u(1) = 0 \end{cases}$$

Rem 1 In $B^X_\varepsilon(0)$, $\varepsilon \ll 1$, only trivial sol by IFT

Rem 2 $-u'' - u^3 = 0$, $f(t) = -t^3$ has negative derivative

Consider the functional $L : H^1_0([0,1]) \longrightarrow \mathbb{R}$

$$u \longrightarrow L(u) = \frac{1}{2}\int (u')^2 - \frac{1}{4}\int u^4$$

u solves $(*)$ in weak sense

$$\Updownarrow$$

$$\int u'\varphi' - \int u^3 \varphi = 0 \qquad \forall \varphi \in H^1_0([0,1])$$

$$\Updownarrow$$

$$dL(u)[\varphi] = 0 \qquad \forall \varphi \in H^1_0([0,1])$$

$(*)$ corresponds to $dL(u)[\varphi] = 0 \quad \forall \varphi$

stationary point of L

One might look for minima of L

Problem: $\nexists$ minimum of L : $L(t u_0) = \frac{1}{2} t^2 \int u_0'^2 - \frac{t^4}{4}\int u_0^4 \xrightarrow{t \to \infty} -\infty$

Other strategy: Consider $F(u) = \frac{1}{2}\int (u')^2$

$$M := \left\{ u \in H^1_0 : \frac{1}{4}\int u^4 = 1 \right\} = \{ u : G(u) = 0 \}$$

where $G(u) = \frac{1}{4}\int u^4 - 1$

Look for minima of F along $\{G=0\}$

$$\min_{u:\, G(u)=0} F(u)$$

Assume we find u_0 sol of constrained minim. problem.
Clearly $u_0 \neq 0$

Apply Lagrange multiplier theorem

- $F : H^1_0 \to \mathbb{R}$ of class C^1 ✓
- $G : H^1_0 \to \mathbb{R}$ " " " ✓ (CHECK IT!)
- $dG(u_0)[\varphi] = \int u_0^3 \varphi$ surjective (take $\varphi = t\, u_0$)

$\leadsto$ $\exists\, \lambda \in \mathbb{R}$: $dF(u_0) = \lambda\, dG(u_0)$

$\iff$ $dF(u_0)[\varphi] = \lambda\, dG(u_0)[\varphi] \quad \forall \varphi \in H^1_0$

$\iff$ $\int u_0' \varphi' = \lambda \int u_0^3 \varphi \quad \forall \varphi \in H^1_0$

$\iff$ u_0 weak sol of $\begin{cases} -u_0'' = \lambda u_0^3 \\ u_0(0) = u_0(1) = 0 \end{cases}$

Note that for $\varphi = u_0 \Rightarrow \int u_0'^2 = \lambda \int u_0^4 \Rightarrow \lambda > 0$

then scale: $v = \alpha u_0$

$$\begin{cases} -v'' = -\alpha u_0'' = \alpha \lambda u_0^3 = \dfrac{\lambda}{\alpha^2} v^3 \\ v(0) = v(1) = 0 \end{cases}$$

$\leadsto$ for $\alpha^2 = \lambda$ we get a sol of $(*)$

It is left to check the existence of the minimum of F on $\{G=0\}$

Let $\{u_k\}$ minimizing seq: $(u_k)_k \subseteq \{G=0\}$

$F(u_k) \longrightarrow \inf_{G=0} F(u)$

$\rightsquigarrow \frac{1}{4}\int u_k^4 = 1$, moreover:

$\int (u_k')^2 \le C \overset{\text{Poincaré}}{\Longrightarrow} \int u_k^2 \le C$

$\rightsquigarrow \|u_k\|_{H_0^1} \le C \Longrightarrow \exists\ u_{k_j} \xrightarrow{H_0^1} u$

$u_{k_j} \xrightarrow{C^0} u$

$\Rightarrow 1 = \lim_{j\to\infty} \frac{1}{4}\int u_{k_j}^4 = \frac{1}{4}\int u^4 \quad \rightsquigarrow u \in \{G=0\}$

Finally

$\inf_{G=0} F \le \frac{1}{2}\int (u')^2 \le \underline{\lim}_{k_j} \frac{1}{2}\int (u_{k_j}')^2 = \inf_{G=0} F(u)$

$\rightsquigarrow$ u solves the minimization problem!

lower semicontinuity of norm here from weak conv. $u_{k_j} \rightharpoonup u$