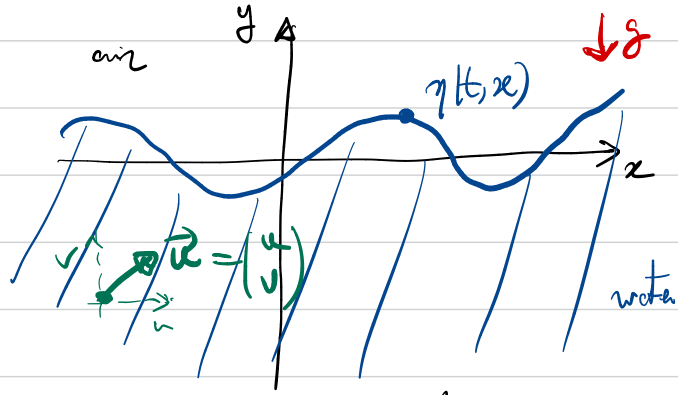
STOKES WAVE FOR WATER WAVES

Take a fluid $\left|\begin{array}{l}\text{incompressible}\\ \text{irrotational}\end{array}\right.$ under effect of gravity g

which at time t occupies a domain with ∞ depth

$\eta(t,x)$ = free surface
profile of waves
(it changes with time)



Fluid in $D_\eta = \{ (x,y) \in \mathbb{R}\times\mathbb{R} : \; y \leq \eta(t,x) \}$

eq of motions: Euler eq for velocity field with
bd conditions:
each particle of fluid has a velocity field $\vec{u} : D_\eta \to \mathbb{R}^2$

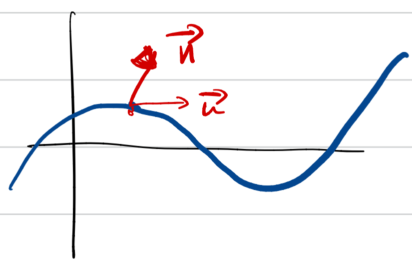
$$\begin{cases} \operatorname{div} \vec{u} = 0 \\ \operatorname{rot} \vec{u} = 0 \\ \partial_t \vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \underbrace{\nabla P}_{\text{fluid pressure}} = -g\,\vec{e_y} \end{cases} \qquad \vec{e_y} \to \binom{0}{1}$$

\+ 3 bd conditions:

(1) Kinematic: free surface moves with fluid

$$\eta_t = \sqrt{1+(\partial_x\eta)^2}\;\vec{u}\cdot\vec{n} \qquad \text{at } y=\eta(t,x)$$

$\vec{n}$ is the normal at D_η



free surface is parametrized by $\binom{x}{\eta(t,x)}$, so
its normal $\vec{n} = \binom{-\partial_x\eta}{1} / \|\cdot\|$

(ii) Dynamic: balance of forces at free surface

$$P = \underbrace{P_{atm}}_{\text{atmospheric pressure}} \quad \text{at free surface}$$

(iii) bottom: at the bottom, fluid move just horizontally

$$\lim_{y \to -\infty} v(x,y) = 0 \qquad \vec{u}(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

As $\operatorname{rot} \vec{u} = 0$, $\exists$ velocity potential: $\Phi : D_\eta \to \mathbb{R}$

$$\vec{u} = \nabla \Phi \qquad \vec{u}(t,x,y) = \nabla \Phi(t,x,y)$$

changes with time

Φ fulfills:

$$\begin{cases} \Delta \phi = 0 & \text{in } D_\eta \\ \lim_{y \to -\infty} \partial_y \phi = 0 & \\ \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + P + gy = 0 & \text{in } D_\eta \end{cases}$$

Bernoulli eq

from Euler with $u = \nabla \Phi$

$\vec{u} = \nabla \phi \rightsquigarrow 0 = \operatorname{div} \vec{u} = \Delta \phi$

$0 = \lim_{y \to -\infty} v = \lim_{y \to -\infty} \partial_y \Phi$

__Rem__ If we know $\phi \rightsquigarrow$ we know $\vec{u}$

Define $\psi(t,x) := \phi(t, x, \eta(t,x))$

trace of Φ at free surface

Zakharov's key obs

$$\begin{cases} \Delta \phi = 0 & \text{in } D_\eta \\ \phi = \psi & \text{at } y = \eta(t,x) \\ \lim_{y \to -\infty} \partial_y \phi = 0 & \end{cases}$$

Elliptic problem with Dirichlet – Neumann bd conditions

If we know η and ψ we can determine $\phi(t,x,y)$ and then $\vec{u}(t,x,y)$

So we write original eq in terms of $\eta(t,x)$ and $\psi(t,x)$

We introduce Dirichlet-Neumann op : $G(\eta)[\psi]$

it maps the Dirichlet datum ψ in the Neumann data $\partial_{\vec{n}}\Phi$

$$G(\eta)[\psi] = \sqrt{1+|\partial_x\eta|^2}\;\partial_{\vec{n}}\Phi\Big|_{y=\eta(t,x)} \qquad \left[\partial_{\vec{n}} = \vec{n}\cdot\nabla\right]$$

$$\vec{n} = \begin{pmatrix}-\partial_x\eta\\ 1\end{pmatrix}\Big/\|\cdot\|$$

$$= \partial_y\phi(t,x,\eta(t,x)) - (\partial_x\eta)(t,x)(\partial_x\phi)(t,x,\eta(t,x))$$

Rem $\psi \to G(\eta)\psi$ depends on ψ through Φ :
solve the elliptic problem with ψ and η and compute ϕ
But: $\psi \to G(\eta)\psi$ linear since ψ Dirichlet data

With this op:

$$\eta_t = G(\eta)\psi$$

It is a bit more involved, but not difficult to evaluate the Bernoulli eq at the free surface and

$$\partial_t\psi = -g\eta - \frac{1}{2}|\partial_x\psi|^2 + \frac{1}{2}\frac{\left(G(\eta)\psi + \partial_x\eta\,\partial_x\psi\right)^2}{1+(\partial_x\eta)^2} - P_{atm}$$

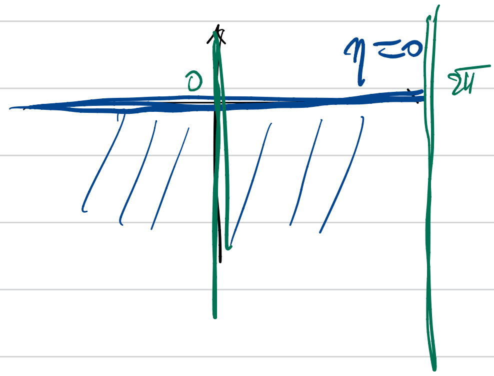
WATER WAVES EQ IN ZAKHAROV VARIABLES:

$$\begin{cases} \partial_t\eta = G(\eta)\psi \\ \partial_t\psi = -g\eta - \frac{1}{2}(\partial_x\psi)^2 + \frac{1}{2}\dfrac{\left(G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)\right)^2}{1+(\partial_x\eta)^2} - P_{atm} \end{cases}$$

<u>Advantage:</u> not anymore a free bd problem!

<u>Disadvantage:</u> $G(\eta)\psi$ is very complicated.
pure gravity WW is a fully nonlinear system

<u>Compute $G(0)\psi$</u>



$$G(0)\psi = \left(\partial_y \phi - (\partial_x \eta)\partial_x \phi\right)\Big|_{y=0}$$
$$= \partial_y \phi \,\Big|_{y=0}$$

We need to solve:
$$\begin{cases} \Delta \phi = 0 \\ \phi = \psi \quad \text{at} \quad y = 0 \\ \partial_y \phi \to 0 \quad y \to -\infty \end{cases}$$

Since we are interested in fluids periodic in x, so look for $\psi(x) = \psi(x+2\pi)$, $\phi(x,y) = \phi(x+2\pi, y)$

Sep of variables: Fourier series in x:

$$\phi(x,y) = \sum_{k \in \mathbb{Z}} \hat{\phi}_k(y)\, e^{ikx}$$

$$\rightsquigarrow \quad \Delta\phi = 0 \quad \rightsquigarrow \quad -k^2 \hat{\phi}_k(y) + \partial_y^2 \hat{\phi}_k(y) = 0 \quad \forall k$$

$$\rightsquigarrow \quad \hat{\phi}_k(y) = \alpha_k e^{ky} + \beta_k e^{-ky}$$

Now impose BC:

<u>Bottom</u> $\quad 0 = \lim_{y \to -\infty} \partial_y \hat{\phi}_k(y) = \lim_{y \to -\infty} k\alpha_k e^{ky} - k\beta_k e^{-ky}$

$$\Leftrightarrow \begin{cases} \beta_k = 0, & k > 0 \\ \alpha_k = 0 & k < 0 \end{cases}$$

$$\rightsquigarrow \quad \hat{\phi}_k(y) = \begin{cases} \alpha_k \, e^{ky} , & k \geq 0 \\ \beta_k \, e^{-ky} , & k < 0 \end{cases}$$

DIRICHLET: $\quad \phi(x,0) = \psi(x) \iff \hat{\phi}_k(0) = \hat{\psi}_k$

$$\rightsquigarrow \quad \hat{\phi}_k(y) = \begin{cases} \hat{\psi}_k \, e^{ky} , & k \geq 0 \\ \hat{\psi}_k \, e^{-ky} , & k < 0 \end{cases}$$

$$\rightsquigarrow \quad \phi(x,y) = \sum_{k \geq 0} \hat{\psi}_k \, e^{ky} \, e^{ikx} + \sum_{k<0} \hat{\psi}_k \, e^{-ky} \, e^{ikx}$$

$$\rightsquigarrow G(0)\psi = (\partial_y \phi)\big|_{y=0} = \sum_{k \geq 0} k \, \hat{\psi}_k \, e^{ikx} + \sum_{k<0} (-k) \, \hat{\psi}_k \, e^{ikx}$$

$$= \sum_{k \in \mathbb{Z}} |k| \, \hat{\psi}_k \, e^{ikx} = |D| \psi$$

FOURIER MULTIPLIER

Traveling waves for water waves

$$\begin{cases} \eta_t = G(\eta)[\psi] \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{1}{2} \dfrac{\left(G(\eta)[\psi] + \eta_x \psi_x \right)^2}{1 + \eta_x^2} - P_{atm} \end{cases}$$

We look for small in size traveling waves, periodic in x, i.e. solutions of the form

$$\eta(t,x) = \check{\eta}(x+ct), \qquad \begin{pmatrix} \check{\eta} \\ \check{\psi} \end{pmatrix}(\cdot) \; 2\pi\text{-periodic}$$
$$\psi(t,x) = \check{\psi}(x+ct)$$

Such solutions fulfill the eq

$$c\,\breve{\eta}_x = G(\breve{\eta})[\breve{\psi}]$$

$$c\,\breve{\psi}_x = -g\,\breve{\eta} - \frac{(\breve{\psi}_x)^2}{2} + \frac{1}{2}\frac{\left(G(\breve{\eta})[\breve{\psi}] + \breve{\eta}_x\,\breve{\psi}_x\right)^2}{1+\breve{\eta}_x^2} - P_{atm}$$

<u>Rem</u> $G(\eta + \text{const}) = G(\eta)$, so $\eta = \breve{\eta} + \frac{P_{atm}}{g}$ fulfills the same eq with $P_{atm} = 0$.

Look for zeros of

$$F(c,\eta,\psi) = \begin{pmatrix} -c\,\eta_x + G(\eta)\psi \\ c\,\psi_x + g\eta + \frac{\psi_x^2}{2} - \frac{1}{2}\frac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+\eta_x^2} \end{pmatrix}$$

<u>Rem</u> $F(c,0,0) = 0$ $\forall c$! Bifurcation problem!

We apply Crandall–Rabinowitz to bifurcate from trivial solution

We need to look for c^* s.t. $d_{(\eta,\psi)} F(c^*, 0)$ is not invertible and with 1-dim kernel

<u>PROBLEM</u>: in general kernel not 1-dim, but true if we impose further conditions on profiles,

η · even function

ψ odd function

<u>FACT 1</u>: if η_{even}, ψ_{odd} $\Rightarrow$ $G(\eta_{even})\psi_{odd}$ is odd

$$\Rightarrow \begin{pmatrix} \eta_{even} \\ \psi_{odd} \end{pmatrix} \longmapsto F\begin{pmatrix} \eta_{even} \\ \psi_{odd} \end{pmatrix} = \begin{pmatrix} F_1 \text{ odd} \\ F_2 \text{ even} \end{pmatrix}$$

FACT 2 If s suff. large, then

$$H^s(\mathbb{T}_x) \times H^s(\mathbb{T}_x) \longrightarrow H^{s-1}(\mathbb{T}_x)$$
$$\eta \quad , \quad \psi \qquad\qquad G(\eta)\psi$$

is C^∞ and moreover

$$d_{\binom{\eta}{\psi}}(G(\eta)\psi)\left[\binom{\hat\eta}{\hat\psi}\right] = G(\eta)\hat\psi + G'(\eta)[\hat\eta]\psi$$

Build functional space to solve the problem

$$F : \mathbb{R} \times H^s_{even}(\mathbb{T}_x) \times H^s_{odd}(\mathbb{T}_x) \longrightarrow \begin{pmatrix} H^{s-1}_{odd}(\mathbb{T}_x) \\ H^{s-1}_{even}(\mathbb{T}_x) \end{pmatrix}$$
$$c \in \mathbb{R}, \quad \eta \in H^s_{even}(\mathbb{T}_x), \quad \psi \in H^s_{odd}(\mathbb{T}_x)$$

FACT 3: $F \in C^2$ provided s suff large

$$H^s_{even}(\mathbb{T}_x) = \left\{ \eta(x) = \sum_{n\geq 0} \eta_n \cos(nx) \; ; \; \|\eta\|_s^2 := \sum_{n\geq 0} (1+n)^{2s} |\eta_n|^2 < \infty \right\}$$

$$H^s_{odd}(\mathbb{T}_x) = \left\{ \psi(x) = \sum_{n\geq 1} \psi_n \sin(nx) \; ; \; \|\psi\|_s^2 = \sum_{n\geq 1} n^{2s} |\psi_n|^2 < \infty \right\}$$

Rem $\|\partial_x \psi\|^2_{L^2} = \sum_{n\geq 1} n^2 |\psi_n|^2 \leq \sum n^{2s} |\psi_n|^2$ if $s\geq 1$

So if we control $\|\psi\|_s$, we control $\|\partial_x^k \psi\|$ $\forall k \leq s$

To apply Crandall–Rabinowitz, we need to check:

1) $\exists\, c^*$: $d_{\binom{\eta}{\psi}} F(c^*, 0, 0)$ has 1-dim kernel

2) $\operatorname{Im} d_{\binom{\eta}{\psi}} F(c^*, 0, 0)$ is closed with codim 1

3) $\partial_{c,\binom{\eta}{\psi}} F(c^*, 0, 0)\left[\binom{\eta^*}{\psi^*}\right] \notin R$, $R =$ range of $d_{\binom{\eta}{\psi}} F(c^*, 0, 0)$

$\binom{\eta^*}{\psi^*} \in \ker d_{\binom{\eta}{\psi}} F(c^*, 0, 0)$

We verify them:

1.) $$d_{\binom{\eta}{\psi}} F(c, 0, 0)\begin{bmatrix}\hat\eta \\ \hat\psi\end{bmatrix} = \begin{pmatrix} -c\,\hat\eta_x + |D|\,\hat\psi \\ c\,\hat\psi_x + g\,\hat\eta \end{pmatrix}$$

using that

•) $$d_{\binom{\eta}{\psi}} \big(G(\eta)\psi\big)\begin{bmatrix}\hat\eta \\ \hat\psi\end{bmatrix}\Big|_{(\eta,\psi)=(0,0)} = G(\eta)\,\hat\psi + G'(\eta)[\hat\eta]\,\psi\,\Big|_{\substack{\eta=0\\ \psi=0}}$$
$$= G(0)\,\hat\psi + 0 = |D|\,\hat\psi$$

•) nonlinearity is quadratic

We want c^* so that kernel has dim $= 1$

0 th Fourier mode

$$d_{\binom{\eta}{\psi}} F(c, 0, 0)\begin{bmatrix}\hat\eta \\ \hat\psi\end{bmatrix} = \begin{pmatrix} \sum_{n\geq 1} cn\,\eta_n \sin(nx) + n\,\psi_n \sin nx \\ \sum_{n\geq 1} cn\,\psi_n \cos(nx) + g\,\eta_n \cos(nx) \end{pmatrix} + \begin{pmatrix} 0 \\ g\,\eta_0 \end{pmatrix}$$

$$\hat\eta(x) = \sum \hat\eta_n \cos(nx) \Rightarrow \hat\eta_x(x) = \sum -n\,\hat\eta_n \sin(nx)$$
$$\hat\psi(x) = \sum \hat\psi_n \sin(nx) \qquad |D|\hat\psi = \sum n\,\hat\psi_n \sin(nx)$$

$$= \begin{pmatrix} 0 \\ g\,\eta_0 \end{pmatrix} + \sum_{n\geq 1} \begin{pmatrix} (cn\,\eta_n + n\,\psi_n)\sin(nx) \\ (cn\,\psi_n + g\,\eta_n)\cos(nx) \end{pmatrix}$$

We look for c so that we cannot invert.

When can we invert? Given f, g we want to solve

$$d_{\binom{\eta}{\psi}} F(c,0,0)\begin{bmatrix}\hat{\eta}\\ \hat{\psi}\end{bmatrix} = \begin{pmatrix} \underset{\in H^{s-1}_{odd}}{g} \\ \underset{\in H^{s-1}_{even}}{f}\end{pmatrix} = \sum_{n\geq 1}\begin{pmatrix} g_n \sin(nx) \\ f_n \cos(nx)\end{pmatrix} + \begin{pmatrix}0\\ f_0\end{pmatrix}$$

It is invertible $\iff \begin{cases} \begin{pmatrix}0\\ g\eta_0\end{pmatrix} = \begin{pmatrix}0\\ f_0\end{pmatrix} \\ \forall n\geq 1 \quad \begin{pmatrix} cn\,\eta_n + n\,\psi_n \\ cn\,\psi_n + g\,\eta_n\end{pmatrix} = \begin{pmatrix} g_n \\ f_n\end{pmatrix}\end{cases}$

on the 0th mode we can always invert, when $n\geq 1$ we need to solve

$$\underbrace{\begin{pmatrix} cn & n \\ g & cn\end{pmatrix}}_{M_n(c)}\begin{pmatrix}\eta_n\\ \psi_n\end{pmatrix} = \begin{pmatrix} g_n\\ f_n\end{pmatrix} \qquad \forall n\geq 1$$

If we solve the system $\forall n\geq 1 \rightsquigarrow$ we invert the op.

If 1 of $M_n(c)$ not invertible $\rightsquigarrow$ we have a kernel

If 1! of $M_n(c)$ has 1-dim kernel $\rightsquigarrow d_{\binom{\eta}{\psi}}F(c,0,0)$ has 1-dim kernel

Fix $\bar{n}$: $\det M_{\bar{n}}(c) = c^2\bar{n}^2 - \bar{n}g = 0$

$$\iff c^2\bar{n}^2 = \bar{n}g$$

$$\iff c = \pm\sqrt{\frac{g}{\bar{n}}}$$

So fix n_0 and choose $c_{n_0} = \sqrt{g/n_0}$, then

$M_{n_0}(c_{n_0})$ has 1-dim kernel.

We need to check that it is the only matrix with non trivial kernel, i.e.

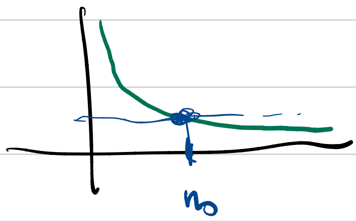
$$M_n(c_{n_0}) \text{ invertible } \quad \forall n \neq n_0$$

$$M_n(c_{n_0}) \text{ invertible } \iff \det M_n(c_{n_0}) \neq 0 \quad \forall n \neq n_0$$

$$\iff c_{n_0}^2 n^2 - ng \neq 0 \quad \forall n \neq n_0$$

$$\iff \underbrace{c_{n_0}}_{= \sqrt{\frac{g}{n_0}}} \neq \pm \sqrt{\frac{g}{n}} \quad \forall n \neq n_0$$

But this is true since $n \mapsto \sqrt{\frac{g}{n}}$ is injective on $(0, \infty)$



So fix $n_0 \in \mathbb{N}$ and compute the kernel of $M_{n_0}(c_{n_0})$

$$\begin{pmatrix} c_{n_0} n_0 & n_0 \\ g & c_{n_0} n_0 \end{pmatrix} \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} c_{n_0} \\ -\frac{g}{n_0} \end{pmatrix}$$

$$c_{n_0} n_0 \eta_{n_0} + n_0 \psi_{n_0} = 0$$

$$\rightsquigarrow \quad \psi_{n_0} = -\frac{c_{n_0}^2 \cancel{n_0}}{\cancel{n_0}} = -\frac{g}{n_0}$$

$$\rightsquigarrow \ker = \operatorname{span} \begin{pmatrix} c_{n_0} \cos(n_0 x) \\ -\frac{g}{n_0} \sin(n_0 x) \end{pmatrix}$$

2) range check $R := \mathrm{Im}\, d_{(\eta,\psi)} F(c^*, 0, 0)$ is closed and with codim $= 1$.

$c^* = c_{n_0} = \sqrt{\frac{g}{n_0}}$

Compute the range: take $\begin{pmatrix} g \\ f \end{pmatrix} \in \begin{pmatrix} H^{s-1}_{odd} \\ H^{s-1}_{even} \end{pmatrix}$, look for

$\begin{pmatrix} \eta \\ \psi \end{pmatrix} \in H^s_{even} \times H^s_{odd}$ st.

$$d_{(\eta,\psi)} F(c^*,0,0) \begin{pmatrix} \eta \\ \psi \end{pmatrix} = \begin{pmatrix} g \\ f \end{pmatrix} \iff \begin{cases} \begin{pmatrix} 0 \\ g\eta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ f_0 \end{pmatrix} \\ \begin{pmatrix} c^* n & n \\ g & c^* n \end{pmatrix} \begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ \psi_n \end{pmatrix}, n \geq 1 \end{cases}$$

2 cases: $n = n_0$ and $n \neq n_0$

If $n \neq n_0$, invert $M_n(c^*)$ (we know $\det M_n(c^*) \neq 0$)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\leadsto \begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \frac{1}{c^{*2} n^2 - ng} \begin{bmatrix} c^* n & -n \\ -g & c^* n \end{bmatrix} \begin{pmatrix} g_n \\ f_n \end{pmatrix}$$

CHECK REGULARITY: if $(g, f) \in H^{s-1}_{odd} \times H^{s-1}_{even}$, do we have $(\eta, \psi) \in H^s_{even} \times H^s_{odd}$?

$$\leadsto |\eta_n| = \frac{1}{|c^{*2} n^2 - ng|} \left| c^* n\, g_n - n f_n \right|$$

$$\leq C_1 \frac{1}{|c^{*2}n^2 - ng|}\left(|n||g_n| + |n||f_n|\right)$$

for n suff. large

$$\leq C \frac{1}{n^2}\left(|n|\,|g_n| + |n|\,|f_n|\right)$$

$$\leq C\left(\frac{|g_n|}{n} + \frac{|f_n|}{n}\right)$$

$(a+b)^2 = a^2+b^2+2ab$

$\oplus$

$2ab \leq a^2+b^2$

$$\rightsquigarrow \|\eta\|_s^2 = \sum_n |\eta_n|^2 n^{2s} \leq$$

$$\leq C\sum_n n^{2s}\left(\frac{|g_n|^2}{n^2} + \frac{|f_n|^2}{n^2}\right)$$

$$\leq C\left(\sum_n n^{2(s-1)}|g_n|^2 + \sum_n n^{2(s-1)}|f_n|^2\right)$$

$$\leq C\left(\|g\|_{s-1}^2 + \|f\|_{s-1}^2\right)$$

Similarly

$$|\psi_n| \leq \frac{C_1}{|c^{*2}n^2 - ng|}\left(|g_n| + |n f_n|\right)$$

$|n| \gg 1$

$$\leq C\left(\frac{|g_n|}{n^2} + \frac{|f_n|}{n}\right)$$

$$\rightsquigarrow \|\psi\|_s^2 = \sum_{n\geq 1} n^{2s}|\psi_n|^2 \leq C\sum_{n\geq 1} n^{2s}\left(\frac{|g_n|^2}{n^4} + \frac{|f_n|^2}{n^2}\right)$$

$$\leq C\left(\|g\|_{s-2}^2 + \|f\|_{s-1}^2\right) \quad \checkmark$$

So for $n \neq n_0$ we can invert. What about $n = n_0$?

$$\begin{pmatrix} c^* n_0 & n_0 \\ g & c^* n_0 \end{pmatrix} \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} g_{n_0} \\ f_{n_0} \end{pmatrix}$$

$$\rightsquigarrow \quad c^* n_0 \eta_{n_0} + n_0 \psi_{n_0} = g_{n_0}$$

multiply by $c^* = \sqrt{\frac{g}{n_0}}$ $\rightsquigarrow$ $\underbrace{c^{*2} n_0}_{g} \eta_{n_0} + c^* n_0 \psi_{n_0} = g_{n_0} c^*$

$$g \eta_{n_0} + c^* n_0 \psi_{n_0} = f_{n_0}$$

Same eq: to solve we need $f_{n_0} = c^* g_{n_0}$ $\rightsquigarrow \operatorname{span}\left\langle \begin{pmatrix} 1 \\ c^* \end{pmatrix} \right\rangle$

(ok since we know $\dim \operatorname{Im} M_{n_0}(c^*) = 1$)

So we find that

$$R = \operatorname{Im} d_{\binom{\eta}{\psi}} F(c^*, 0, 0) = \sum_{n \neq n_0} \begin{pmatrix} g_n \sin(nx) \\ f_n \cos(nx) \end{pmatrix} + \left\langle \begin{pmatrix} \sin(n_0 x) \\ c^* \cos(n_0 x) \end{pmatrix} \right\rangle$$

i.e. $R = \left\langle \begin{pmatrix} c^* \sin(n_0 x) \\ - \cos(n_0 x) \end{pmatrix} \right\rangle^{\perp}$ $\perp$ in $L^2 \times L^2$

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle := \int_0^{2\pi} u_1 u_2 + v_1 v_2$$

$\rightsquigarrow$ R is closed and $\operatorname{codim} = 1$

3) Transversality condition

$$\left(\partial_{c, \binom{\eta}{\psi}} F\right)(c^*, 0, 0) \begin{bmatrix} \eta^* \\ \psi^* \end{bmatrix} \notin R$$

element of kernel

$$\left(\partial_{c, \binom{\eta}{\psi}} F\right)(c^*, 0, 0) \begin{bmatrix} \eta^* \\ \psi^* \end{bmatrix} = \begin{bmatrix} -\eta^*_x \\ \psi^*_x \end{bmatrix}, \quad \begin{pmatrix} \eta^* \\ \psi^* \end{pmatrix} = \begin{pmatrix} c^* \cos(n_0 x) \\ -\frac{g}{n_0} \sin(n_0 x) \end{pmatrix}$$

So we set:

$$\left(\partial_{c,\binom{\eta}{\psi}} F\right)(c^*,0,0)\begin{bmatrix}\eta^* \\ \psi^*\end{bmatrix} = \begin{bmatrix} c^* n_0 \sin(n_0 x) \\ -\frac{g}{n_0}\, n_0 \cos(n_0 x)\end{bmatrix} = \begin{bmatrix} c^* n_0 \sin(n_0 x) \\ -g \cos(n_0 x)\end{bmatrix} \cancel{\in} R$$

$$\Leftarrow \left\langle \begin{bmatrix} c^* n_0 \sin(n_0 x) \\ -g \cos(n_0 x)\end{bmatrix}, \begin{bmatrix} c^* \sin(n_0 x) \\ - \cos(n_0 x)\end{bmatrix} \right\rangle \neq 0$$

$$= \int_0^{2\pi} \left(c^* n_0 \sin(n_0 x) \cdot c^* \sin(n_0 x) + g \cos^2(n_0 x)\right) dx$$

$$= c^{*2} n_0 \int_0^{2\pi} \sin^2(n_0 x)\, dx + g \int_0^{2\pi} \cos^2(n_0 x)\, dx$$

$$= 2\left(c^{*2} n_0 + g\right) \int_0^{2\pi} \sin^2(n_0 x)\, dx \neq 0$$

! DONE!

Thm Let $s > \frac{5}{2}$ and $n_0 \in \mathbb{N}$, then $\exists\ \varepsilon_0 > 0$ and C^1 functions

$$c_\varepsilon : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$$

$$\begin{pmatrix} \check{\eta}_\varepsilon \\ \check{\psi}_\varepsilon \end{pmatrix} : (-\varepsilon_0, \varepsilon_0) \to H^s_{even} \times H^s_{odd}$$

s.t.

$$\begin{pmatrix} \check{\eta}_\varepsilon (x - c_\varepsilon t) \\ \check{\psi}_\varepsilon (x - c_\varepsilon t) \end{pmatrix} \quad \text{solves} \quad WW$$

Moreover:

$$c_\varepsilon = c_{n_0} + O(\varepsilon)$$

$$\begin{pmatrix} \check{\eta}_\varepsilon(x) \\ \check{\psi}_\varepsilon(x) \end{pmatrix} = \varepsilon \begin{pmatrix} c^* \cos(n_0 x) \\ -\frac{g}{n_0} \sin(n_0 x) \end{pmatrix} + O(\varepsilon^2)$$