ODE'S IN BANACH SPACES

"Teschl, Topics in lin ...., 7.6"

(I) $\begin{cases} \dot{x}(t) = f(t,x) \\ x(t_0) = x_0 \end{cases}$

- $f : J \times V \to X$ Banach
- $J \subset \mathbb{R}$, $V \subseteq X$ open
- $x_0 \in X$

Def a sol of (I) is a curve $x : I_{t_0} \to X$ $\in \mathcal{C}^1$ with $I_{t_0} \ni t_0$ and so that $x(t_0) = x_0$

$$\frac{d}{dt} x(t) = dx(t)[1] = f(t, x(t))$$

Main problem: dependence of the sol on the initial data

Basic properties

Thm (Cauchy-Lip) $f : \mathcal{U} \subseteq X \to X$ Lipschitz with Lip. constant L, i.e. $\| f(x) - f(y) \|_X \le L \|x - y\|$ $\forall x, y \in \mathcal{U}$, then $\forall x_0 \in \mathcal{U}$, $\exists \delta = \delta(x_0, L)$ and $\exists!$ function

$$x : (-\delta, \delta) \to X \quad \text{sol of} \quad \begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$$

proof Like in fin dim: contraction + integral formulation

Lemma $f \in \mathcal{C}^0$, $x \in \mathcal{C}^1$ is a sol of (I)

$\Updownarrow$

$$x \in \mathcal{C}^0 \quad \text{and} \quad x(t) = x_0 + \int_{t_0}^{t} f(x(s))\, ds$$

We denote by $x(\cdot, x_0)$ the sol of (I) with
initial datum x_0

Q: $\mathcal{U} \longrightarrow C^1$ flow map: how regular?
$x_0 \longrightarrow x(\cdot, x_0)$

We can also consider

$$(*) \begin{cases} \frac{d}{dt} x(t) = F(t, x, \lambda) & , (t_0, x_0, \lambda) \in I \times \mathcal{U} \times \Lambda \\ x(t_0) = x_0 \end{cases}$$

$I \subset \mathbb{R}$, $\mathcal{U} \subset X$, $\Lambda \subset \Sigma$; Λ = parameters

Thm $I \subseteq \mathbb{R}$, $\mathcal{U} \subseteq X$ Banach, $\Lambda \subset \Sigma$ Banach

If $F \in C^\ell(I \times \mathcal{U} \times \Lambda; X)$, $\ell \geq 1$ then
the problem $(*)$ has a unique sol

$$t \longmapsto x(t, t_0, x_0, \lambda) \in C^\ell(I_1 \times I_2 \times \mathcal{U}_1 \times \Lambda_1, \mathcal{U})$$

where
$I_1, I_2 \subseteq I$
$\mathcal{U}_1 \subseteq \mathcal{U}$
$\Lambda_1 \subseteq \Lambda$
are open sets containing any point $(t_0, x_0, \lambda_0) \in I \times \mathcal{U} \times \Lambda$

proof Add t and λ to the dependent variables
i.e. we consider $y = \begin{pmatrix} \tau \\ x \\ \lambda \end{pmatrix} \in I \times X \times \Lambda$

$$\dot{y} = \frac{d}{dt}\begin{pmatrix} \tau \\ x \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ F(\tau, x, \lambda) \\ 0 \end{pmatrix} \rightsquigarrow \dot{y} = \mathcal{F}(y)$$

It is also useful to rescale the time (in order to get a neighb where all solutions are defined on a same interval of time)

Let $\varepsilon > 0$ arbitrary and define

$$z(t) = y(\varepsilon t + t_0) - y_0 \iff y(t) = y_0 + z\left(\frac{t - t_0}{\varepsilon}\right)$$

$$z(0) = y(t_0) - y_0 = 0$$

$$\frac{d}{dt} z = \varepsilon \frac{d}{dt} y(\varepsilon t + t_0) = \varepsilon F(y(\varepsilon t + t_0)) = \varepsilon F(z(t) + y_0)$$

Therefore

$y(t)$ solves

$$\begin{cases} \dot{y} = F(y) \\ y(t_0) = y_0 \end{cases} \iff \begin{cases} \dot{z} = \varepsilon F(z + y_0) \\ z(0) = 0 \end{cases}$$

It is equivalent to find zeroes of

$$\mathcal{G} : D \times \mathcal{U}_0 \times \mathbb{R} \longrightarrow \mathcal{C}([-1,1], Y)$$
$$(z, y_0, \varepsilon) \longrightarrow \frac{d}{dt} z - \varepsilon F(z + y_0)$$

$$D = \{ z \in C^1([-1,1], B^Y_\delta(0)) \text{ with } z(0) = 0 \}$$

with δ so small so that $\mathcal{U}_0 + B^Y_\delta(0) = \{ y_0 + z \in \mathcal{U}_0 + B^Y_\delta(0) \}$ is contained in $\mathcal{U}$ ($\rightsquigarrow F(z_0 + y)$ well defined)

To apply IFT: we need

- $\mathcal{G}(0, y_0, 0) = 0$
- $\mathcal{G} \in C^1(D \times \mathcal{U}_0 \times \mathbb{R}; \mathcal{C}([-1,1]; Y))$

$z \mapsto \frac{d}{dt} z$ is $C^1 \iff$ bounded : $\|\frac{d}{dt} z\|_{C^0} \leq \|z\|_{C^1}$

$$\shortparallel$$
$$\|z\|_D$$

$D \to C^0([-1,1];Y)$ it is of class C^1

$z(t) \to F(z(t)+y_0)$

Lemma [T, Lemma 7.30] $I \subseteq \mathbb{R}$ compact, $\mathcal{U} \subseteq X$ open, $f \in C^r(\mathcal{U}, Y)$

Then the composition op

$$F: C(I, \mathcal{U}) \to C(I, Y)$$
$$z(t) \to F(z(t)) := f(z(t))$$

is of class C^r.

In particular $dF(z(t))[h(t)] = df(z(t))[h(t)]$

using this lemma: $z(t) \to F(z(t)+y_0) \in C^r$
$: D \to C^0$

Invertibility of $d_z G(0, y_0, 0)[h(t)] = \frac{d}{dt} h(t)$

We want $d_z G(0,y_0,0)$ invertible: $\forall g \in C^0$, find $h \in C^1$ with $h(0)=0$ with $Ah = g$

$\Leftrightarrow \frac{d}{dt} h = g \Leftrightarrow h(t) = h(0) + \int_0^t g(s)ds = \int_0^t g(s) ds$

$\rightsquigarrow h \in C^1(I, \mathcal{U})$

$\xrightarrow{\text{IFT}}$ $\exists!$ map $z_{(y_0,\varepsilon)} \in C^1(\mathcal{U}_0 \times (-\varepsilon_0, \varepsilon_0); D)$, so that

$$G(z_{(y_0,\varepsilon)}, y_0, \varepsilon) \equiv 0$$

Now fix $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and note that

$z_{(y_0,\varepsilon)} \in C^1([-1,1], Y) \quad \forall y_0 \in \mathcal{U}_0$

$\rightsquigarrow y(t, y_0, \varepsilon) := y_0 + z_{(y_0,\varepsilon)}\left(\frac{t-t_0}{\varepsilon}\right)$

is a curve defined $\forall y_0 \in \mathcal{U}_0$ for time

$\left|\frac{t-t_0}{\varepsilon}\right| \leq 1 \Rightarrow |t-t_0| \leq \varepsilon$, solving $\dot{y} = F(y)$

Moreover: $(t, y_0) \longrightarrow y(t, y_0, \varepsilon) \in C^1\left((t_0-\varepsilon, t_0+\varepsilon) \times \mathcal{U}_0, Y\right)$

This proves the result for $r=1$

For $r>1$ we use induction: Assume that $F \in C^{r+1}$ and let $y(t, y_0)$ be the sol of class C^r of $\begin{cases} \dot{y} = F(y) \\ y(0) = y_0 \end{cases}$

$$\rightsquigarrow \begin{cases} \frac{d}{dt} y(t, y_0) = F(y(t, y_0)) \\ y(0, y_0) = y_0 \end{cases}$$

take the diff in y_0 VARIATION EQUATION

$$\begin{cases} \frac{d}{dt} d_{y_0} y(t, y_0) = d_y F(y(t, y_0)) \underbrace{d_{y_0} y(t, y_0)}_{B(t) \in \mathcal{L}(\mathcal{U}, Y)} \\ d_{y_0} y(0, y_0) = \mathbb{1} \end{cases}$$

$\rightsquigarrow B(t, y_0)$ solves an eq in Banach space $\mathcal{L}(\mathcal{U}, Y)$

$$\begin{cases} \frac{d}{dt} B(t, y_0) = d_y F(y(t, y_0)) B(t, y_0) \\ B(0, y_0) = \mathbb{1} \end{cases}$$

$B(t)$ solves an eq with a vector field of class C^r

$\rightsquigarrow B(t, y_0)$ is of class C^r

$\rightsquigarrow y_0 \to y(t, y_0)$ is of class C^{r+1}

Finally $\frac{d}{dt} y(t, y_0) = F(y(t, y_0)) \in C^r$ in time $\rightsquigarrow t \to y(t, y_0) \in C^{r+1}$ $\square$