SPECRAL THEORY

Motivation: If $\dim X < \infty$, then we say usually that

$$\sigma(A) = \{\lambda \text{ eigenv: } Ax = \lambda x, x \neq 0\} \subseteq \mathbb{C}$$

What about $\dim X = +\infty$?

EXAMPLES

$$x = (x_1, x_2, x_3, \dots)$$

(1) $\ell^2(\mathbb{N})$, $(Tx) = (\alpha_1 x_1, \alpha_2 x_2, \dots)$

with $\{\alpha_j\}_{j \geq 1}$, $\alpha_j \in \mathbb{C}$, $\sup_j |\alpha_j| < +\infty$.

Then T is bd. Does T have eigenvalues?

$$Tx = \lambda x \iff \begin{pmatrix} \alpha_1 x_1 \\ \alpha_2 x_2 \\ \alpha_3 x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \vdots \end{pmatrix}$$

$\rightarrow$ $\alpha_j x_j = \lambda x_j \quad \forall j$

$\rightarrow$ $x = \vec{e}_j = (0, 0, 0, \dots, 0, \underset{j^{th}}{1}, 0, \dots)$

$$T\vec{e}_j = \alpha_j \vec{e}_j$$

$\Rightarrow$ $(\alpha_j)_{j \geq 1}$ are all eigenvalues with eigenvectors $\vec{e}_j$

(2) $L^2[0,1]$; $(Tf)(t) = t\,f(t)$

T is bd , $\|T\| \leq \|t\|_{L^\infty[0,1]} \leq 1$

Does T have eigenvalues? Look for $f \in L^2$:

$Tf = \lambda f \iff t\,f(t) = \lambda f(t)$ a.e. t

$\iff (t-\lambda) f(t) = 0$ a.e. t

$\Rightarrow f(t) = 0$ a.e.

No EIGENVALUES, NO EIGENVECTORS!

In inf dim, we need to improve the definition of spectrum. How?

In dim $X < +\infty$, $\lambda \in \rho(A) \iff \ker(A-\lambda) = 0$

$\iff \operatorname{Im}(A-\lambda) = X$

NULL + RANK THM $\iff A - \lambda$ invertible

If dim $X = +\infty$ we can have operators with

$\ker A = 0$, not surjective: $R(x_1, x_2, \ldots) = (0, x_1, x_2)$

$\ker A \neq 0$, surjective: $L(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$

Def (Resolvent set) X Banach, $T \in \mathcal{L}(X)$

$$\rho(T) = \{ \lambda \in \mathbb{C} : \ (T-\lambda)^{-1} \text{ exists and it is bd} \}$$

Def (Spectrum)

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

Rem $\sigma(T) = \{ \lambda \in \mathbb{C} : \ \ker(T-\lambda) \neq \{0\}$ or $\operatorname{Im}(T-\lambda) \neq X \}$

Indeed if $\ker(T-\lambda) = 0$ & $\operatorname{Im}(T-\lambda) = X$
$\rightsquigarrow$ $T-\lambda$ bijective and bd $\rightsquigarrow$ by inverse mapping thm $(T-\lambda)^{-1}$ is bd! So $\lambda \in \rho(T)$

We can further split the spectrum in diff. components:

$$\sigma_p = \{ \lambda \in \mathbb{C} : \ \ker(T-\lambda) \neq \{0\} \}$$ POINT SPECTRUM
EIGENVALUES!

$$\sigma_r = \{ \lambda \in \mathbb{C} : \ \ker(T-\lambda) = 0, \ \overline{\operatorname{Im} T-\lambda} \neq X \}$$ RESIDUAL SET
($\operatorname{Im}(T-\lambda)$ not dense in X)

$$\sigma_c = \{ \lambda \in \mathbb{C} : \ \ker(T-\lambda) = 0, \ \operatorname{Im} T-\lambda \text{ dense in } X \}$$ CONTINUOUS SPECTRUM
($\operatorname{Im} T-\lambda$ not closed
$\Updownarrow$
$\nexists c : \ \|(T-\lambda)x\| \geq c\|x\| \ \forall y$)

EX (1) $\ell^2(\mathbb{N})$, $Tx = (\alpha_1 x_1, \alpha_2 x_2, \dots)$

We saw that each $\alpha_j \in \sigma(T)$, actually to $\sigma_p(T)$

What about $\lambda \notin (\alpha_j)_{j\geq 1}$? is it in $\rho(T)$?

$\lambda \in \rho(T) \iff T-\lambda$ is inv. with bd inverse

Given y, $\exists !\, x : (T-\lambda)x = y \iff (\alpha_j - \lambda)x_j = y_j \quad \forall j$

$$\iff x_j = \frac{y_j}{\alpha_j - \lambda}$$

So if $\lambda \notin (\alpha_j)_{j\geq 1}$, $\vec{x}$ is well defined by

NOT FINISHED! $(T-\lambda)^{-1}$ must be bounded!

$$\|(T-\lambda)^{-1}y\|^2 = \sum_j \frac{|y_j|^2}{|\alpha_j - \lambda|^2}$$

If $\exists\, (\alpha_{j_k})$ with $\alpha_{j_k} \to \lambda$, then we have a problem!

If $\lambda \in \overline{(\alpha_j)_{j\geq 1}}$, then $\exists\, \alpha_{j_k} \to \lambda$

then take $y^{(k)} := \vec{e}_{j_k}$ and

$$\begin{cases} \|y^{(k)}\| = 1 \quad \forall k \\ \|(T-\lambda)^{-1} y^{(k)}\| = \dfrac{1}{|\alpha_{j_k} - \lambda|} \xrightarrow{k\to\infty} +\infty \end{cases}$$

$\rightsquigarrow (T-\lambda)^{-1}$ NOT BOUNDED

If $\lambda \in \overline{\{\alpha_j\}}_{j\geq 1} \Rightarrow \lambda \in \sigma(T)$

If $\lambda \notin \overline{(\alpha_j)}_{j\geq 1} \Rightarrow \operatorname{dist}(\lambda, \overline{(\alpha_j)}_{j\geq 1}) > \delta > 0$

$\rightsquigarrow (T-\lambda)^{-1}$ is bounded

Hence $\sigma(T) = \overline{(\alpha_j)}_{j\geq 1}$

$\sigma_p(T) = (\alpha_j)_{j\geq 1}$

$\sigma_c(T) = \overline{(\alpha_j)}_{j\geq 1} \setminus \sigma_p(T)$

(If $\lambda \in$ —, $\exists\, \alpha_{j_k} \to \lambda$, then $\|(T-\lambda)e_{j_k}\| = |\alpha_{j_k}-\lambda| \to 0$

cannot be $\|(T-\lambda)x\| \geq c\|x\| \ \forall x$!

(2) $L^2[0,1]$, $(Tf)(t) = t f(t)$

$(T-\lambda)f = g \iff (t-\lambda)f(t) = g(t)$ a.e.

$\iff f(t) = \dfrac{g(t)}{t-\lambda}$

Again if $\lambda \notin [0,1]$, then $|t-\lambda| \geq \delta > 0$, so $f \in L^2$

But if $\lambda \in [0,1]$, then take $g(t) = 1$ and

$f(t) = \dfrac{1}{t-\lambda} \notin L^2$

$\Rightarrow \sigma(T) = [0,1]$

Q: which is the nature of $\sigma(T)$?

Rem $T_1 : \ell^2 \to \ell^2$, $T_1 x = (\alpha_1 x_1, \alpha_2 x_2, \dots)$

$T_2 : L^2 \to L^2$, $(Tf)(t) = t f(t)$

So take $(\alpha_j)_{j \geq 1}$ the rationals in $[0,1]$

$$\sigma(T_1) = [0,1] = \sigma(T_2)$$

but the nature of the spectrum is different!

Def (Resolvent function) Given $\lambda \in \rho(A)$

$$R_\lambda(A) := (A - \lambda)^{-1}$$

$R_\lambda(A) \in \mathcal{L}(X)$

FIRST RESOLVENT IDENTITY: $\forall \lambda, \mu \in \rho(A)$

$$R_\lambda(A) - R_\mu(A) = (\lambda - \mu) R_\mu(A) R_\lambda(A)$$

SECOND RESOLVENT IDENTITY: $\forall \lambda \in \rho(A) \cap \rho(B)$

$$R_\lambda(A) - R_\lambda(B) = R_\lambda(A)(B - A) R_\lambda(B)$$

EXERCISE: prove them!

Properties of spectrum and resolvent

Lemma $T \in \mathcal{L}(X)$, then $\rho(T)$ is open and if $\lambda \in \mathbb{C}$, $\|T\| < |\lambda| \Rightarrow \lambda \in \rho(T)$

proof Take $\lambda_0 \in \rho(T)$ and $\lambda \in \mathbb{C}$: $|\lambda - \lambda_0| < \varepsilon$ then we want $\lambda \in \rho(T)$ provided ε small enough

$$T - \lambda = T - \lambda_0 + \lambda_0 - \lambda = \underbrace{(T - \lambda_0)}_{\text{invertible}}\left(\mathbb{1} + \underbrace{(\lambda_0 - \lambda)(T - \lambda_0)^{-1}}_{\text{small}}\right)$$

Invert by Neumann series, provided

$|\lambda_0 - \lambda| \, \|(T - \lambda_0)^{-1}\| < 1 \rightsquigarrow$ ok if $|\lambda_0 - \lambda| < \varepsilon < \dfrac{1}{\|(T-\lambda_0)^{-1}\|}$

$\Rightarrow \lambda \in \rho(T)$

Take $\lambda \in \mathbb{C}$: $|\lambda| > \|T\|$:

$$T - \lambda = \underbrace{\lambda\left(\frac{T}{\lambda} - \mathbb{1}\right)}_{\text{invert by Neumann series as } \frac{\|T\|}{|\lambda|} < 1}$$

$\Rightarrow \lambda \in \rho(T)$ $\square$

Prop $T \in \mathcal{L}(X)$, then $\sigma(T)$ is a nonempty compact set of $\mathbb{C}$, with

$$\sigma(T) \subseteq \{ z : |z| \le \|T\| \} \qquad (*)$$

proof $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is closed

and $(*)$ holds by previous point $\rightsquigarrow$ $\sigma(T)$ closed & bd $\Downarrow$ compact

$\sigma(T)$ not empty: B.C. $\sigma(T)$ is empty, hence

$R_\lambda(T)$ is well defined $\forall \lambda \in \mathbb{C}$

Take $\ell \in (\mathcal{L}(X))^*$ and put

$$f(\lambda) := \ell(R_\lambda(T)) \;; \quad \mathbb{C} \to \mathbb{C}$$

CLAIM $f(\lambda)$ analytic & bd function

•) $f(\lambda)$ analytic: we write it as a converg. power series around each $\lambda_0 \in \rho(T)$

Take $\lambda_0 \in \rho(T)$ and $0 < \varepsilon < 1/\|R_{\lambda_0}(T)\|$

then $\forall \lambda$ with $|\lambda - \lambda_0| < \varepsilon$

$$R_\lambda(T) = (T-\lambda)^{-1} = \left[(T-\lambda_0)\left(\mathbb{1} + (\lambda-\lambda_0) R_{\lambda_0}(T)\right)\right]^{-1}$$

$$= \sum (\lambda-\lambda_0)^k R_{\lambda_0}(T)^{k+1} (-1)^k$$

and the series is absolutely convergent

$$\rightsquigarrow \quad f(\lambda) = \sum_{k=0}^{\infty} (\lambda-\lambda_0)^k \, \ell\left(R_{\lambda_0}(T)^{k+1}\right) (-1)^k$$

is absolutely convergent in

a) $f(\lambda)$ bd $\quad T-\lambda = -\lambda\left[-\frac{T}{\lambda}+\mathbb{1}\right]$

$$\Rightarrow (T-\lambda)^{-1} = -\frac{1}{\lambda}\sum_{k\geq 0}\left(\frac{T}{\lambda}\right)^k$$

$$\rightsquigarrow R_\lambda(T) = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{T^k}{\lambda^k}$$

so for $|\lambda| > M\|T\|$ we have

$$\|R_\lambda(T)\| \leq \frac{1}{M\|T\|}\sum\frac{1}{M^k} < \frac{c}{M} \qquad \forall\, |\lambda| > M\|T\|$$

$$\rightsquigarrow |f(\lambda)| = |\ell(R_\lambda(T))| \leq \|\ell\|_{\mathcal{L}(X)^*}\|R_\lambda(T)\| < \frac{c_1}{M}$$

for $|\lambda| > M\|T\|$ ✓

Claim proved $\Rightarrow$ by Liouville thm of complex analy

$f(\lambda) \equiv$ const $\leq \frac{c}{M}$, but M is arbitrarily large

$\Rightarrow f(\lambda) \equiv 0 \Rightarrow \ell(R_\lambda(T)) \equiv 0 \quad \forall\, \ell\in\mathcal{L}(X)^*$

$\Rightarrow R_\lambda(T) \equiv 0 \quad ↯ \quad \square$

We know a first bound: $\sigma(T) \subseteq \{|\lambda| \leq \|T\|\}$

We can improve the bound:

Def (spectral radius) $A\in\mathcal{L}(X)$,

$$r(A) := \lim_{n\to\infty}\|A^n\|^{1/n} \qquad (*)$$

Rem $r(A) \leq \|A\|$, as $\|A^n\| \leq \|A\|^n$

Prop The limit (*) exists and

$$r(A) = \max\{ |z| : z \in \sigma(A) \}$$

proof We start proving the limit exists. Take $\varepsilon > 0$ and choose $p \in \mathbb{N}$:

$$\|A^p\|^{1/p} \le \left(\inf_{\ell} \|A^\ell\|^{1/\ell}\right) + \varepsilon \qquad (**)$$

For any $n \ge p$, write $n = kp + m$, $0 \le m \le p-1$

$$\rightsquigarrow \|A^n\| \le \|A^p\|^k \|A^m\| \le \|A^p\|^k \underbrace{\left(1 + \|A\| + \dots + \|A\|^{p-1}\right)}_{M}$$

$$\Rightarrow \|A^n\|^{1/n} \le M^{1/n} \|A^p\|^{k/n}$$

$$\overset{(**)}{\le} \underbrace{M^{1/n}}_{\xrightarrow{n\to\infty} 1} \left(\inf_{\ell} \|A^\ell\|^{1/\ell} + \varepsilon\right)^{kp/n}$$

$\frac{kp}{n} + \frac{m}{n} = 1$

$\downarrow$

1 as $n \to \infty$

$$\rightsquigarrow \overline{\lim_{n\to\infty}} \|A^n\|^{1/n} \le \inf_{\ell} \|A^\ell\|^{1/\ell} + \varepsilon$$

$$\rightsquigarrow \overline{\lim_{n\to\infty}} \|A^n\|^{1/n} \le \underline{\lim_{\ell\to\infty}} \|A^\ell\|^{1/\ell} + \varepsilon \qquad \forall \varepsilon > 0$$

$$\rightsquigarrow \lim_{n\to\infty} \|A^n\|^{1/n} \text{ exists.}$$

Let us prove that for $|\lambda| > r(A) \Rightarrow A - \lambda$ invertible

$(A - \lambda) = -\lambda\left(\frac{A}{\lambda} + \mathbb{1}\right)$. To invert by Neumann we need that $\sum \frac{A^k}{\lambda^k}$ converges: ok if $\frac{\|A^k\|}{\lambda^k} < a < 1$

this is the sharpest condition and this is what we check! Pick $\varepsilon > 0$ so that $r(A) + \varepsilon < |\lambda|$

For any k suff. large

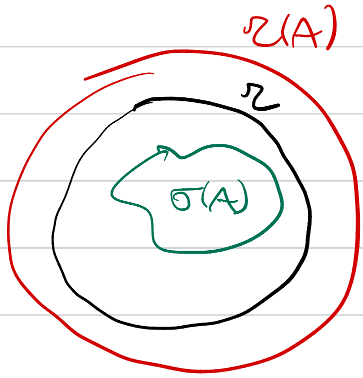
$$\frac{\|A^k\|}{|\lambda|^k} \overset{\text{by def of } r(A)}{\le} \frac{(r(A)+\varepsilon)^k}{|\lambda|^k} = \Bigg| \underbrace{\frac{r(A)+\varepsilon}{\lambda}}_{<1} \Bigg|^k$$

$\Rightarrow$ the series for $R_A(\lambda)$ converges in the operator norm, so $R_A(\lambda)$ is bd.

This proves that $\sigma(A) \subseteq \{|z| \le r(A)\}$

B.C. Assume that $\sigma(A) \subset \{|z| \le r(A)\}$.

Since $\sigma(A)$ is compact, $\exists\ 0 < r < r(A)$
$\sigma(A) \subset \{|z| \le r\}$, $\{|z| = r\} \subseteq \rho(A)$



Fix $k \in \mathbb{N}$, take $\ell_k \in \mathcal{L}(X)^*$ s.t.
$\|\ell_k\| = 1$ & $\ell_k(A^k) = \|A^k\|$

We know that $(A - \lambda)^{-1}$ is analytic in $|z| \ge r$, so is $\ell_k((A-\lambda)^{-1}) = \sum (-1)^n \frac{\ell_k(A^n)}{\lambda^{n+1}}$

By classical Cauchy theorem

$$\frac{1}{2\pi i} \oint_{|z|=r} \ell_k((A - z)^{-1})\, z^k = (-1)^k \ell_k(A^k)$$

$$\Rightarrow |\ell_k(A^k)| = \|A^k\| \le r^{k+1} \sup_{|z|=r} \|\ell_k(A-z)^{-1}\| \underset{\substack{\|\ell_k\|=1\ \forall k \\ z \to (A-z)^{-1} \text{ continuous}}}{\le} r^{k+1} C \quad \forall k$$

(C uniform in k)

$$\Rightarrow \|A^k\|^{1/k} \le r^{\frac{k+1}{k}} C^{1/k} \quad \forall k \Rightarrow r(A) \le r \quad \lightning$$

Relations between $\sigma(T)$ and $\left| \begin{array}{l} \sigma(T') \\ \sigma(T^*) \end{array} \right.$ adjoint in Hilbert space

Thm X Banach, $T \in \mathcal{L}(X) \Rightarrow \sigma(T) = \sigma(T')$
H Hilbert, $T \in \mathcal{L}(H) \Rightarrow \sigma(T^*) = \{ \lambda \mid \bar{\lambda} \in \sigma(T) \}$

proof It will follow from the following property:

$$T \in \mathcal{L}(X) \text{ invertible } \Leftrightarrow T' \in \mathcal{L}(X^*) \text{ invertible}$$

Indeed $T - \lambda$ invertible $\Leftrightarrow$ $T' - \lambda$ invertible
$\lambda \in \rho(T) \Leftrightarrow \lambda \in \rho(T')$

If H Hilbert, use $(T - \lambda)^* = T^* - \bar{\lambda}$ $\square$

Prop $T \in \mathcal{L}(X)$ invertible $\Leftrightarrow$ $T' \in \mathcal{L}(X^*)$ invertible

proof $\Rightarrow)$ If $ST = TS = \mathbb{1} \Rightarrow S'T' = T'S' = \mathbb{1}$

$\Leftarrow)$ If X reflexive, trivial since $(T')' = T$
otherwise we must show that T bijective

$\underline{T \text{ into}}$: $\ker T = {}^{\perp}(\operatorname{Im} T') = {}^{\perp}X^* = \{0\}$ (Identities II)

$\underline{T \text{ onto}}$: $\overline{\operatorname{Im} T} = {}^{\perp}(\ker T') = {}^{\perp}\{0\} = X$ (Identities III)

It is enough to check $\operatorname{Im} T$ closed. As $\ker T = 0$, enough
$\|Tx\| \geq c\|x\| \quad \forall x \in X$

T' invertible $\rightsquigarrow T'$ onto. By open map theorem, $\exists\, c > 0$ st
$T'(B_1^{X^*}(0)) \supseteq B_c^{X^*}(0)$

$\Rightarrow \quad \forall x^* \in X^*, \ \|x^*\|_{X^*} = 1, \quad \exists\, y^* \in B_1^{Y^*}(0)$ with

$$T' y^* = x^* \frac{c}{2}$$

$\Rightarrow \forall x \in X:$
$$\frac{c}{2} |x^*(x)| = |T' y^* (x)| = |y^* (Tx)|$$
$$\leq \|y^*\| \, \|Tx\| \qquad \forall x^* \in B_1^{X^*}(0)$$

$$\Rightarrow \qquad \frac{c}{2} \|x\| \leq \|Tx\|$$

$\square$

Lemma $\quad \lambda \in \sigma_p(T) \iff \mathrm{Im}\,(T' - \lambda)$ not dense

(in particular if $\ker(T' - \lambda) = 0$, then $\lambda \in \sigma_r(T')$)

proof $\quad \lambda \in \sigma_p(T) \iff \ker(T - \lambda) \neq \{0\}$
$$\| \ \text{Identities II}$$
$$^{\perp}(\mathrm{Im}\,(T' - \lambda))$$

$\leadsto \quad \mathrm{Im}\,(T' - \lambda)$ not dense (exercise: check it!)

$\square$

EXAMPLES

(1) Volterra operator $\quad T: C([0,1]) \to C([0,1])$
$$f(t) \longrightarrow (Tf)(t) = \int_0^t f(s)\,ds$$

Note that $\|T\| \leq 1$ & T compact operator

$\leadsto \sigma(T) \subseteq \{z : |z| \leq 1\}$

$\sigma(T) \subseteq \{z : |z| \leq r(T)\}, \qquad r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$

$$(T^2 f)(t) = T(Tf)(t) = \int_0^t (Tf)(t_1)\,dt_1 = \int_0^t\int_0^{t_1} f(s)\,ds\,dt_1$$

$$= \int_0^t f(s)(t-s)\,ds$$

by induction, prove that $(T^n f)(t) = \int_0^t f(s) \frac{(t-s)^{n-1}}{(n-1)!}\,ds$

$$|(T^n f)(t)| \leq \frac{\|f\|_{L^\infty}}{(n-1)!}\int_0^t |t-s|^{n-1}\,ds \overset{t\in[0,1]}{\leq} \frac{\|f\|_{L^\infty}}{n!}$$

$\Rightarrow \|T^n f\|_{L^\infty} \leq \|f\|_{L^\infty}/n!$

$\Rightarrow \|T^n\| \leq \frac{1}{n!} \rightsquigarrow r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = 0$

$\sigma(T) \subseteq \{0\}$

$\sigma(T)$ ~~not~~ compact empty set $\Rightarrow \sigma(T) = \{0\}$

(2) $L, R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$L(x_1, x_2, \dots) = (x_2, x_3, \dots)$
$R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$

Recall: $L' = R$ and $\|L\| = \|R\| = 1$

$\rightsquigarrow \sigma(R) = \sigma(L) \subseteq \{|z| \leq 1\}$.

<u>Left shift</u>: Take $|\lambda| < 1$. Then $Lx = \lambda x$

$\Leftrightarrow$
$x_2 = \lambda x_1$
$x_3 = \lambda x_2$
$x_4 = \lambda x_3$
$\vdots$

$\rightsquigarrow \vec{x}_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots)$ solves the system

$\|\vec{x}_\lambda\|^2 = \sum_{n\geq 0} |\lambda|^{2n} < +\infty$ for $|\lambda| < 1$

$\rightsquigarrow \vec{x}_\lambda$ eigenvector

$\forall\ |\lambda| < 1 \implies \lambda \in \sigma_p(L)$

$\rightsquigarrow \{|\lambda| < 1\} \subseteq \sigma(L) \subseteq \{|z| \leq 1\}$

$\sigma(L)$ closed $\implies \sigma(L) = \{|\lambda| \leq 1\}$.

<u>Right shift</u> $\quad \sigma(R) = \sigma(L) = \{|\lambda| \leq 1\}$

Let $|\lambda| < 1$: eigenvalues? $\quad Rx = \lambda x \iff \begin{matrix} 0 = \lambda x_1 \\ x_1 = \lambda x_2 \\ \vdots \end{matrix}$

If $\lambda \neq 0$ then $x = 0 \rightsquigarrow \ker(R - \lambda) = \{0\}$

From proposition, $\forall\ |\lambda| < 1$: $\lambda \in \sigma_p(L) \implies \left| \begin{array}{l} \operatorname{Im}(R-\lambda) \text{ not dense} \\ \ker(R-\lambda) = \{0\} \end{array} \right.$

$\rightsquigarrow \lambda \in \sigma_r(R)$

<u>EXERCISE</u>: Discuss the "nature" of the spectrum at the boundaries: $|\lambda| = 1$ & $\lambda = 0$.

SPECTRUM OF SELFADJOINT OPERATORS

From now, H Hilbert, $T \in \mathcal{L}(H)$,
T selfadjoint iff $T = T^*$: $\langle Tx, y \rangle = \langle x, Ty \rangle\ \forall x, y \in H$
Recall also $(\lambda T)^* = \bar{\lambda} T^*$

Prop $\quad T \in \mathcal{L}(H)$, $T = T^*$. Then

(i) $\sigma(T) \subseteq \mathbb{R}$

(ii) $\sigma_r(T) = \emptyset$

(iii) If $\lambda_1, \lambda_2 \in \sigma_p(T)$, $\lambda_1 \neq \lambda_2 \implies \ker(T - \lambda_1) \perp \ker(T - \lambda_2)$

We will need the following auxiliary results:

Lemma $T = T^*$, and take $\lambda \in \mathbb{C}$. Then

$$\|(T-\lambda)u\| \geq |\operatorname{Im}\lambda| \, \|u\| \quad \forall u \in H$$

proof

$$\langle (\lambda - T)u, (\lambda - T)u \rangle = \langle u, (\bar\lambda - T)(\lambda - T)u \rangle$$

$$= \langle u, \left(|\lambda|^2 - (\bar\lambda + \lambda)T + T^2 \right) u \rangle$$

$$= \langle u, \left(|\lambda|^2 - 2\operatorname{Re}\lambda\, T + T^2 \right) u \rangle$$

$$= \langle u, \left(|\operatorname{Im}\lambda|^2 + (T - \operatorname{Re}\lambda)^2 \right) u \rangle$$

$$= |\operatorname{Im}\lambda|^2 \|u\|^2 + \langle u, (T - \operatorname{Re}\lambda)^2 u \rangle$$

$T - \operatorname{Re}\lambda$ selfadjoint

$$= |\operatorname{Im}\lambda|^2 \|u\|^2 + \underbrace{\|(T - \operatorname{Re}\lambda)u\|^2}_{\geq 0}$$

$$\geq |\operatorname{Im}\lambda|^2 \|u\|^2$$

$\square$

Rem

$$T : L^2([0,1]) \to L^2([0,1])$$
$$f \longmapsto (Tf)(t) = t f(t)$$

$T = T^*$, $\sigma(T) = [0,1]$, $\sigma_p(T) = \emptyset$ (direct compt)

$\sigma_r(T) = \emptyset$ (by prop)

$\rightsquigarrow \sigma(T) = \sigma_c(T)$

EX check directly that for $\lambda \in [0,1]$, $\operatorname{Im}(T-\lambda)$ dense but not all L^2.

EX what happens if $T : C([0,1]) \to C([0,1])$?

proof of proposition

(i) Claim: $\lambda \in \mathbb{C} \setminus \mathbb{R} \Rightarrow T-\lambda$ invertible

from previous lemma: $\left| \begin{array}{l} \ker(T-\lambda) = \{0\} \\ \operatorname{Im}(T-\lambda) \text{ closed} \end{array} \right.$

To conclude need $\overline{\operatorname{Im} T-\lambda} = H$. We use that, analogously

Here $\perp$ is the standard orthogonal

$$\{0\} = \ker(T-\bar{\lambda}) = \ker((T-\lambda)^*) = (\operatorname{Im} T-\lambda)^\perp$$

$$\Rightarrow \overline{\operatorname{Im}(T-\lambda)} = H \rightsquigarrow \operatorname{Im}(T-\lambda) = H$$

$\rightsquigarrow$ $T-\lambda$ is bijective and thus invertible

(ii) B.C. $\exists \lambda \in \sigma_R(T) \subseteq \mathbb{R}$ $\Leftarrow$ $\left| \begin{array}{l} \overline{\operatorname{Im}(T-\lambda)} \subsetneq H \\ \ker T-\lambda = \{0\} \end{array} \right.$

As above

$$\{0\} = \ker(T-\lambda) \overset{\substack{T=T^* \\ \lambda \text{ real}}}{=\!=} \ker((T-\lambda)^*) = (\operatorname{Im}(T-\lambda))^\perp$$

$\rightsquigarrow$ $\overline{\operatorname{Im}(T-\lambda)}$ is dense $\lightning$

(iii) Assume $Tu_1 = \lambda_1 u_1$, $Tu_2 = \lambda_2 u_2$, then

$$\lambda_1 \langle u_1, u_2\rangle = \langle Tu_1, u_2\rangle = \langle u_1, Tu_2\rangle = \lambda_2 \langle u_1, u_2\rangle$$

$\rightsquigarrow$ $\lambda_1 \neq \lambda_2$ implies $u_1 \perp u_2$

$\square$

Thm (Weyl's CRITERION) $T \in \mathcal{L}(H)$, $T = T^*$, then

$$\lambda \in \sigma(T) \iff \exists \underbrace{\{x_n\}_{n\geq 1}}_{\text{Weyl sequence}} \subset H, \quad \begin{cases} \|x_n\| = 1 \quad \forall n \\ \|(T-\lambda)x_n\| \xrightarrow{n\to\infty} 0 \end{cases}$$

proof $\Leftarrow$) BC. $\lambda \in \rho(T)$, then $(T-\lambda)^{-1}$ bd

$$1 = \|x_n\| = \|(T-\lambda)^{-1}(T-\lambda)x_n\| \leq C \, \|(T-\lambda)x_n\| \to 0 \quad ↯$$

$\Rightarrow$) we know that $\sigma_r(T) = \emptyset$

so $\lambda \in \sigma_p(T) \Rightarrow$ take $x_n \in \ker(T-\lambda) \; \forall n$ ✓

$\lambda \in \sigma_c(T) \rightsquigarrow \begin{cases} \operatorname{Im}(T-\lambda) \text{ dense} \\ \ker(T-\lambda) = \{0\} \end{cases}$

Let $c := \inf_{\|x\|=1} \|(T-\lambda)x\|$. BC Assume that $c > 0$

$\rightsquigarrow \|(T-\lambda)x\| \geq c\|x\| \Rightarrow \begin{cases} \operatorname{Im}(T-\lambda) \text{ closed} \\ \ker(T-\lambda) = \{0\} \end{cases}$

$\rightsquigarrow H = \overline{\operatorname{Im} T-\lambda} = \operatorname{Im} T-\lambda \rightsquigarrow T-\lambda$ invertible ↯

Hence $c = 0$ $\square$

Lemma $T \in \mathcal{L}(H)$, $T = T^*$. Assume that $\exists \delta > 0$:

$$\langle Tx, x\rangle \geq \delta \|x\|^2 \quad \forall x \in H$$

$\Rightarrow T$ invertible

proof T into : trivial

T onto : $\delta \|x\|^2 \leq \langle Tx, x\rangle \leq \|Tx\| \|x\|$

$\rightsquigarrow \quad \delta \|x\| \leq \|Tx\|$

Im T closed & ker T = 0

$\mathrm{Im}\, T = \overline{\mathrm{Im}\, T} = {}^{\perp}\ker T^* = {}^{\perp}\ker T = \{0\}^{\perp} = H$

T bijective $\rightsquigarrow$ T invertible

The next important result is the following localization result:

Prop $T \in \mathcal{L}(H)$, $T = T^*$, then $\sigma(T) \subseteq [m, M]$ with

$$m := \inf_{\|x\|=1} \langle Tx, x\rangle \quad , \quad M = \sup_{\|x\|=1} \langle Tx, x\rangle.$$

Moreover, $m, M \in \sigma(T)$

proof We do the proof for M (m by exercise)
We know $\sigma(T) \subseteq \mathbb{R}$. If $\lambda > M$, then

$$\langle (\lambda - T)x, x\rangle \geq \underbrace{(\lambda - M)}_{>0}\|x\|^2 \quad \forall\, x \in H.$$

By previous lemma $\lambda - T$ is invertible $\rightsquigarrow \lambda \in \rho(T)$
We prove that $M \in \sigma(T)$. Define the bilinear form

$$a(x,y) := \langle (M - T)x, y\rangle$$

It is symmetric (T selfadjoint)
non negative (by def of M)

We apply Cauchy - Schwartz to $a(x,y)$

$$|a(x,y)| \leq a(x,x)^{1/2} \, a(y,y)^{1/2}$$

$$\Rightarrow \langle (M-T)x, y\rangle \leq \langle (M-T)x, x\rangle^{1/2} \, \langle (M-T)y, y\rangle^{1/2} \quad (\ast)$$

Now we have that

$$\|(M-T)x\| = \sup_{\|y\| \leq 1} |\langle (M-T)x, y\rangle| \overset{(\ast)}{\leq} \langle (M-T)x, x\rangle^{1/2} \, \|M-T\|^{1/2}$$

Now just take $(x_n)_{n\geq 1}$, $\|x_n\| = 1$ so that $\langle Tx_n, x_n\rangle \to M$

But then $\langle (M-T)x_n, x_n\rangle = M \underbrace{\|x_n\|^2}_{=1} - \langle Tx_n, x_n\rangle \to 0$

So $(x_n)_{n\geq 1}$ is Weyl seq for $M-T \Rightarrow M \in \sigma(T)$

$\square$

<u>Cor 1</u> $T \in \mathcal{L}(H)$, $T = T^*$. Then at least one among $\|T\|, -\|T\|$ belong to $\sigma(T)$.

<u>proof</u> Recall $\|T\| = \sup_{\|x\| = 1} |\langle Tx, x\rangle|$

$$= \max\Big(\underbrace{\sup \langle Tx, x\rangle}_{M}, \; - \underbrace{\inf \langle Tx, x\rangle}_{m}\Big)$$

$\Rightarrow \|T\| = M \in \sigma(T)$

$- \|T\| = m \in \sigma(T)$

$\square$

<u>Cor 2</u> $T = T^*$, then $r(T) = \|T\|$

<u>proof</u> $r(T) = \max\{|\lambda| \in \sigma(T)\} = \|T\|$

<u>Cor 3</u> $T = T^*$ and $\sigma(T) = \{0\} \Rightarrow T = 0$

<u>Rem</u> Volterra operator has $\sigma(T) = 0$ and it is not selfadjoint.