PROJECTIONS

Def $P \in \mathcal{L}(H)$, H Hilbert space, **projection** if

$$P^2 = P$$

P is orthogonal projection if $P^* = P$

Rem 1 P project $\Rightarrow$ Ran P is closed

Indeed $x_n \to x$, $(x_n)_n \subseteq \operatorname{Ran} P$

$\Rightarrow$ $x_n = P y_n$ $\qquad$ P continuous

$\leadsto$ $P x_n = P^2 y_n = P y_n = x_n$ $\Rightarrow$ $\underset{\underset{x_n}{\parallel}}{P x_n} \to \underset{\underset{x}{\parallel}}{P x}$, i.e. $x_n \to x$

$\leadsto$ $x \in \operatorname{Ran} P$

Rem 2 $P \neq 0$ orthog. proj $\Rightarrow$ $\|P\| = 1$

Indeed $\|P\|^2 = \|P^* P\| = \|P^2\| = \|P\|$

Rem 3 P orth proj $\Rightarrow$ $H = \operatorname{Ran} P \oplus (\operatorname{Ran} P)^\perp$

Indeed $x = \underbrace{Px} + \underbrace{(\mathbb{1} - P)x}$; $\langle Px, (\mathbb{1}-P)x \rangle = \langle (\mathbb{1}-P)^* P x, x \rangle = \langle (P - P^2)x, x \rangle = 0$

$V \subset H$, $\quad V^\perp = \{ x \in H : (x, v) = 0 \ \forall v \in V \}$

$\hookrightarrow$ orth. complement of V

P orth. project $\Longleftrightarrow$ $H = V \oplus V^\perp$

given a closed lin sub V of H, associate to it an orth. projection

__Thm__ V closed lin sub of H, Hilbert, then

$$H = V \oplus V^\perp$$

In particular $\forall x \in H$, $\exists!\ v \in V$ st $x - v \perp V$
$\Downarrow$
$x - v \in V^\perp$

i.e. $\exists!$ decomposition $x = v + v^\perp$ with $v \in V$, $v^\perp \in V^\perp$

__proof__ Thm 5.4.5 of Bogadev – Smolyanov

Given $V \subseteq H$, closed, define

$$P_V \equiv P : H \longrightarrow V$$
$$x \longmapsto P_V x = v$$

i.e. the unique element of V st. $x - v \in V^\perp$

__Lemma__ P_V is orth. projection. and $V = \{x : P_V x = x\}$
$V^\perp = \ker P_V$

__proof__ $\underline{P_V \text{ is linear}}$ by the uniqueness of decomposition

given $x \in H$, $\exists!$ v st. $x - v \in V^\perp$, $x = v + v^\perp$

$$\lambda x = \underbrace{\lambda v}_{\in V} + \underbrace{\lambda v^\perp}_{\in V^\perp} \quad \& \quad \lambda x - \lambda v = \lambda v^\perp \in V^\perp$$

$\Rightarrow \quad P(\lambda x) = \lambda v = \lambda P x$

similarly one has $P(x+y) = Px + Py$ (exercise)

<u>$P^2 = P$</u> $\quad P^2 x = P(Px)$

and Px is the ! vector in V s.t $x - Px \in V^\perp$

then $P(Px)$ is the ! vector in V s.t. $Px - P(Px) \in V^\perp$

clearly Px fulfills this last requirement

$\rightsquigarrow$ by uniqueness $P^2 x = Px$

<u>$P^* = P$</u> $\quad (Px, y) = (\underbrace{\underset{\substack{\cap \\ V}}{Px}, \underset{\substack{\cap \\ V^\perp}}{\underbrace{y - Py}}}_{\Rightarrow 0}) + (Px, Py)$

$= (Px, Py)$

$= (x, Py) + (\overbrace{\underbrace{Px - x}_{\in V^\perp}, \underset{\substack{\cap \\ V}}{Py}}^{\to\, = 0})$

$= (x, Py)$

<u>P continuous</u> $\quad$ by previous remark $\|P\|^2 = \|P\| \Rightarrow \|P\| = \begin{cases} 0 \\ 1 \end{cases}$

<u>$\ker P = V^\perp$</u> $\quad x \in \ker P \iff Px = 0 \iff x \in V^\perp$

<u>$\operatorname{Ran} P = \{x : Px = x\}$</u> $\quad (\subseteq)\ y \in \operatorname{Ran} P \Rightarrow \exists x \in H : Px = y$

$$\Rightarrow \underset{\substack{\| \\ Px}}{P^2 x} = Py$$

$$Px = y$$

$(\supseteq)$ trivial

<u>Rem</u> $\quad H = V \oplus V^\perp = \operatorname{Ran} P \oplus \ker P$

CONVERGENCE OF OPERATORS

Def $(T_n)_{n\geq 1} \subset \mathcal{L}(X)$, X Banach. Then $T_n \to T$

- uniformly if $\|T_n - T\|_{\mathcal{L}(X)} \to 0$
  (in the op. topology)
- strongly if $T_n x \to Tx \quad \forall x \in X$
  (pointwise convergence)
- weakly if $\forall \ell \in X^*$, $\forall x \in X$
  $\ell(T_n x) \to \ell(Tx)$ (dual convergence)

Rem H Hilbert weakly $\iff$ $\langle T_n x, y\rangle \to \langle Tx, y\rangle$
$\forall x, y \in H$
(by Riesz)

Rem uniform $\Rightarrow$ strong $\Rightarrow$ weak

EXERCISE $\ell^2(\mathbb{N})$, $\|\ \|_2$. Then prove that

(1) $T_n(x_1, x_2, x_3, \dots) = \left(\frac{1}{n} x_1, \frac{1}{n} x_2, \dots\right)$

$T_n \to 0$ uniformly

(2) $S_n(x_1, x_2, \dots) = (\underbrace{0, \dots, 0}_{n\text{-times}}, x_{n+1}, x_{n+2}, \dots)$

$S_n \to 0$ strongly, but not uniformly

(3) $W_n(x_1, x_2, \dots) = (\underbrace{0, \dots, 0}_{n\text{-times}}, x_1, x_2, \dots) \rightsquigarrow W_n \xrightarrow{\text{weakly}} 0$
not strongly

Inverses by Newmann series

Assume to have $A \in \mathcal{L}(X)$ and consider

$$(\mathbb{1} - A)u = f \qquad , \quad u, f \in X$$

Can we solve it? i.e. $\forall f \in X$, $\exists!$ $u \in X$ solving it?

If $\|A\| < 1$, then we can invert by Newmann series

Prop If $A \in \mathcal{L}(X)$ with $\|A\| < 1$, then

$\mathbb{1} - A$ is invertible and

$$(\mathbb{1} - A)^{-1} = \sum_{k \geq 0} A^k = \mathbb{1} + A + A^2 + \dots$$

(analogous to $\frac{1}{1-x} = \sum_{k \geq 0} x^k$, $|x| < 1$)

proof $S_N := \sum_{k=0}^{N} A^k$ is a Cauchy seq in $\sum_{k \geq 1} A^k$

$(\mathcal{L}(X), \|\ \|_o)$, this is Banach sp.:

$$\|S_M - S_N\| = \left\| \sum_{k=N+1}^{M} A^k \right\| \leq \sum_{k=N+1}^{M} \|A\|^k = s_M - s_N$$

$$s_N = \sum_{k=0}^{N} \|A\|^k$$

$\downarrow 0$ as $M, N \to \infty$

$S_N \to S$ in $(\mathcal{L}(X), \|\ \|_{op})$

" $\sum_{k \geq 0} A^k$

Consider $(\mathbb{1} - A) S_N = (\mathbb{1} - A) \sum_{k=0}^{N} A^k$

$$= \sum_{k=0}^{N} A^k - \sum_{k=0}^{N} A^{k+1}$$

$$= \sum_{k=0}^{N} A^k - \sum_{k=1}^{N+1} A^k$$ telescopic series

$$= \mathbb{1} - A^{N+1}$$

$$\rightsquigarrow \quad (\mathbb{1} - A) S_N - \mathbb{1} = - A^{N+1}$$

$$\| (\mathbb{1} - A) S_N - \mathbb{1} \| = \| A^{N+1} \| \leq \underbrace{\| A \|}_{\|A\| < 1}{}^{N+1} \xrightarrow{N \to \infty} 0$$

$$\rightsquigarrow \quad (\mathbb{1} - A) S_N - \mathbb{1} \longrightarrow 0 \quad \Rightarrow \quad (\mathbb{1} - A) S_N \longrightarrow \mathbb{1}$$

$$\downarrow$$

$$(\mathbb{1} - A) S = \mathbb{1}$$

Similarly one proves $S(\mathbb{1} - A) = \mathbb{1}$

$$\rightsquigarrow \quad S = (\mathbb{1} - A)^{-1}$$

Application $D \in \mathcal{L}(X)$, invertible (D^{-1} exists $\& \in \mathcal{L}(X)$)

$D + A$ is invertible provided $\|A\| < \dfrac{1}{\|D^{-1}\|}$

Indeed $D + A = D(\mathbb{1} + D^{-1} A)$

$$\rightsquigarrow \quad (D + A)^{-1} = \underbrace{(\mathbb{1} + D^{-1} A)^{-1}}_{\text{this exists by Neumann series}} D^{-1}$$

provided $\| D^{-1} A \| < 1 \quad \Leftarrow \quad \|D^{-1} A\| \leq \|D^{-1}\| \, \|A\| < 1$

Lemma $\{$ Invertible operators $\}$ is open in $\mathcal{L}(X)$