A GLANCE ON TOPOLOGICAL METHODS

(Teschl, Nirenberg)

Topological methods to solve the eq $f(x)=y$ in bd domain

fin dim $\rightsquigarrow$ Brouwer

inf dim $\rightsquigarrow$ Schauder

Thm (Brouwer's fixed point thm) Let $\mathcal{U}$ open set with $\overline{\mathcal{U}}$ homeomorphic to $\overline{B_1(0)} \subseteq \mathbb{R}^n$ and $f: \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ continuous

$$\Rightarrow \exists\, x \in \overline{\mathcal{U}} : f(x) = x$$

What about ∞-dim version of Brouwer's thm?

Let $F: B_1^{\ell^2}(0) \longrightarrow \ell^2(\mathbb{N})$

$$x = (x_1, x_2, \dots) \longrightarrow \left(\sqrt{1-\|x\|^2},\ x_1,\ x_2,\ \dots\right)$$

F continuous, $\forall\, x \in \overline{B_1^{\ell^2}(0)}$ we have $\|F(x)\| = 1$

so $F\left(\overline{B_1^{\ell^2}(0)}\right) \subseteq \partial B_1(0) = \{x \in \ell^2 : \|x\| = 1\}$

Assume $\exists$ x fixed point in the ball:

$$x = F(x) \Rightarrow \|x\| = \|F(x)\| = 1$$

$$\Rightarrow x_1 = [F(x)]_1 = 0$$

$$x_2 = [F(x)]_2 = x_1 = 0$$

$$\vdots$$

$$x_j = 0 \qquad \forall j$$

$\Rightarrow x = 0$, but $F(0) = (1, 0, \dots)$ $\lightning$

Brouwer's thm fails in ∞-dim space!

Need extra assumption: F is compact perturb of $\mathbb{1}$

Def $F : \mathcal{U} \subseteq X \to X$, X Banach, F is compact if it is continuous and $\forall B \subseteq \mathcal{U}$, B bounded, $\overline{F(B)}$ is compact

Denote $\mathcal{K}(\mathcal{U}, X)$ the set of compact maps

Rem •) If $F \in \mathcal{L}(X)$, this def reduces to the "old" one

•) $f : \mathbb{R} \to \mathbb{R}$ continuous, $F : C^0([0,1]) \to C^0([0,1])$

$$u \longmapsto F(u)(t) = \int_0^t f(u(s))\, ds$$

Then F compact (CHECK IT using Ascoli-Arzelà)

Properties of compact maps

Prop Let $(F_j)_{j \geq 1}$, $F_j : \mathcal{U} \to X$ compact $\forall j$ and such that $F_j \to F$ in the sup norm (i.e. $\sup_{x \in \mathcal{U}} \| F_j(u) - F(u) \| \to 0$) to some $F : \mathcal{U} \to X$ continuous

Then F is compact.

proof Let $B \subseteq U$ bd.

claim $\forall \varepsilon > 0 \;\; \exists N_\varepsilon \in \mathbb{N} : \;\; \overline{F(B)} \subseteq \bigcup_{i=1}^{N_\varepsilon} B_\varepsilon(y_i)$

This is true for $\overline{F_j(B)}$ since this set is compact.

Now, given $\varepsilon > 0$, take j : $\| F - F_j \|_\infty \leq \varepsilon/2$

As $\overline{F_J(B)}$ compact, $\exists$ $\frac{\varepsilon}{2}$-net:

$$\overline{F_J(B)} \subseteq \bigcup_{i=1}^{N_\varepsilon} B_{\frac{\varepsilon}{2}}(y_i)$$

We claim it is ε-net for $\overline{F(B)}$. Indeed $\forall$ any $x \in B$ we have $F(x)$ belongs to $B_\varepsilon(y_a)$ for some y_a. Indeed take y_a s.t. $F_J(x) \in B_{\varepsilon/2}(y_a)$.

Then $\| F(x) - y_a \| \leq \| F(x) - F_J(x) \| + \| F_J(x) - y_a \|$

$$\leq \| F - F_J \|_\infty + \frac{\varepsilon}{2} = \varepsilon$$

$\rightsquigarrow$ $F(B) \subseteq \bigcup_{i=1}^{N} B_\varepsilon(y_i)$ $\rightsquigarrow$ $\overline{F(B)} \subseteq \bigcup_{i=1}^{N} B_\varepsilon(y_i)$

Cor $F: X \to X$ continuous with $\sup_{x \in X} \|F(x)\| < +\infty$ and $\exists (F_J)_J$ s.t. $F_J \to F$ in the sup norm and $\forall J$: $\dim(\operatorname{Im} F_J(B)) < \infty$ $\forall B$ bd

$\Rightarrow$ F compact

Rem The converse approximation problem, for linear maps
- in Hilbert spaces, it is true
- In Banach space, it is false

Dropping linearity, converse is valid in Banach

Prop $F: \mathcal{U} \subseteq X \to X$, $\mathcal{U}$ open bounded and F compact. Then $\forall \varepsilon > 0$ $\exists$ F_ε continuous s.t.

(i) $\| F - F_\varepsilon \|_\infty \leq \varepsilon$

(ii) $\dim(\operatorname{Im} F_\varepsilon(\mathcal{U})) < +\infty$

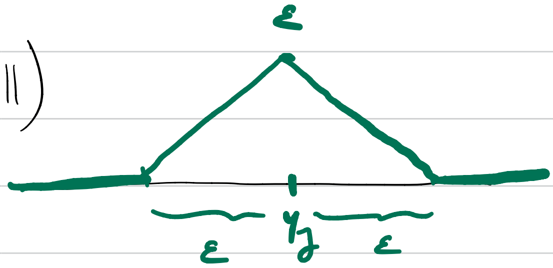
proof F compact $\Rightarrow$ $\forall \varepsilon > 0$, $\exists\, y_1, \dots, y_p$:

$$\overline{F(\mathcal{U})} \subset \bigcup_{i=1}^{p} B_\varepsilon(y_i)$$

and we choose $(y_i)_{i=1,\dots,p} \in \overline{F(\mathcal{U})}$

Let $\varphi_j^\varepsilon(y) := \max(0, \varepsilon - \|y - y_j\|)$

$\varphi_j^\varepsilon : X \to \mathbb{R}$



IF $y \in \overline{F(\mathcal{U})}$, $\exists\, j$: $\varphi_j^\varepsilon(y) \neq 0$

$$\rightsquigarrow \quad \psi_j^\varepsilon(y) := \frac{\varphi_j^\varepsilon(y)}{\sum_a \varphi_a^\varepsilon(y)}$$

•) is well defined for $y \in \overline{F(\mathcal{U})}$

•) $\sum_j \psi_j^\varepsilon(y) = 1$

•) $\psi_j^\varepsilon(y) \neq 0$ iff $\|y - y_j\| \leq \varepsilon$

Put $$F_\varepsilon(x) := \sum_{a=1}^{p} \psi_a^\varepsilon(F(x))\, y_a$$

•) F_ε continuous

•) $\operatorname{Im} F_\varepsilon \subseteq \operatorname{span}(y_1, \dots, y_p)$ $\rightsquigarrow$ $\operatorname{Im} F_\varepsilon \subseteq$ finite dim

•) $$F(x) - F_\varepsilon(x) = \sum_{a=1}^{p} \psi_a^\varepsilon(F(x))\, F(x) - F_\varepsilon(x)$$

$$= \sum_{a=1}^{p} \psi_a^\varepsilon(F(x)) \big(F(x) - y_a \big)$$

$$\Rightarrow \|F(x) - F_\varepsilon(x)\| \leq \sum_{a=1}^{p} \underbrace{\psi_a^\varepsilon(F(x))\, \|F(x) - y_a\|}_{\neq 0 \text{ only if } \|F(x) - y_a\| \leq \varepsilon}$$

$$\leq \underbrace{\sum_{a=1}^{p} \psi_a^\varepsilon(F(x))}_{=1}\, \varepsilon = \varepsilon$$

$\rightsquigarrow \|F - F_\varepsilon\|_\infty \leq \varepsilon$ $\square$

Lemma $F : \mathcal{U} \subseteq X \to X$, $\mathcal{U}$ open bd and F compact, then $\mathbb{1} + F$ is closed (it maps closed set into closed sets)

proof $B \subseteq \mathcal{U}$ closed, $(x_n)_n \subseteq B$

Want $(\mathbb{1} + F)(B)$ closed : assume $x_n + F(x_n) \to y$,

want to prove $y = x + F(x)$ for some $x \in B$.

Since F compact, $\exists\ (x_{n_k})$ with $F(x_{n_k}) \to \overline{y}$

so $x_{n_k} = \underbrace{x_{n_k} + F(x_{n_k})}_{\downarrow\, y} - \underbrace{F(x_{n_k})}_{\downarrow\, \overline{y}} \to y - \overline{y}$

$\Rightarrow$ $y - \overline{y} \in B$ since B closed

Call $x := y - \overline{y}$, then

$$x + F(x) = y - \overline{y} + F(y - \overline{y})$$

(F cont.) $$= y - \overline{y} + \lim_{k \to +\infty} F(x_{n_k}) = y$$

$\square$

Leray – Schauder fixed point theorem

Def The <u>convex hull</u> of V is

$$\operatorname{conv}(V) = \bigcap_{\substack{A \text{ convex} \\ A \supseteq V}} A$$

Thm (Leray-Schauder) Let D be a closed, convex, bounded subset of a Banach space X and

$$F : D \longrightarrow D \quad \text{compact.}$$

Then F has a fixed point

proof Given $n > 0$ arbitrary, let F_n be an approximation of F with

- $\|F - F_n\|_\infty < 1/n$
- $\text{Im } F_n \subseteq \underbrace{\text{span} \langle y_1, \dots, y_{N(n)} \rangle}_{N_n}$ with $(y_a) \subseteq F(D) \subseteq D$ (as in the previous proof)

Moreover
$$\begin{cases} F_n(x) = \sum \psi_a^n (F(x)) \, y_a & \forall x \in D \\ \sum \psi_a^n (x) = 1 \end{cases}$$

$\Rightarrow \quad F_n(D) \subseteq \text{conv} \{ (y_a) \} \subseteq \text{Conv}(D) \subseteq D$

$\rightsquigarrow \quad F_n : D \to D \cap N_n$

$\rightsquigarrow \quad F_n|_{N_n \cap D} : N_n \cap D \to \underbrace{D \cap N_n}_{\text{closed \& convex \& fin dim}}$

Brouwer's fixed point thm :

$\forall n \quad \exists x_n : \quad F_n(x_n) = x_n$

$\rightsquigarrow \quad (x_n)_n \subset D$ bd $\rightsquigarrow \left(F(x_n) \right)_n$ has subseq convergent :

$F(x_{n_k}) \longrightarrow z \in D$ (as D closed)

Then $x_{n_k} - F(x_{n_k}) = F_{n_k}(x_{n_k}) - F(x_{n_k})$

$\Rightarrow \| \underbrace{x_{n_k} - F(x_{n_k})}_{\downarrow z} \| \leq \| F_{n_k} - F \|_\infty \leq \frac{1}{n_k} \to 0$

$\Rightarrow x_{n_k} \longrightarrow z$

As F continuous, $\underbrace{F(x_{n_k})}_{\downarrow z} \longrightarrow F(z)$ =

F has fixed point!

$\square$

There is another form of Schauder thm:

Thm K convex, compact set and $f: K \to K$ continuous. Then $\exists\ x \in K$ with $f(x) = x$

<u>EX</u> 1. prove this form (deduce it from the other one)

Applications of Schauder Theorem

Peano theorem Let $f \in C(I \times \mathcal{U}, \mathbb{R}^n)$ with $I \subseteq \mathbb{R}$, $\mathcal{U} \subseteq \mathbb{R}^n$ open and consider

$$(P) \begin{cases} \dot{x} = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$ with $t_0 \in I$, $x_0 \in \mathcal{U}$.

Then (P) has at least 1 sol
$x(t) \in C^1((t_0-\varepsilon, t_0+\varepsilon), \mathbb{R}^n)$

proof integral formulation: $x(t)$ is a sol iff

$$x(t) = x_0 + \int_{t_0}^{t} f(t, x(t)) = F(x(t))$$

The map $F : C([0,1]) \longrightarrow C([0,1])$ is compact.

CHECK IT : (Ascoli–Arzelà)

Let $M := \sup_{t \in J,\ \|x - x_0\| \le \rho} \| f(t,x) \|$

and consider $D := \{ y(t) \in C^0([t_0-\delta, t_0+\delta], \mathbb{R}^n) : \sup_t \| y(t) - x_0 \| \le \rho \}$

Then ·) D closed and convex (it is a ball)

·) Does $F(D) \subseteq D$? Let $x(t) \in D$, then

$$\| F(x(t)) - x_0 \| = \left\| \int_{t_0}^{t} f(t, x(t))\, dt \right\|$$

$$\le \delta M \le \rho$$

provided δ is chosen suff. small!

$\rightsquigarrow$ by Schauder's theorem $\exists$ fixed point of F
it is a solution of (P)

Sturm-Liouville problem

$$(D)\begin{cases} -u'' = f(x, u(x)) \\ u(0) = u(1) = 0 \end{cases}$$

Thm Let $f: \Omega \times \mathbb{R} \to \mathbb{R}$ continuous and bounded. Then (D) has a classical solution.

proof Consider the auxiliary linear problem

$$\begin{cases} -u'' = g \\ u(0) = u(1) = 0 \end{cases}$$

By Sturm-Liouville theory, the sol op

$$T: L^2 \to L^2 \qquad \text{is compact}$$
$$g \mapsto Tg$$

Consider $F(u) := T[f(x, u(x))]$

•) F continuous: f is C^0 and bd, $u \mapsto f(x,u)$ is a Nemitzki operator, continuous $L^2 \to L^2$

•) F compact: ok because T is compact

•) $F: D \to D$ with D closed and convex;

Put $M := \sup\limits_{x \in \Omega,\, t \in \mathbb{R}} |f(x,t)|$

$$D := \{ u \in L^2 : \|u\|_{L^2} \le R \}$$

Then $\| F(u) \|_{L^2} \leq C_1 \| f(x, u(x)) \|_{L^2} \leq C_1 M \leq R$

provided R large enough

$\Longrightarrow$ $\exists\, u$ fixed point in L^2 of $u = F(u)$

i.e. $u = T[f(x, u(x))]$

$\rightsquigarrow$ $f : L^2 \to L^2$, $T : L^2 \to H^1_0$ so $u \in H^1_0$

so u is weak sol $\Rightarrow$ $u(0) = u(1) = 0$.

Then $u \in H^1_0 \hookrightarrow C([0,1])$ and so $-u'' = f(x, u(x)) \in C^0$

$\Longrightarrow$ $u \in C^2([0,1])$ and $u(0) = u(1) = 0$ $\to$ classical solution.

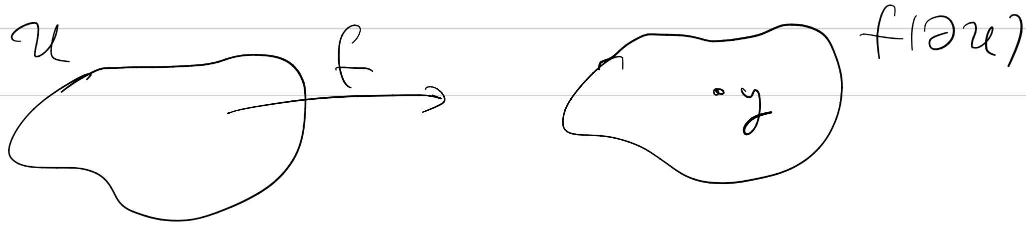
Topological degree theory

Goal: build a function "counting solutions" continuous and stable under perturbations

Take $\mathcal{U} \subset \mathbb{R}^n$ open - bounded, $y \in \mathbb{R}^n$

$$D_y(\mathcal{U}; \mathbb{R}^n) = \{ f : \overline{\mathcal{U}} \to \mathbb{R}^n : \text{continuous}, \ y \notin f(\partial \mathcal{U}) \}$$

($y \notin f(\partial\mathcal{U})$ because we want stability under perturbations)



EXERCISE $D_y(\mathcal{U};\mathbb{R}^n) \subseteq C^0(\mathcal{U};\mathbb{R}^n)$ open

Def A degree map is a function

$$\deg(\cdot,\mathcal{U};y) : D_y(\mathcal{U};\mathbb{R}^n) \longrightarrow \mathbb{R}$$
$$f \longrightarrow \deg(f,\mathcal{U},y)$$

s.t.

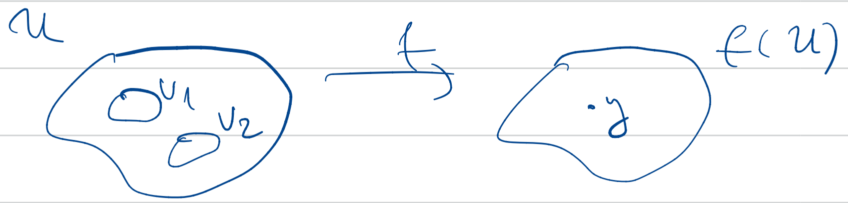
(D1) $\deg(f,\mathcal{U},y) = \deg(f-y,\mathcal{U},0)$ invariance under translation

(D2) $\deg(\mathbb{1}_{\mathbb{R}^n},\mathcal{U},y) = \begin{cases} 1 & \text{if } y\in\mathcal{U} \\ 0 & \text{if } y\notin\mathcal{U} \end{cases}$ normalization

(D3) $\mathcal{U}_1,\mathcal{U}_2 \subseteq \mathcal{U}$ open, $\mathcal{U}_1\cap\mathcal{U}_2=\emptyset$,

$y \notin f(\overline{\mathcal{U}} \setminus (\mathcal{U}_1\cup\mathcal{U}_2))$ additivity

$\Rightarrow \deg(f,\mathcal{U},y) = \deg(f,\mathcal{U}_1,y) + \deg(f,\mathcal{U}_2,y)$



(D4) If $h(t,x)$ is admissible homotopy, i.e.

·) $h \in C([0,1]\times\overline{\mathcal{U}};\mathbb{R}^n)$

·) $h(t,x) \neq y \quad \forall (t,x)\in[0,1]\times\partial\mathcal{U}$ $\left(\Rightarrow h(t,\cdot)\in D_y(\mathcal{U};\mathbb{R}^n) \;\; \forall t\right.$

Then $\deg(h(t,\cdot),\mathcal{U},y)$ does not depend on t

If degree exists, it has immediately the following properties

Prop Let deg be given, then

(i) $\deg(f, \emptyset, y) = 0$

(ii) If $y \notin f(\overline{\mathcal{U}} \setminus \bigcup_{i=1}^{n} \mathcal{U}_i)$, $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ for $i \neq j$

$$\Rightarrow \deg(f, \mathcal{U}, y) = \sum_{i=1}^{n} \deg(f, \mathcal{U}_i, y)$$

(iii) $f, g \in D_y(\mathcal{U}; \mathbb{R}^n)$ and

$$\operatorname{dist}(f(x), g(x)) < \operatorname{dist}(y, f(\partial \mathcal{U})) \quad \forall x \in \partial \mathcal{U}$$

$$\Rightarrow \deg(f, \mathcal{U}, y) = \deg(g, \mathcal{U}, y)$$

In particular, if $f = g$ on $\partial \mathcal{U}$

$$\Rightarrow \deg(f, \mathcal{U}, y) = \deg(g, \mathcal{U}, y)$$

(iv) if $\deg(f, \mathcal{U}, y) \neq 0 \Rightarrow \exists\, x \in \mathcal{U} : f(x) = y$

(v) The map
$$\begin{aligned} D_y(\mathcal{U}, \mathbb{R}^n) &\longrightarrow \mathbb{R} \\ f &\longrightarrow \deg(f, \mathcal{U}, y) \end{aligned}$$
is locally constant

for $f \in D_y(\mathcal{U}; \mathbb{R}^n)$
$$\begin{aligned} \mathbb{R}^n \setminus f(\partial \mathcal{U}) &\longrightarrow \mathbb{R} \\ y &\longrightarrow \deg(f, \mathcal{U}, y) \end{aligned}$$
is locally constant

So they are both continuous and constant on connected components proof (i) Use (D3) with $\mathcal{U}_1 = \mathcal{U}$, $\mathcal{U}_2 = \emptyset$

(ii) (D3) + induction

(iii) $h(t,\cdot) = (1-t)f + tg$

check h is admissible: $h(t,x) \neq y \quad \forall (t,x) \in [0,1] \times \partial \mathcal{U}$

i.e. $\quad \operatorname{dist}(y, h(t, \partial\mathcal{U})) > 0 \qquad \forall \ t \in [0,1]$

$$\operatorname{dist}(y, h(t,\partial\mathcal{U})) = \operatorname{dist}(y, f(\partial\mathcal{U})) - \| h(t,\cdot) - f \|_{L^\infty(\partial\mathcal{U})}$$

$$\geq \operatorname{dist}(y, f(\partial\mathcal{U})) - \| f - g \|_{L^\infty(\partial\mathcal{U})} > 0$$

$\overset{(D4)}{\Longrightarrow}$

$$\underset{\substack{\| \\ \deg(f,\mathcal{U},y)}}{\deg(h(0,\cdot),\mathcal{U},y)} = \underset{\substack{\| \\ \deg(g,\mathcal{U},y)}}{\deg(h(1,\cdot),\mathcal{U},y)}$$

(iv) B.C. $y \notin f(\mathcal{U})$, we also know $y \notin f(\partial\mathcal{U}) \Rightarrow y \notin f(\overline{\mathcal{U}})$

(D3) with $U_1 = U_2 = \emptyset$ gives

$$\deg(f,\mathcal{U},y) = \underbrace{\deg(f,\mathcal{U}_1,y)}_{=0} + \underbrace{\deg(f,\mathcal{U}_2,y)}_{=0 \text{ by (i)}} = 0 \quad ⚡$$

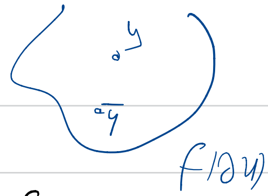
(v) $\circ$) $f \in D_y(\mathcal{U},\mathbb{R}^n)$, $g \in C(\overline{\mathcal{U}},\mathbb{R}^n)$ with $\|f-g\|_{L^\infty} \ll \operatorname{dist}(y, f(\partial\mathcal{U}))$

Then $g \in D_y(\mathcal{U};\mathbb{R}^n)$

Put $h(t,\cdot) = (1-t)f + tg$ is admissible homotopy

$\rightsquigarrow \quad \deg(f,\mathcal{U},y) = \deg(g,\mathcal{U},y)$

•) take $\bar y$ with $|\bar y - y| < \operatorname{dist}(y, f(\partial U))$



put $y(t) := (1-t)\,y + t\,\bar y \rightsquigarrow \operatorname{dist}(y(t), f(\partial U)) > \delta > 0$

$\rightsquigarrow\ f - y(t) \in D_0(U; \mathbb{R}^n) \quad \forall t \in [0,1]$

"$\{g : 0 \notin g(\partial U)\}$"

$\rightsquigarrow\ \deg(f, U, y) \overset{(D1)}{=} \deg(f-y, U, 0)$

$\|f - y(t) - (f-y)\| \longrightarrow \quad = \deg(f - y(t), U, 0) \quad \forall t \in [0,1]$

$\|y(t) - y\| \le \operatorname{dist}(0, f(\partial U) - y) \quad = \deg(f - \bar y, U, 0)$

$= \deg(f, U, \bar y)$ ☒

FACT: if y is a regular value of $f \in C^1$ then (D1 – D4) determine deg uniquely

Def •) $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$, $y \in \mathbb{R}^n$ is a regular value of f if $\forall x \in f^{-1}(y)$ $Jf(x)$ is invertible (if $y \notin f(U)$, it is regular)

•) y is critical value if y not regular

Thm (Sard) $f \in C^1$, the set of critical values has measure 0

Cor U open bd, $y \notin f(\partial U)$ regular value, then

$$\#\{x \in U : f(x) = y\} < +\infty$$

and if $f^{-1}(y) = \{x_1, \dots, x_n\}$ then $\exists\ U_{x_i}$ neighb of x_i and $\mathcal{U}_y$ neigh of y: $\left| \begin{array}{l} f: \mathcal{U}_{x_i} \to \mathcal{U}_y \text{ is bijective} \\ f^{-1}(\mathcal{U}_y) = \bigcup_{i=1}^{n} \mathcal{U}_{x_i} \end{array} \right.$

proof $f^{-1}(y)$ closed in $\overline{\mathcal{U}}$, so compact. By inverse function thm, f loc diffeo around x $\forall x : f(x) = y$
$\rightsquigarrow$ $f^{-1}(y)$ is compact set of isolated points, so it has finitely many points

☒

Thm $f \in D_y(\mathcal{U}; \mathbb{R}^n)$, $f \in C^1$, y reg value, then any degree map has the form

$$\deg(f, \mathcal{U}; y) = \sum_{x \in f^{-1}(y) \cap \mathcal{U}} \operatorname{sgn}(\det Jf(x))$$

(agreement: $\sum_{x \in \emptyset} = 0$)

What about y critical value and $f \in C^0$?

Idea $\cdot)$ critical value: pick y_1, y_2 regular values with $|y - y_1|, |y - y_2| \ll 1$ $\rightsquigarrow$ $\deg(f, \mathcal{U}, y_1) = \deg(f, \mathcal{U}, y_2)$

Put $\deg(f, \mathcal{U}, y) := \lim_{n \to \infty} \deg(f, \mathcal{U}, y_n)$

with $y_n \to y$, y_n regular value $\forall n$

$\cdot)$ $f \in C^0$: approximate with C^1 fct: take $(f_n)_n \subseteq C^1(\mathcal{U}; \mathbb{R}^n) \cap C^0(\overline{\mathcal{U}}; \mathbb{R}^n)$ with $f_n \to f$ uniformly in $\overline{\mathcal{U}}$.

Put $\deg(f, \mathcal{U}, y) := \lim_{n \to \infty} \deg(f_n, \mathcal{U}, y)$

lim well defined because $g \to \deg(g, \mathcal{U}, y)$ is loc. const.

Thm (Brouwer's fixed point) Let $\mathcal{U}$ open with $\overline{\mathcal{U}}$ homeomorphic to $\overline{B_1(0)} \subseteq \mathbb{R}^n$ and $f: \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ continuous.
Then $\exists x \in \overline{\mathcal{U}}$: $f(x) = x$

Leray – Schauder degree

Look for a degree for maps $\mathbb{1} + F$, F compact, which satisfies (D1 – D4)

Idea: approximate F with fin dim range maps F_ε acting on finite dimensional space X_ε

Consider the Brouwer degree $\deg\left((\mathbb{1} + F_\varepsilon)_{|X_\varepsilon}\right)$ and take the limit

We will need the following

Lemma $f : \overline{\mathcal{U}} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ continuous, $m < n$, $\mathcal{U}$ open, bd, $y \in \mathbb{R}^m \setminus (\mathbb{1} + f)(\partial \mathcal{U})$ then

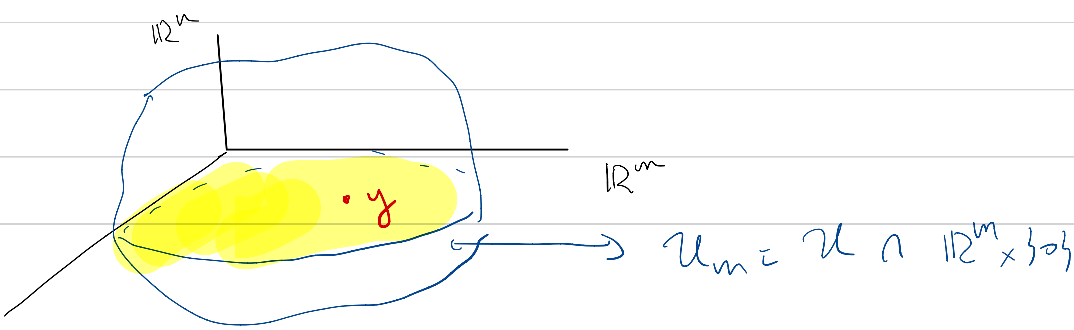
$$\deg(\mathbb{1} + f, \mathcal{U}, y) = \deg\left((\mathbb{1} + f)_{|\mathcal{U}_m}, \mathcal{U}_m, y\right)$$

where $\mathcal{U}_m = \mathcal{U} \cap \mathbb{R}^m = \mathcal{U} \cap (\mathbb{R}^m \times \underbrace{\{0\}}_{n-m \text{ times}})$

Rem •) $\mathbb{R}^m \simeq \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$

$$f : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \begin{pmatrix} f(x) \\ 0 \end{pmatrix} \begin{array}{l} \} \ m \text{ - components} \\ \} \ n-m \text{ components} \end{array}$$

•) $y \in \mathbb{R}^m$, $y \simeq \begin{pmatrix} y \\ 0 \end{pmatrix} \in \left(\mathbb{1} + \begin{pmatrix} f \\ 0 \end{pmatrix}\right)(\partial \mathcal{U})$



$\circ)$ Let $x \in \overline{\mathcal{U}}$ s.t. $x + f(x) = y \Leftrightarrow x = y - f(x) \in \mathbb{R}^m$

$\rightsquigarrow (\mathbb{1} + f)(x) = (\mathbb{1} + f)|_{\mathcal{U}_m}(x) = y$

$\rightsquigarrow x \in (\mathbb{1} + f)^{-1}_{|\mathcal{U}_m}(y)$

$\Rightarrow (\mathbb{1} + f)^{-1}(y) = \left((\mathbb{1} + f)_{|\mathcal{U}_m}\right)^{-1}(y)$

<u>proof</u> we assume $f \in C^1$ and y reg. value. enough to prove

$$\text{sign} \det d(\mathbb{1} + f)(x) = \text{sign} \det d(\mathbb{1} + f)_{|\mathcal{U}_m}(x)$$

$\forall x \in (\mathbb{1} + f)^{-1}(y) = \left((\mathbb{1} + f)_{|\mathcal{U}_m}\right)^{-1}(y)$

Recall $f(x) = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}$

$$\det d(\mathbb{1}+f)(x) = \det \begin{array}{c} \left[\begin{array}{c|c} \mathbb{1}_m + \partial_j f_i & \partial_j f_i \\ \hline 0 & \mathbb{1}_{n-m} \end{array}\right] \begin{array}{l} \}\, m \\ \}\, n-m \end{array} \\ \underbrace{\hphantom{\mathbb{1}_m + \partial_j f_i}}_{m} \; \underbrace{\hphantom{\mathbb{1}_{n-m}}}_{n-m} \end{array}$$

develop w.r.t. last $n-m$ rows:

$$= \det (\mathbb{1}_m + f)(x)$$

$\square$

Define $\mathcal{K}(\mathcal{U}, X) = \{F \in \mathcal{C}^0(\mathcal{U}; X),\ F \text{ compact}\}$

$\mathcal{F}(\mathcal{U}, X) = \{F \in \mathcal{C}^0(\mathcal{U}, X),\ \operatorname{Im} F \text{ fin dim}\}$

$D_y(\mathcal{U}, X) = \{F \in \mathcal{K}(\overline{\mathcal{U}}, X) : y \notin (\mathbb{1}+F)(\partial \mathcal{U})\}$

Rem $F \in D_y(\mathcal{U}, X) \Rightarrow \operatorname{dist}\left(y, (\mathbb{1}+F)(\partial\mathcal{U})\right) > 0$

For $F \in K(\overline{\mathcal{U}}, X)$, approximate it by $F_1 \in \mathcal{F}(\mathcal{U}, X)$ with

$$\| F - F_1 \|_\infty < \frac{1}{2} \operatorname{dist}(y, (\mathbb{1}+F)(\partial\mathcal{U})) = \rho/2$$

$\Rightarrow \operatorname{dist}(y, (\mathbb{1}+F_1)(\partial\mathcal{U})) > 0 \rightarrow F_1 \in \mathcal{F}_y(\mathcal{U}, X)$

Next, take $X_1 \subset X$ fin dim sub of X with

$$F_1(\mathcal{U}) \subset X_1 \quad , \quad y \in X_1$$

set $\mathcal{U}_1 := \mathcal{U} \cap X_1$, then $F_1 \in \mathcal{F}_y(\mathcal{U}_1, X_1)$

We put

$$\left[\deg(\mathbb{1}+F, \mathcal{U}, y) := \deg(\mathbb{1}+F_1, \mathcal{U}_1, y) \right]$$

LERAY - SCHAUDER DEGREE

Prop LS degree is well posed. Pick $F_2 \in \mathcal{F}(\overline{\mathcal{U}}, X)$ with $\|F_2 - F\|_\infty < \rho/2$, X_2 as before.

Put $X_0 := X_1 + X_2$, $\mathcal{U}_0 = \mathcal{U} \cap X_0$

By reduction thm $(y \in X_1, y \in X_2)$

$$\deg(\mathbb{1}+F_1, \mathcal{U}_0, y) = \deg(\mathbb{1}+F_1, \mathcal{U}_1, y)$$
$$\deg(\mathbb{1}+F_2, \mathcal{U}_0, y) = \deg(\mathbb{1}+F_2, \mathcal{U}_2, y)$$

Put $\quad H(t) = \mathbb{1} + (1-t) F_1 + t F_2$

It is admissible homotopy since

$$\| H(t) - (\mathbb{1} + F) \|_\infty \leq \| F_1 - F \|_\infty + \| F_2 - F \| < \rho$$

$$\leadsto \deg(\mathbb{1} + F_1, \mathcal{U}_1, y) = \deg(\mathbb{1} + F_1, \mathcal{U}_0, y)$$
$$= \deg(\mathbb{1} + F_2, \mathcal{U}_0, y)$$
$$= \deg(\mathbb{1} + F_2, \mathcal{U}_2, y)$$

$\boxtimes$

Thm $\mathcal{U}$ open, bdd, $\mathcal{U} \subseteq X$, $F \in D_y(\overline{\mathcal{U}}, X)$, $y \in X$
the LS degree fulfills (D1) – (D4), so the additional properties

$\boxtimes$