$\chi_t \le u \le \psi_t$. It remains to show that (P_t) has a solution for $t = T^*$. Take a sequence $t_k \to T^*, t_k < T^*$. Problems (P_k) corresponding to values $t = t_k$ have solutions u_k. From the preceding construction it is easy to check that the u_k converge to some u^* which solves (P_{T^*}).

This completes the proof of Theorem 2.5.

Remarks 2.10

(i) Improving Theorem 2.5, it has been shown [**AmaH**] that for all $t < T^*(P_t)$ has at least two distinct solutions (the second one is found by degree-theoretic arguments). See Problems (1) and (2) below for another multiplicity result in this direction. For an extensive discussion of elliptic equations with jumping nonlinearities we refer to [**Fu**].

(ii) A geometric description of the range of a differential operator more in the spirit of Theorem 2.4 can be found in [**McS**].

5

Bifurcation results

The structure of the solution set of a nonlinear functional equation can be very complicated and often it could be convenient to assume a "genetic" point of view, seeking for when new solutions are generated, near a given one, after a small perturbation. A convenient device consists in finding (or introducing) a parameter λ, and studying an equation $F(\lambda, u) = 0$ which possesses a fixed solution for all values of the parameter. An interesting phenomenon is when there is a "branching" of new solutions of $F(\lambda, u) = 0$ in correspondence with some value of the parameter. This is the object of the "Bifurcation theory" we will discuss in this chapter in its more elementary aspects.

1 Introduction

Let X, Y be two Banach spaces. We are interested in studying equations of the type

$$F(\lambda, u) = 0 \tag{1.1}$$

where

$$F : \mathbb{R} \times X \to Y$$

is a map depending on a *real* parameter λ.

As we will see in the next chapter, equations like (1.1) model a broad class of problems arising in applications, where the parameter λ often has a physical interpretation: it can be the intensity of the loading in some elasticity problems, the Rayleigh number in hydrodynamics, and so on.

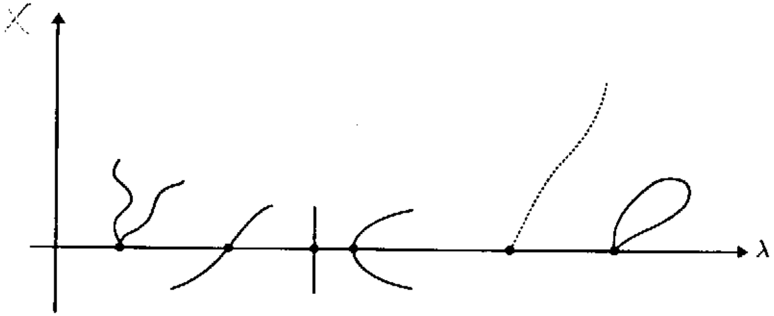


Figure 5.1 This and the following bifurcation diagrams are to be interpreted as suggestions only.

In this chapter we will always assume that $F \in C^2(\mathbb{R} \times X, Y)$ and that

$$F(\lambda, 0) = 0 \text{ for all } \lambda \in \mathbb{R}.$$

If this is true, then (1.1) has for all λ the solution $u = 0$, which will be referred to as the *trivial solution*.

$\mathcal{S} = \{(\lambda, u) \in \mathbb{R} \times X : u \neq 0, F(\lambda, u) = 0\}$ will denote the set of *non-trivial* solutions of (1.1).

It can happen that for some values of the parameter there are one or more solutions of (1.1) that branch off from the trivial one. These values of λ are called the *bifurcation points* of (1.1) (Figure 5.1). More precisely, we give the following definition:

Definition 1.1 We say that λ^* is a *bifurcation point* for F (from the trivial solution) if there is a sequence $(\lambda_n, u_n) \in \mathbb{R} \times X$ with $u_n \neq 0$ and $F(\lambda_n, u_n) = 0$ such that

$$(\lambda_n, u_n) \to (\lambda^*, 0).$$

Another, equivalent, way to define a bifurcation point, is to require that $(\lambda^*, 0)$ belong to the closure (in $\mathbb{R} \times X$) of $\mathcal{S}$, that is, that in any neighbourhood of $(\lambda^*, 0)$ there is a point $(\lambda, u) \in \mathcal{S}$.

Let us begin our discussion by stating a result that follows immediately from the Implicit Function Theorem.

Proposition 1.2 *A necessary condition for λ^* to be a bifurcation point for F is that the partial derivative $F_u(\lambda^*, 0)$ is not invertible.*

Proof. If $F_u(\lambda^*, 0) \in \text{Inv}(X, Y)$ then Theorem 2.2.3 applies and there exists a neighbourhood $\Theta \times V$ of $(\lambda^*, 0)$ such that

$$F(\lambda, u) = 0, (\lambda, u) \in \Theta \times V, \Longleftrightarrow u = 0.$$

Therefore λ^* is not a bifurcation point for F.

An interesting case is when $X = Y$ and

$$F(\lambda, u) = \lambda u - G(u). \tag{1.2}$$

In such a case $F_u(\lambda^*, 0) = \lambda^* I - G'(0)$ and Proposition 1.2 becomes the following.

Proposition 1.3 *If λ is a bifurcation point for F of the form* (1.2) *then λ belongs to the spectrum $\sigma[G'(0)]$ of $G'(0)$.*

It is quite natural to ask whether or not Proposition 1.3 can be inverted:

if $\lambda \in \sigma[G'(0)]$, is λ a bifurcation point for F ?

We anticipate that, in this generality, the answer to the preceding question is negative.

The particular case when F has the form (1.2) with $G = A \in L(X)$ is particularly enlightening. Note that, if G is linear, then $G'(0) = A$ and the relationships between the bifurcation points for

$$F = \lambda I - A \tag{1.3}$$

and the spectrum $\sigma(A)$ of A can be established in a precise fashion.

First of all, it is clear that the eigenvalues of A are bifurcation points for $F = (1.3)$.

Moreover, the following result can readily be proved.

Proposition 1.4 *Let $A \in L(X)$ and $F(\lambda, u) = \lambda u - A(u)$. Then λ^* is a bifurcation point for F if and only if λ^* belongs to the closure of the eigenvalues of A.*

Remarks 1.5

(i) As a consequence of Proposition 1.4, we deduce that *in general, there might be points λ belonging to the spectrum of A that are not bifurcation points for F of the form* (1.3).

(ii) From Proposition 1.4 it also follows that λ^* can be a bifurcation point for $\lambda I - A$ without being an eigenvalue of A.

We have seen that when F has the form (1.3) all the eigenvalues of A are bifurcation points for F. The following example shows that in the nonlinear case a value λ^* can be an eigenvalue of $G'(0)$ without being a bifurcation for $F(\lambda, u) = \lambda u - G(u)$.

Example 1.6 Let $X = Y = \mathbb{R}^2$, and consider the application $G : X \to X$ defined by

$$G(x,y) = (x + y^3, y - x^3).$$

The value $\lambda^* = 1$ is an eigenvalue of $g'(0) = I$, but it is not a bifurcation for

$$F : (x,y) \to \lambda(x,y) - G(x,y).$$

For let (x,y) be a solution of $F = 0$. From

$$\left.\begin{aligned} \lambda x &= x + y^3, \\ \lambda y &= y - x^3, \end{aligned}\right\}$$

it follows that

$$x^4 + y^4 = 0,$$

and hence $(x,y) = (0,0)$. Therefore $F = 0$ has only the trivial solution and there are no bifurcation points for F.

2 Some elementary examples

In this section we will discuss a couple of simple examples, where the existence of bifurcation points can be proved in a rather elementary way.

As a first example, let us consider the boundary-value problem

$$\frac{d^2u}{dt^2} + \lambda(u - u^3) = 0, \quad t \in [0,\pi], \tag{2.1}$$

$$u(0) = u(\pi) = 0. \tag{2.1'}$$

For all values of the parameter $\lambda \in \mathbb{R}$ (2.1)–(2.1′) has the *trivial solution* $u(t) \equiv 0$. To find other possible solutions we can work in the phase plane and argue as follows. Multiplying (2.1) by

$$p = \frac{du}{dt}$$

one finds immediately that any solution of (2.1) satisfies the energy relationship

$$\frac{1}{2}p^2 + \lambda(\frac{u^2}{2} - \frac{u^4}{4}) = c \ (= \text{constant}). \tag{2.2}$$

The integral curves in the phase plane (u,p) are represented in Figure 5.2.

We can distinguish between two families of integrals, which are separated by curves that pass through the singular points $(1,0)$ and $(-1,0)$. In particular, the closed curves correspond to periodic solutions of (2.1) and we are interested in the arcs of those closed integrals that start from

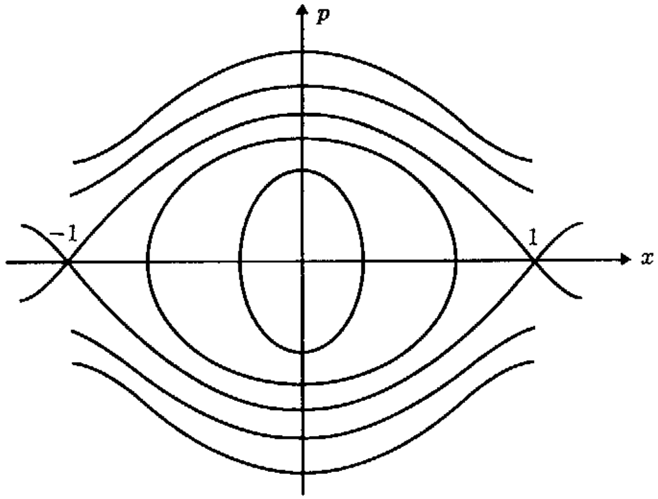


Figure 5.2

points (0, p) and again reach the p-axis after a time equal to π. From the symmetry, such a time is an integer multiple of the semiperiod T.

Let Γ_ξ be an integral curve crossing the u-axis at $u = \xi$, with $0 < \xi < 1$ (see Figure 5.3).

Putting $p = 0$ and $u = \xi$ in (2.2) we can find the value of c corresponding to Γ_ξ:

$$c = \lambda(\frac{1}{2}\xi^2 - \frac{1}{4}\xi^4). \tag{2.3}$$

Let $T(\lambda,\xi)$ be the semiperiod of Γ_ξ. From the symmetry, $T(\lambda,\xi)$ is given by

$$T(\lambda,\xi) = 2\int_\gamma \frac{du}{p},$$

where γ denotes the arc of Γ_ξ with $p \geq 0$.

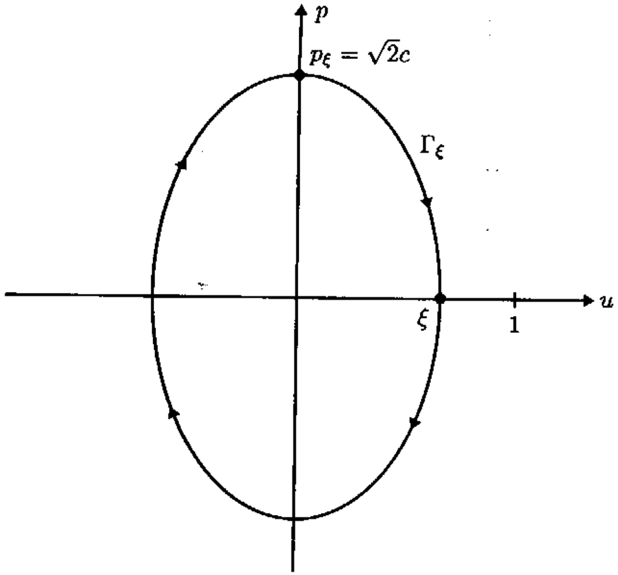


Figure 5.3

Taking into account (2.2) and (2.3) one finds with easy calculations

$$\begin{aligned} T(\lambda,\xi) &= 2\int_0^\xi \frac{\mathrm{d}x}{\sqrt{[2c-\lambda(x^2-\frac{1}{2}x^4)]}} \\ &= 2\int_0^1 \frac{\xi\,\mathrm{d}y}{\sqrt{[2c-\lambda(\xi^2y^2-\frac{1}{2}\xi^4y^4)]}} \\ &= 2\int_0^1 \frac{\xi\,\mathrm{d}y}{\sqrt{[2\lambda(\frac{1}{2}\xi^2-\frac{1}{4}\xi^4)-\lambda(\xi^2y^2-\frac{1}{2}\xi^4y^4)]}} \\ &= \frac{2}{\sqrt{\lambda}}\int_0^1 \frac{\mathrm{d}y}{\sqrt{[1-\frac{1}{2}\xi^2-y^2(1-\frac{1}{2}\xi^2y^2)]}}\,. \end{aligned} \tag{2.4}$$

Let us note explicitly that (2.4) allows us to extend $T(\lambda,.)$ at $\xi = 0$ by setting

$$T(\lambda,0) = \frac{2}{\sqrt{\lambda}}\int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y^2)}} = \frac{\pi}{\sqrt{\lambda}}. \tag{2.5}$$

As anticipated before, Γ_ξ gives rise to a solution of the boundary-value

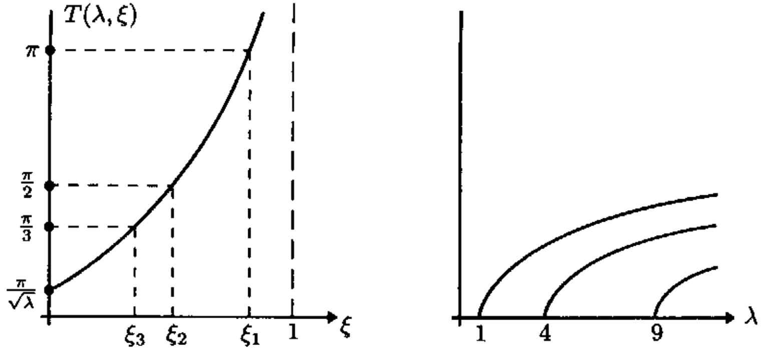


Figure 5.4

problem (2.1)–(2.1′) whenever ξ is such that

$$mT(\lambda,\xi) = \pi \tag{2.6}$$

for some $m \in \mathbb{N}$.

For example, if ξ is such that $T(\lambda,\xi) = \pi$, then Γ_ξ corresponds to a positive solution of (2.1) with $u(0) = 0$ and $p(0) = u'(0) = p_\xi$ (see Fig.5.3), $u(\pi) = 0$ and $p(\pi) = u'(\pi) = -p_\xi$.

To discuss equation (2.6) we first deduce from (2.4) that, for any fixed λ, there result

$$\frac{\partial T(\lambda,\xi)}{\partial \xi} > 0,$$
$$T(\lambda,\xi) \to +\infty \text{ as } \xi \uparrow 1.$$

Taking (2.5) also into account, we get that

(a) if $\lambda < 1$ then $T(\lambda,\xi) > \pi$ for all ξ, and (2.1)–(2.1′) has only the trivial solution $u \equiv 0$,

(b) if $\lambda = 1$ then $T(\lambda,0) = \pi$ and $\xi = 0$ is the only solution of (2.6), and hence (2.1)–(2.1′) has again the trivial solution only,

(c) if $1 < \lambda < 4$, then (2.6) (with $m = 1$) has a unique solution $\xi \neq 0$ which corresponds to a positive solution u_ξ of (2.1)–(2.1′). (More precisely, as $\lambda \downarrow 1$ the solution $\xi = \xi(\lambda)$ as well as p_ξ tends to 0; correspondingly (2.1)–(2.1′) has a family $u_\lambda = u_{\xi(\lambda)}$ of (positive) solutions, depending continuously on λ, such that $\|u_\lambda\|_{C^1} \to 0$ as $\lambda \downarrow 1$, and we can say that the (positive) solutions of (2.1)–(2.1′) bifurcate from the trivial solution at the value $\lambda = 1$ of the parameter).

The discussion can be carried over showing that (see Figure 5.4)

(d) if $k^2 < \lambda < (k+1)^2$ then (2.6) has k solutions $\xi_1, \ldots, \xi_k \neq 0$,

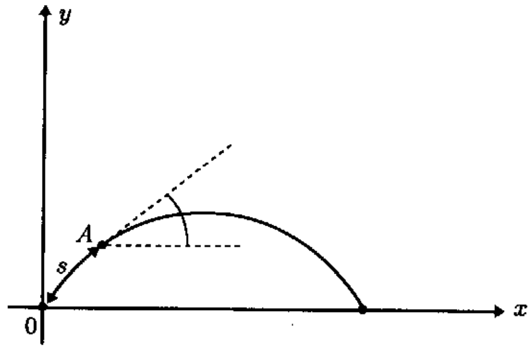


Figure 5.5

satisfying

$$hT(\lambda,\xi_h)=\pi \quad (h=1,\ldots,k).$$

Each ξ_h corresponds to a solution of (2.1)–(2.1′) with precisely $h-1$ nodes in $(0,\pi)$. In particular we can say that there is a continuous family of solutions u_λ of (2.1)–(2.1′), with $k-1$ nodes in $(0,\pi)$, such that $\|u_\lambda\|_{C^1}\to 0$ as $\lambda \downarrow k^2$.

It is worth noting that, from the abstract point of view, the boundary-value problem (2.1)–(2.1′) gives rise to a functional equation of the type (1.1). Here X is the Banach space of $C^2(0,\pi)$ functions vanishing at $t=0$ and $t=\pi$, $Y=C(0,\pi)$, and $F:\mathbb{R}\times X\to Y$ is given by (see also Example 2.1.5)

$$F(\lambda,u)=\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\lambda(u-u^3).$$

There results

$$F_u(\lambda,0):u\to\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\lambda u,$$

and the values $\lambda_k=k^2$ are precisely the eigenvalues of the linear problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\lambda u=0,\ u(0)=u(\pi)=0.$$

As a second example, we consider the buckling problem for an elastic beam of length L.

We suppose that one edge of the beam is hinged, while the other one is variable on the x-axis. The beam is compressed at the free edge by a force of intensity $K>0$. Denote by $(x(s),y(s))$ the coordinates of a point A on the beam, as a function of the length s of the arc $0A$, and let $\phi(s)$ be the angle between the tangent to the beam at A and the x-axis. See Figure 5.5.

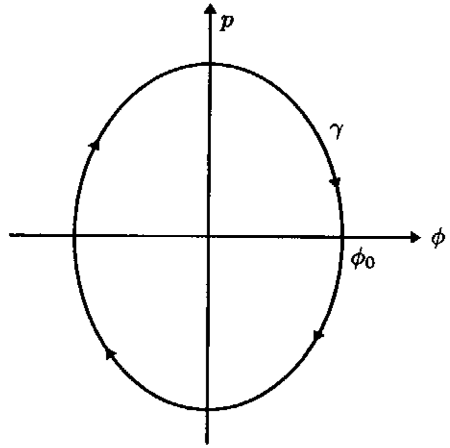


Figure 5.6

In accordance with the Euler–Bernoulli theory, the curvature of the beam at any point is proportional to the momentum of the applied force. Then there result

$$Ky=-\kappa\frac{\mathrm{d}\phi}{\mathrm{d}s}\quad(\kappa=\text{constant}),\ \frac{\mathrm{d}y}{\mathrm{d}s}=\sin\phi, \tag{2.7}$$

together with the boundary conditions

$$y(0)=y(L)=0. \tag{2.7′}$$

From (2.7)–(2.7′) we deduce

$$\left.\begin{aligned}&\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}+\lambda\sin\phi=0,\quad s\in[0,L],\\&\phi'(0)=\phi'(L)=0,\end{aligned}\right\} \tag{2.8}$$

where $\lambda=K/\kappa>0$.

To study (2.8) we can proceed as before. We refer to Figure 5.6. If we let $p=\mathrm{d}\phi/\mathrm{d}s$, conservation of energy yields

$$\frac{1}{2}p^2-\lambda\cos\phi=c=-\lambda\cos\phi_0,$$

and the semiperiod of the (closed) curve passing through $(\phi_0,0)$ is given by

$$T(\lambda,\phi_0)=2\int_\gamma\frac{\mathrm{d}\phi}{p}$$

$$=\frac{2}{\sqrt{\lambda}}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{(1-\omega^2\sin^2\theta)}},$$

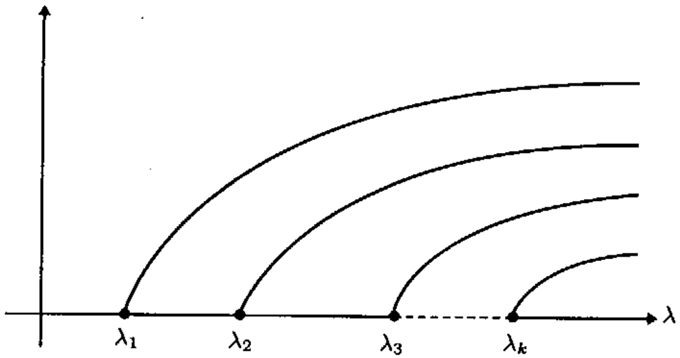


Figure 5.7

where

$$\omega = \sin\frac{\phi_0}{2}.$$

An arc of a closed curve joining two points of the x-axis corresponds to a solution of (2.8) whenever

$$hT(\lambda, \phi_0) = L,$$

for some $h \in \mathbb{N}$.

Since $T(\lambda, .)$ is still strictly increasing and $T(\lambda, \phi_0) \to +\infty$ as $\phi_0 \uparrow \pi$, we deduce:

(a) For $0 < \lambda \le \pi^2/L^2$ (2.8) has only the trivial solution $\phi \equiv 0$.

(b) For $k^2\pi^2/L^2 < \lambda \le (k+1)^2\pi^2/L^2$ (2.8) has k nontrivial solutions $\phi_1, \dots, \phi_k$. In addition, for fixed $k = 1, 2, \dots$, there exists a continuous family ϕ_λ of nontrivial solutions of (2.8) whose C^1 -norm tends to zero as $\lambda \downarrow \lambda_k = k^2\pi^2/L^2$.

The bifurcation diagram is drawn in Figure 5.7.

As for the preceding case, we can see that the functional equation corresponding to (2.8) is

$$F(\lambda, \phi) := \frac{d^2\phi}{ds^2} + \lambda \sin\phi = 0$$

whith $\phi \in X$, the Banach space of functions of $C^2(0, L)$ such that $\phi'(0) = \phi'(L) = 0$.

The linearized equation $F_\phi(\lambda, 0)\psi = 0$ becomes

$$\frac{d^2\psi}{ds^2} + \lambda\psi = 0, \psi'(0) = \psi'(L) = 0, \tag{2.9}$$

whose positive eigenvalues are just

$$\lambda_k = \frac{k^2\pi^2}{L^2},\ k = 1, 2, \dots.$$

Note that $\lambda = 0$ would also be a bifurcation point of $F(\lambda, \phi) = 0$, F given by (2.9). Indeed, for $\lambda = 0$, the equation $F(0, \phi) = 0$ has the family of nontrivial solutions ϕ =const. However, these solutions correspond to the trivial solution $y \equiv 0$ of the physical problem (2.7).

3 The Lyapunov–Schmidt reduction

In this section we dicuss a general procedure, introduced by Lyapunov [**Ly 1–2**] and Schmidt [**Schm**], which will be a basic tool hereafter, but can also be useful in several other situations. It is a method we have already used in Sections 3.2 and 4.1, for the specific problem studied there.

Let $F \in C^2(\mathbb{R} \times X, Y)$ be such that

$$F(\lambda, 0) = 0.$$

According to Proposition 1.2 the possible bifurcation points for F are the values λ^* such that $F_u(\lambda^*, 0)$ is not invertible. We set

$$L = F_u(\lambda^*, 0),$$
$$V = \mathrm{Ker}(L),$$
$$R = R(L),$$

and suppose that

(a) *V has a topological complement W in X.*

This means that there exists a closed subspace W of X such that

$$X = V \oplus W \tag{3.1}$$

and any $u \in X$ can be written in the form

$$u = v + w,\ v \in V,\ w \in W. \tag{3.2}$$

On the range R of L we assume

(b) *R is closed and has a topological complement Z in Y.*

This means that $Y = Z \oplus R$, with Z closed and such that $Z \cap R = \{0\}$. For example, (a) and (b) hold true when V is finite-dimensional and R has finite codimension, that is, when L is a *Fredholm operator*.

Next, let P and Q denote the conjugate projections onto Z and R, respectively.

Using (3.2) and applying P and Q one finds that $F(\lambda, u) = 0$ is equivalent to the system

$$PF(\lambda, v + w) = 0, \tag{3.3'}$$

$$QF(\lambda, v + w) = 0. \tag{3.3''}$$

For later use, it is also convenient to set

$$F(\lambda, u) = Lu + \varphi(\lambda, u).$$

Using (3.2) and recalling that $Lv = 0$, one has

$$F(\lambda, u) = Lw + \varphi(\lambda, v + w).$$

Recalling that $Lw \in R$, we get $QLw = Lw$. Then (3.3″) becomes

$$Lw + Q\varphi(\lambda, v + w) = 0. \tag{3.4}$$

We set

$$\Phi(\lambda, v, w) = Lw + Q\varphi(\lambda, v + w)$$

and note that $\Phi \in C^2(\mathbb{R} \times V \times W, R)$. Moreover

$$\Phi_w(\lambda^*, 0, 0) : w \to Lw + Q\varphi_u(\lambda^*, 0)w.$$

Since, by definition, $\varphi(\lambda, u) = F(\lambda, u) - Lu$, there results

$$\varphi_u(\lambda^*, 0) = F_u(\lambda^*, 0) - L = 0. \tag{3.5}$$

In other words, $\varphi_u(\lambda^*, 0)$ is the zero mapping in $L(X, Y)$ and therefore it follows that $\Phi_w(\lambda^*, 0, 0) = L|_W$.

We remark that the restriction $L|_W$ of L to W, as a map from W to R, is injective and surjective, Since R is closed, $(L|_W)^{-1}$ is continuous from R to W, namely

$$L|_W \in \text{Iso}\,(W, R). \tag{3.6}$$

Hence $\Phi_w(\lambda^*, 0, 0) \in \text{Iso}\,(W, R)$, the Implicit Function Theorem applies to Φ and (3.4) can be uniquely solved, locally, with respect to w. To be precise, there exist

(i) a neighbourhood Λ of λ^*,
(ii) a neighbourhood $\mathcal{V}$ of $v = 0$ in V,
(iii) a neighbourhood $\mathcal{W}$ of $w = 0$ in W, and
(iv) a function $\gamma \in C^2(\Lambda \times \mathcal{V}, \mathcal{W})$,

such that the unique solutions of (3.3″) in $\Lambda \times \mathcal{V} \times \mathcal{W}$ are given by $(\lambda, v, \gamma(\lambda, v))$.

In particular, for future reference, we remark that results

$$\gamma(\lambda, 0) = 0 \text{ for all } \lambda \in \Lambda. \tag{3.7}$$

Moreover one has

$$\gamma_v(\lambda^*, 0) = 0. \tag{3.8}$$

To see this, we can use the Implicit Function Theorem or else we can take into account that

$$L\gamma(\lambda, v) + Q\varphi(\lambda, v + \gamma(\lambda, v)) = 0 \quad \text{for all } (\lambda, v) \in \Lambda \times \mathcal{V}.$$

Differentiating with respect to v at $(\lambda^*, 0)$, and letting $\Gamma = \gamma_v(\lambda^*, 0)$, we

find

$$L\Gamma x + Q\varphi_u(\lambda^*, \gamma(\lambda^*, 0))[x + \Gamma x] = 0 \text{ for all } x \in V.$$

Since $\gamma(\lambda^*, 0) = 0$ and using (3.5) we get that $L\Gamma x = 0$ for all $x \in V$, and hence $\Gamma x \in V \cap W$. Thus $\Gamma x = 0$ for all $x \in V$.

After these preliminaries, we can substitute

$$w = \gamma(\lambda, v) \tag{3.9}$$

in (3.3′) getting

$$P(F(\lambda, v + \gamma(\lambda, v))) = 0. \tag{3.10}$$

The equation (3.10) in the unknowns $(\lambda, v) \in \Lambda \times \mathcal{V}$ is called *the bifurcation equation* and, together with (3.9), is equivalent (in $\Lambda \times \mathcal{V} \times \mathcal{W}$) to the initial equation $F(\lambda, u) = 0$.

Obviously, the preceding reduction is useful if the bifurcation equation is simpler than $F = 0$. This is the case when L is a Fredholm operator: if $\dim(V) = p$ and $\text{codim}(R) = \dim(Z) = q$, then (3.10) is a system of q equations in the unknowns $(\lambda, v) \in \mathbb{R} \times \mathbb{R}^p$.

4 Bifurcation from the simple eigenvalue

In Section 1 we saw that the possible bifurcation points of $F(\lambda, u) = 0$ are those λ^* such that $F_u(\lambda^*, 0)$ is not invertible. To find sufficient conditions for λ^* to be a bifurcation point, some restrictions are in order. In this section we will study the case in which $L = F_u(\lambda^*, 0)$ is a Fredholm map with index zero and with one-dimensional kernel. In the case of equations like

$$G(u) = \lambda u,$$

with $G \in C^1(X, X)$, $G(0) = 0$ and $G'(0)$ compact, this corresponds to the case when λ^* is a *simple* eigenvalue of $G'(0)$.

Let us take a map $F \in C^2(\mathbb{R} \times X, Y)$ satisfying $F(\lambda, 0) = 0$ for all λ. We note explicitly that the condition $F \in C^2$ could be weakened assuming that $F \in C^1(\mathbb{R} \times X, Y)$ and has mixed partial derivative $F_{u,\lambda}$, see Remark 4.3 (i).

The hypothesis that $L = F_u(\lambda^*, 0)$ satisfies assumptions (a) and (b) of Section 3 needs to be specified here.

Keeping the notation of the preceding section, we set $V = \text{Ker}(L)$, $R = R(L)$ and let W and Z denote complementary subspaces of V in X and R in Y, respectively.

We will say that L (or F) satisfies Assumption (I) if

(I-i) V is one-dimensional: $\exists u^* \in X, u^* \neq 0$ such that $V = \{tu^* : t \in \mathbb{R}\}$,

(I-ii) R is closed and codim $(R) = 1$.

According to (I-ii) Z is one-dimensional and there exists a linear functional $\psi \in Y^*, \psi \neq 0$, such that

$$R = \{y \in Y : \langle \psi, y \rangle = 0\}.$$

We also use symbols P and Q to denote the projections onto Z and R, respectively.

With this notation, the bifurcation equation (see (3.10)) becomes

$$\langle \psi, F(\lambda, tu^* + \gamma(\lambda, tu^*)) \rangle = 0. \tag{4.1}$$

It is convenient to set $\lambda = \lambda^* + \mu$, and

$$\beta(\mu, t) = \langle \psi, F(\lambda^* + \mu, tu^* + \gamma(\lambda^* + \mu, tu^*)) \rangle.$$

Note that β is a real-valued function defined in a neighbourhood U of $(0,0) \in \mathbb{R} \times \mathbb{R}$ and is of class C^2 there, because F and γ are C^2.

The following properties of β will be used later (subscripts denote partial derivatives):

(β1) $\beta(\mu, 0) = 0$ for all μ; in particular,

(β2) $\beta_\mu(0,0) = \beta_{\mu,\mu}(0,0) = 0$;

(β3) $\beta_t(0,0) = 0$.

To prove (β1) we note that

$$\beta(\mu, 0) = \langle \psi, F(\lambda^*, \gamma(\lambda^* + \mu, 0)) \rangle.$$

Since $\gamma(\lambda, 0) \equiv 0$ (see (3.7)) and $F(\lambda^*, 0) = 0$, (β1) and (β2) follow.

Next, to prove (β3) we differentiate β with respect to t yielding

$$\beta_t(\mu, t) = \langle \psi, F_u(\lambda^* + \mu, tu^* + \gamma(\lambda^* + \mu, tu^*))[u^* + \gamma_v(\lambda^* + \mu, tu^*)u^*] \rangle.$$

Letting $t = 0$ and taking into account that $\gamma(\lambda^* + \mu, 0) = 0$, we find

$$\beta_t(\mu, 0) = \langle \psi, F_u(\lambda^* + \mu, \gamma(\lambda^* + \mu, 0))[u^* + \gamma_v(\lambda^* + \mu, 0)u^*] \rangle$$

$$= \langle \psi, F_u(\lambda^* + \mu, 0)[u^* + \gamma_v(\lambda^* + \mu, 0)u^*] \rangle. \tag{4.2}$$

Since $\gamma_v(\lambda^*, 0) = 0$ (see (3.8)), we infer

$$\beta_t(0,0) = \langle \psi, F_u(\lambda^*, 0)u^* \rangle = \langle \psi, Lu^* \rangle = 0,$$

proving (β3).

Furthermore, from (4.2) it follows that

$$\beta_{t,\mu}(\mu, 0) = \langle \psi, F_{u,\lambda}(\lambda^* + \mu, 0)[u^* + \gamma_v(\lambda^* + \mu, 0)u^*] \rangle$$

$$+ \langle \psi, F_u(\lambda^* + \mu, 0)\gamma_{v,\lambda}(\lambda^* + \mu, 0)[u^*] \rangle.$$

Here and always hereafter, we identify, according to Remark 1.4.4, the mixed derivative such as $F_{u,\lambda}$ or $\gamma_{v,\lambda}$ with linear maps.

Letting $\mu = 0$ one finds

$$\beta_{t,\mu}(0,0) = \langle \psi, F_{u,\lambda}(\lambda^*, 0)[u^* + \gamma_v(\lambda^*, 0)u^*] \rangle$$

$$+ \langle \psi, F_u(\lambda^*, 0)\gamma_{v,\lambda}(\lambda^*, 0)[u^*] \rangle$$

$$= \langle \psi, F_{u,\lambda}(\lambda^*, 0)[u^*] \rangle + \langle \psi, F_u(\lambda^*, 0)\gamma_{v,\lambda}(\lambda^*, 0)[u^*] \rangle.$$

Finally, since $\psi|_R = 0$ and $F_u(\lambda^*, 0)\gamma_{v,\lambda}(\lambda^*, 0)[u^*] \in R$,

$$\langle \psi, F_u(\lambda^*, 0)\gamma_{v,\lambda}(\lambda^*, 0)[u^*] \rangle = 0,$$

and we infer

(β4) $\beta_{t,\mu}(0,0) = \langle \psi, F_{u,\lambda}(\lambda^*, 0)[u^*] \rangle$.

Furthermore, let us remark for future reference that, with direct calculations, one finds

(β5) $\beta_{t,t}(0,0) = \langle \psi, F_{u,u}(\lambda^*, 0)[u^*, u^*] \rangle$.

We are now in position to state the main result of this section.

Theorem 4.1 *Suppose $F \in C^2(\mathbb{R} \times X, Y)$ be such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Let λ^* be such that $L = F_u(\lambda^*, 0)$ satisfies assumption* (I). *Moreover, letting M denote the linear map $F_{u,\lambda}(\lambda^*, 0)$, we assume that*

$$Mu^* \notin R. \tag{4.3}$$

Then λ^* is a bifurcation point for F. In addition the set of non-trivial solutions of $F = 0$ is, near $(\lambda^*, 0)$, a unique C^1 cartesian curve with parametric representation on V.

Proof. According to the preceding discussion we have to solve the equation

$$\beta(\mu, t) = 0,$$

where β is C^2. In order to use the elementary Implicit Function Theorem, we need to "desingularize" β. For this, let us introduce the function

$$h(\mu, t) = \begin{cases} \beta(\mu, t)/t & \text{for } t \neq 0, \\ \beta_t(\mu, 0) & \text{for } t = 0. \end{cases}$$

Using properties (β1–4) it is easy to see that h is C^1, $h(0,0) = 0$ and that

$$h_\mu(0,0) = \beta_{t,\mu}(0,0),$$

$$h_t(0,0) = \frac{1}{2}\beta_{t,t}(0,0).$$

Setting

$$a := h_\mu(0,0) \text{ and } b := h_t(0,0)$$

and using ($\beta 4$) and ($\beta 5$) we find

$$a = \langle \psi, Mu^* \rangle,$$

$$b = \frac{1}{2}\langle \psi, F_{u,u}(\lambda^*, 0)[u^*, u^*] \rangle.$$

In particular, from Assumption (4.3) one deduces

$$a = \langle \psi, Mu^* \rangle \neq 0. \tag{4.4}$$

Therefore the Implicit Function Theorem applies to $h = 0$ yielding a neighbourhood $(-\varepsilon, \varepsilon)$ of $t = 0$ and a unique function $\mu \in C^1(-\varepsilon, \varepsilon)$ such that $\mu(0) = 0$ and $h(\mu(t), t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Since the equation $h(\mu, t) = 0$ is equivalent for $t \neq 0$ to $\beta(\mu, t) = 0$, it follows that the bifurcation equation (4.1) has been solved uniquely by $\mu = \mu(t)$.

Then, according to the results of Section 5.3, one finds that

$$F(\lambda^* + \mu(t), tu^* + \gamma(\lambda^* + \mu(t), tu^*)) = 0 \text{ for all } t \in (-\epsilon, \epsilon).$$

Note that $tu^* + \gamma(\lambda^* + \mu(t), tu^*) \neq 0$ provided $t \neq 0$. Therefore the set $\mathcal{S}$ of nontrivial solutions of $F(\lambda, u) = 0$ is given, in a neighbourhood of $(\lambda^*, 0)$, by the (unique) cartesian curve

$$\left.\begin{aligned} \lambda &= \lambda^* + \mu(t), \\ u &= tu^* + \gamma(\lambda^* + \mu(t), tu^*), \end{aligned}\right\}$$

where $t \in (-\varepsilon, \varepsilon), t \neq 0$. This completes the proof of the theorem.

Theorem 4.1 becomes particularly expressive when $Y = X$ and

$$F(\lambda, y) = \lambda u - G(u),$$

where $G \in C^2(X, X)$ is such that $G(0) = 0$. As already seen in Section 1, the possible bifurcation points of f are points of the spectrum of $G'(0)$. Here we will show that, when $G'(0)$ is compact, any *simple* eigenvalue $\lambda \neq 0$ of $G'(0)$ is in fact a bifurcation point. A statement of this sort will provide a first answer to the question posed in Section 1, after Proposition 1.3.

Theorem 4.2 *Let $G \in C^2(X, X)$ be such that $G(0) = 0$ and such that $G'(0)$ is compact. Suppose that $\lambda^* \neq 0$ is a simple eigenvalue of $G'(0)$, in the sense that*

$$\dim(\mathrm{Ker}(\lambda^* I - G'(0))) = 1, \tag{4.5}$$

$$\mathrm{Ker}(\lambda^* I - G'(0)) \cap R(\lambda^* I - G'(0)) = \{0\}. \tag{4.6}$$

Then λ^* is a bifurcation point for $F(\lambda, u) = \lambda u - G(u)$.

Proof. Here $L = F_u(\lambda^*, 0) = \lambda^* I - G'(0)$, $V = \mathrm{Ker}(\lambda^* I - G'(0)) = \{tu^* : t \in \mathbb{R}\}$ and $R = R(\lambda^* I - G'(0))$. Since $G'(0)$ is compact, assumption (I) follows immediately from (4.5).

Moreover, owing to the specific form of F, one has $M = F_{u,\lambda}(\lambda^*, 0)$ is the identity map, and therefore $a = \langle \psi, u^* \rangle$ where, as before, u^* denotes a vector spanning V. According to (4.6), $u^* \notin R$ and thus $a \neq 0$, proving (4.3). Then Theorem 4.1 applies and the result follows.

Remarks 4.3

(i) As anticipated, F can be assumed of class C^1, with continuous mixed partial derivative $F_{u,\lambda}$. For a proof, which requires some technicality, see [**PA**]. Similarly, one can show that Theorem 4.2 holds provided $G \in C^1(X, X)$.

(ii) The bifurcation results stated in Theorems 4.1 (or 4.2) do not hold, in general, if we only assume that L satisfies (I), namely that the dimension of Ker(L) and the codimension of R(L) are 1 without assuming (4.3) or (4.6). To see this, we can slightly modify Example 1.6. To be precise, let $X = Y = \mathbb{R}^2$ and

$$F(\lambda, x, y) = \begin{pmatrix} \lambda x - y - y^3 \\ \lambda y + x^3 \end{pmatrix}.$$

The same calculations performed in Example 1.6 show that, if

$$F(\lambda, x, y) = 0,$$

then there results

$$y^2 + y^4 + x^4 = 0.$$

Hence $F(\lambda, x, y) = 0$ has the trivial solution, only, and there are no bifurcation points.

Here, the derivative of F with respect to $u = \binom{x}{y}$ evaluated at $(\lambda, 0, 0)$ is the map

$$F_u(\lambda, 0, 0) : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \lambda x - y \\ \lambda y \end{pmatrix}.$$

Then, for $\lambda^* = 0$, we have that L can be identified with the matrix

$$L = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Thus $V = R = \mathrm{span}\{u^*\}$, with $u^* = \binom{1}{0}$, and (I) holds. On the other hand, M is the identity and hence $Mu^* = u^* \in R$. Note that Theorem 4.2 does not apply either. Indeed, the algebraic multiplicity of $\lambda = 0$ is 2, not 1 (in other words, (4.6) is not satisfied).

It is worth noticing that Theorem 4.2 applies only to maps of the specific form $F(\lambda, u) = \lambda u - G(u)$. We mean that when $X = Y$ but F is not in the form $\lambda I - G$, one has to use Theorem 4.1. In such a case, the condition $V \cap R = \{0\}$ does not play any role. To explain this, let us consider the map $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$,

$$F(\lambda, x, y) = \begin{pmatrix} \lambda x - y - y^3 \\ \lambda x + \lambda y + x^3 \end{pmatrix}.$$

As before, for $\lambda = 0$, $L = \left[\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right]$ and $V = R = \text{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, but now the mixed derivative $F_{u,\lambda}(0,0,0)$ is the matrix $M = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $M : \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \notin R$, so that (4.3) holds true and $\lambda^* = 0$ is a bifurcation point for F. With a rather elementary calculation one can solve the system

$$\left.\begin{aligned} \lambda x - y - y^3 = 0, \\ \lambda(x+y) + x^3 = 0, \end{aligned}\right\}$$

showing that the bifurcating branch has equation

$$\left.\begin{aligned} y &= -x^3 + \dots, \\ \lambda &= -x^2 - x^4 + \dots. \end{aligned}\right\}$$

(iii) When $a = \langle \psi, F_{u,\lambda}(\lambda^*, 0)[u^*] \rangle = 0$ several different situations can occur and a more careful analysis is required. In the analytic case, it can be useful to employ the Newton polygon method to solve the bifurcation equation. For more details on this matter we refer to [**VT**]. The Newton polygon method has been extended to differentiable functions by Dieudonné [**D2**].

(iv) Assuming F more regular (say C^∞, for simplicity) one can complete Theorems 4.1 and 4.2 by some calculations which will allow us to specify the behaviour of the bifurcating branch near $(\lambda^*, 0)$ (see Figure 5.8).

Since $\mu(t)$ solves $h(\mu, t) = 0$, there results (see the proof of Theorem 4.1)

$$\mu'(0) = -\frac{h_t(0,0)}{h_\mu(0,0)} = -\frac{b}{a},$$

where (see earlier)

$$a = \langle \psi, F_{u,\lambda}(\lambda^*, 0)[u^*] \rangle,$$

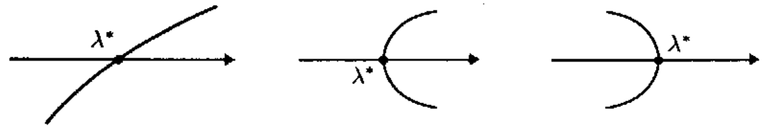
$$b = \frac{1}{2} \langle \psi, F_{u,u}(\lambda^*, 0)[u^*, u^*] \rangle.$$

Therefore if $b \neq 0$ we have

$$\lambda = \lambda^* - \frac{b}{a} t + o(t)$$

and the bifurcating branch $(\lambda, u) \in \mathcal{S}$ can be parametrized (for $|\lambda - \lambda^*|$ small) in the form

$$u = -\frac{a}{b}(\lambda - \lambda^*) u^* + o(\lambda - \lambda^*).$$



(*a*) case $b \neq 0$ (transcritical) (*b*) case $b = 0,\ c > 0$ (supercritical) (*c*) case $b = 0,\ c < 0$ (subcritical)

Figure 5.8

We note that when $b \neq 0$ the equation $F = 0$ has nontrivial solutions both for $\lambda > \lambda^*$ and for $\lambda < \lambda^*$ (*transcritical* bifurcation).

When $b = 0$ one finds

$$2c := \mu''(0) = -\frac{1}{3a} \langle \psi, F_{uuu}(\lambda^*, 0)[u^*]^3 \rangle. \tag{4.7}$$

If $b = 0$ and $c \neq 0$ the bifurcating branch has the form

$$u = \pm \left(\frac{\lambda - \lambda^*}{c} \right)^{1/2} \cdot u^* + O(\lambda - \lambda^*).$$

Note that the preceding formula shows that if $c > 0$ (respectively, $c < 0$) then the bifurcating branch emanates on the right (respectively, left) of λ^* (*supercritical*, respectively *subcritical*, bifurcation).

It is worth remarking that, when $F(\lambda, u) = \lambda u - G(u)$ and G is smooth, the values of b and c are given by the formulas

$$b = -\frac{1}{2a} \langle \psi, G''(0)[u^*, u^*] \rangle$$

and

$$c = \frac{1}{6a} \langle \psi, G'''(0)[u^*]^3 \rangle.$$

Remark 4.4 When $F(\lambda, u) = \lambda u - G(u)$, with G *compact*, that is, $G(u_n)$ is relatively compact in X for any bounded sequence $\{u_n\}$, it is possible to use the Leray–Schauder topological degree and Theorem 4.2 can be greatly improved. Results of this sort are outside the scope of this book and cannot be discussed here. However, owing to their relevance, we shall give a review of the most important ones in a short appendix at the end of this chapter.

Postponing further examples to the next chapter we discuss here some problems related to those studied in Section 2.

Example 4.5 (Sturm–Liouville problems) Let $J = [0, \pi]$, $\alpha \in C^1(J), \beta \in C(J), \alpha, \beta > 0$ on J, $p \in C^2(J \times \mathbb{R} \times \mathbb{R})$ and let a_0, b_0, a_1, b_1 be such that $(a_0^2 + b_0^2)(a_1^2 + b_1^2) \neq 0$.

Consider the Sturm–Liouville b.v.p.

$$-\mathcal{L}u := -\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha\frac{\mathrm{d}}{\mathrm{d}x}u\right) + \beta u = \lambda u + p\left(x, u, \frac{\mathrm{d}u}{\mathrm{d}x}\right), x \in J, \tag{4.8}$$

$$a_0u(0) + b_0u'(0) = a_1u(\pi) + b_1u'(\pi) = 0, \tag{4.8'}$$

where λ is a real parameter.

Setting $X = \{u \in C^2(J) : u \text{ satisfies } (4.8')\}, Y = C(J)$, define $F : \mathbb{R} \times X \to Y$ by

$$F(\lambda, u) = \mathcal{L}u + \lambda u + p(u) \tag{4.9}$$

(as usual we are using the same symbol p to indicate the Nemitski operator associated with the real-valued function p) so that the solutions of (4.8)–(4.8′) are the pairs $(\lambda, u) \in \mathbb{R} \times X$ such that $F(\lambda, u) = 0$.

Suppose that $p = p(x, s, \xi)$ satisfies

$$p(x, 0, 0) \equiv 0, p_s(x, 0, 0) \equiv 0 \text{ and } p_\xi(x, 0, 0) \equiv 0.$$

As a consequence, one has

$$F(\lambda, 0) = 0 \text{ for all } \lambda,$$

$$F_u(\lambda, 0) : u \to \mathcal{L}u + \lambda u.$$

Recall (see subsection 0.6) that the linear problem

$$\left.\begin{aligned} -\mathcal{L}u(x) &= \lambda u(x)\ (x \in J), \\ a_0u(0) + b_0u'(0) &= a_1u(\pi) + b_1u'(\pi) = 0, \end{aligned}\right\} \tag{4.10}$$

has a sequence λ_k of positive, *simple* eigenvalues, such that $\lambda_k \to \infty$ as $k \to \infty$. Let φ_k be an eigenfunction of (4.10) corresponding to λ_k, normalized by

$$\int_0^\pi \varphi_k^2 \mathrm{d}x = 1.$$

Let us apply Theorem 4.1 with $\lambda^* = \lambda_k$ and $u^* = \varphi_k$. According to subsection 0.4, one has

$$V = \mathrm{Ker}[F_u(\lambda_k, 0)] = \mathbb{R}\varphi_k,\ R = R[F_u(\lambda_k, 0)] = \{u \in Y : \int_0^\pi u\varphi_k \mathrm{d}x = 0\}.$$

Therefore (I) holds true. Furthermore we can define ψ by $\langle\psi, u\rangle = \int_0^\pi u\varphi_k \mathrm{d}x$.

Since $F_{u,\lambda}(\lambda_k, 0) : v \to v$, $a = \langle\psi, \varphi_k\rangle = \int_0^\pi \varphi_k^2 dx = 1$, proving (4.3). In conclusion, applying Theorem 4.1 we infer that each λ_k is a bifurcation point for $F =$ (4.9). Hence

> *For each $k = 1, 2, \ldots$, there is a continuous family u_λ of nontrivial solutions of* (4.8)–(4.8′) *such that $\|u_\lambda\|_{C^2} \to 0$ as $\lambda \to \lambda_k$.*

Example 4.6 (Dirichlet Problems) Let Ω be an open bounded domain in $\mathbb{R}^n$ and consider the boundary-value problem

$$\left.\begin{aligned} -\Delta u &= \lambda u + p(x, u, \nabla u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}\right\}$$

where $p \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n)$ satisfies $p(x, 0, 0) = 0$, $p_s(x, 0, 0) = 0$ and $p_\xi(x, 0, 0) = 0$.

Since the discussion does not differ from that of the Sturm–Liouville problem, we will be sketchy.

Let $X = \{u \in C^{2,\alpha}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$, $Y = C^{0,\alpha}(\overline{\Omega})$ and $F(\lambda, u) = \Delta u + \lambda u + p(u)$; one has that $F(\lambda, 0) = 0$ for all λ and $F_u(\lambda, 0)$ is the map $v \to \Delta v + \lambda v$. Hence $F_u(\lambda, 0)$ has a nontrivial kernel provided λ is an eigenvalue of

$$\left.\begin{aligned} -\Delta v &= \lambda v \text{ in } \Omega, \\ v &= 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{4.11}$$

If λ_k is any *simple* eigenvalue of (4.11) with corresponding eigenfunction φ_k, normalized by $\int_\Omega \varphi_k^2 \mathrm{d}x = 1$, then (I) holds true. As before, one has $R = R(F_u(\lambda_k, 0)) = \{u \in Y : \langle\psi, u\rangle = \int_\Omega u\varphi_k \mathrm{d}x = 0\}$; since $F_{u,\lambda}(\lambda_k, 0) : v \to v, a = \langle\psi, \varphi_k\rangle = 1$ and (4.3) holds, too. Therefore, from Theorem 4.1 it follows that

> *any simple eigenvalue of* (4.11) *is a bifurcation point for $F(\lambda, u) = \Delta u + \lambda u + p(u)$.*

We note that, in particular, this result applies when we take the first eigenvalue of (4.11).

To know the behaviour of the bifurcating branch we refer to Remark 4.5. For simplicity, let us take a (smooth) nonlinearity p depending on u only.

Since here $F_{u,u}(\lambda_k, 0) : (v, w) \to p''(0)vw$, (4.4) becomes

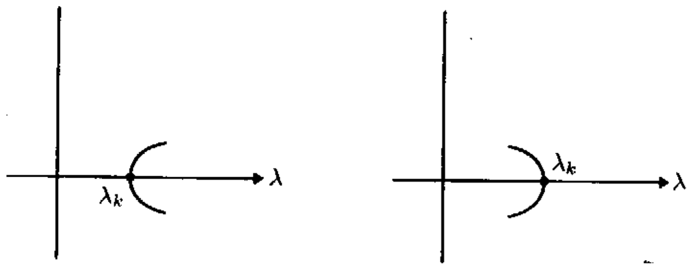
$$b = \frac{1}{2}\langle\psi, F_{u,u}(\lambda^*, 0)[\varphi_k, \varphi_k]\rangle = \frac{1}{2}p''(0)\int_\Omega \varphi_k^3 \mathrm{d}x.$$

If $p''(0) = 0$, one uses (4.7) yielding

$$c = -\frac{1}{3}p'''(0)\int_\Omega \varphi_k^4 \mathrm{d}x.$$

For example, if $p(u) = -u^3$, then $b = 0$ and $c > 0$; hence the bifurcation is *supercritical* that is, occurs for $\lambda > \lambda_k$, while, if $p(u) = u^3$, then $c < 0$ and the bifurcation is *subcritical* (see Figure 5.9).

We end this section with some further remarks on the geometric character of Theorems 4.1–4.2.



(*a*) Bifurcation portrait for $-\Delta u = \lambda u - u^3$

(*b*) Bifurcation portrait for $-\Delta u = \lambda u + u^3$

Figure 5.9

After the Lyapunov-Schmidt reduction, the problem of finding the bifurcation points of $F(\lambda, u) = 0$ is reduced to the search for the zeros of a real-valued C^2 function $\beta = \beta(\mu, t)$, with the properties that

$$\beta(\mu, 0) = 0 \text{ for all } \mu,$$
$$\beta_t(0,0) = 0,$$
$$\beta_{\mu,t}(0,0) \neq 0.$$

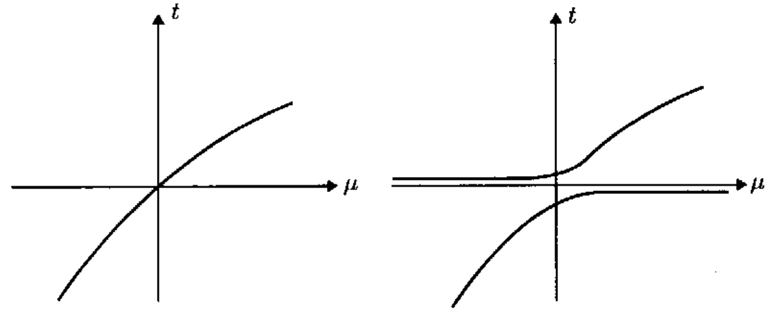
The proof we have carried out led us to find two branches of solutions: that of the trivial zeros, and that of the nontrivial solutions, giving rise to the bifurcation branch. Suppose now that F is perturbed through $\tilde{F}$, with $\|F - \tilde{F}\|_{C^2} < \varepsilon$, with ε small. Perturbing F through $\tilde{F}$ will affect the bifurcation equation in the sense that $\beta = 0$ will be replaced by a perturbed bifurcation equation $\tilde{\beta} = 0$, with $\|\beta - \tilde{\beta}\|_{C^2}$ small. In general it is possible to prove that the zeros of $\tilde{\beta}$ become two branches that do not cross themselves, in general, but are merely "close" (see Figure 5.10).

This kind of perturbation phenomenon arises, for example, when one deals with bifurcation problems from the point of view of *numerical analysis*: approximation or truncation procedures can be viewed as perturbation.

For a discussion of this kind of problems, we refer to the paper by Golubitsky and Schaeffer [**GS**].

5 A bifurcation theorem from a multiple eigenvalue

In this section we will discuss a result dealing with a case in which Ker(L) is, possibly, not one-dimensional.



(*a*) Solutions of $\beta = 0$

(*b*) Solutions of $\tilde{\beta} = 0$

Figure 5.10

For simplicity, we consider an $F \in C^\infty(\mathbb{R} \times X, Y)$ and assume that (a) and (b) of Section 3 hold true. Keeping the notation of §3, we set $L = F_u(\lambda^*, 0)$ and write $X = V \oplus W, Y = Z \oplus R$, with $V = \text{Ker}(L)$ and $R = R(L)$.

Let M denote the linear map $F_{u,\lambda}(\lambda^*, 0)$ (recall Remark 1.4.4) and $\mathcal{B}$ the bilinear map $F_{u,u}(\lambda^*, 0)$; then, setting $\lambda = \lambda^* + \mu$ we find that the equation $F = 0$ becomes

$$Lu + \mu M u + \frac{1}{2}\mathcal{B}(u,u) + \psi(\lambda^* + \mu, u) = 0, \tag{5.1}$$

where ψ is smooth and such that

$$\psi(\lambda, 0) \equiv 0,\ \psi_u(\lambda^*, 0) = 0,\ \psi_{u,u}(\lambda^*, 0) = 0,\ \psi_{\lambda,u}(\lambda^*, 0) = 0. \tag{5.2}$$

We seek solutions of the form $u = \mu(v + w)$, with $v \in V$ and $w \in W$.

Substituting into (5.1) we find

$$\mu L w + \mu^2 M(v+w) + \frac{1}{2}\mu^2 \mathcal{B}[v+w, v+w] + \psi(\lambda^* + \mu, \mu(v+w)) = 0.$$

Then, according to the Lyapunov-Schmidt reduction, the equation $F = 0$ is equivalent to the system

$$\mu^2 PM(v+w) + \frac{1}{2}\mu^2 P\mathcal{B}[v+w, v+w] + P\psi(\lambda^* + \mu, \mu(v+w)) = 0, \tag{5.3'}$$

$$\mu L w + \mu^2 QM(v+w) + \frac{1}{2}\mu^2 Q\mathcal{B}[v+w, v+w] + Q\psi(\lambda^* + \mu, \mu(v+w)) = 0, \tag{5.3''}$$

where P and Q indicate, as usual, the projections onto Z and R.

According to (5.2) we can write

$$\psi(\lambda^* + \mu, \mu(v+w)) = \mu^3 \tilde{\psi}(\mu, v, w)$$

where $\tilde{\psi}$ is smooth.

Hence (5.3')–(5.3″) are equivalent for $\mu \neq 0$ to

$$PM(v+w) + \frac{1}{2}P\mathcal{B}[v+w, v+w] + \mu P\tilde{\psi}(\lambda^* + \mu, \mu(v+w)) = 0, \quad (5.4')$$

$$Lw + \mu QM(v+w) + \frac{1}{2}\mu Q\mathcal{B}[v+w, v+w] + \mu^2 Q\tilde{\psi}(\lambda^* + \mu, \mu(v+w)) = 0. \quad (5.4'')$$

With $\tilde{\Phi} = \tilde{\Phi}(\mu, v, w)$ denoting the left-hand side of (5.4″) there results $\tilde{\Phi}(0, v, 0) = 0$ for all $v \in V$ as well as $\tilde{\Phi}_w(0, v, 0) = L|_W$; hence, for any fixed $v^* \in V$, we can solve (5.4″) uniquely with respect to w in a neighbourhood of $\mu = 0, v = v^*$; from (5.4″) it follows readily that

$$w = \mu\gamma(\mu, v)$$

with γ smooth. Substituting into (5.4') we find the bifurcation equation

$$N(\mu, v) := PM(v + \mu\gamma(\mu, v)) + \frac{1}{2}P\mathcal{B}[v + \mu\gamma(\mu, v), v + \mu\gamma(\mu, v)]$$

$$+\mu P\tilde{\psi}(\lambda^* + \mu, \mu(v + \mu\gamma(\mu, v))) = 0. \quad (5.5)$$

Note that N is smooth. Moreover, let us point out that γ depends on $v^* \in V$. We will show that if v^* can be chosen in a suitable way then (5.5) can be solved, giving rise to a bifurcating branch for $F = 0$. More precisely one has the following.

Theorem 5.1 *Suppose that $V = \mathrm{Ker}(L)$ has a topological complement in X and $R = R(L)$ is closed and has a topological complement in Y. Moreover, letting $M = F_{u,\lambda}(\lambda^*, 0)$ and $\mathcal{B} = F_{u,u}(\lambda^*, 0)$, suppose there exists $v^* \in V$, $v^* \neq 0$, such that*

(a) $PMv^* + \frac{1}{2}P\mathcal{B}(v^*, v^*) = 0$,

(b) *the linear map $S : V \to V$, $Sv = PMv + P\mathcal{B}(v^*, v)$ is invertible.*

Then there is a branch of nontrivial solutions of $F = 0$ bifurcating from $(\lambda^*, 0)$ with equations

$$\left.\begin{aligned} \lambda &= \lambda^* + \mu, \\ u &= \chi(\mu), \end{aligned}\right\} \quad (5.6)$$

where $\chi(0) = 0$ and $\chi'(0) = v^*$.

Proof. From (a) it follows that $N(0, v^*) = 0$; moreover $N_v(0, v^*) = S$, which is invertible by (b). Then the Implicit Function Theorem applies to $N(\mu, v) = 0$. To be precise, there exists $v = v(\mu)$, defined for $|\mu|$ small, such that $v(0) = 0$ and

$$N(\mu, v(\mu)) = 0.$$

Hence we find a bifurcation branch of the form

$$u(\mu) = \mu(v(\mu) + \mu\gamma(\mu, v(\mu))).$$

Setting $\chi(\mu) := \mu(v(\mu) + \mu\gamma(\mu, v(\mu)))$, we get $\chi'(0) = v^*$. As a consequence, $u(\mu) \neq 0$, $|\mu|$ small and > 0, and $u = \chi(\mu)$ gives rise to a branch of nontrivial solutions of $F = 0$. This proves that λ^* is a bifurcation point for $F = 0$ and completes the proof of the theorem.

Remarks 5.2

(i) The branch found in the preceding theorem might not be unique: either because v^* might be not uniquely determined, or because there are other nontrivial solutions of $F = 0$, not in the form $\mu(v + w)$.

(ii) The equation of the bifurcating branch is parametrized with respect to μ and thus indicates that Theorem 5.1 gives rise to a transcritical bifurcation.

Theorem 5.1 can be used to find a sufficient condition for the existence of a bifurcation when $\dim(V) = \dim(Z) = 1$, but Theorem 4.1 does not apply.

Let $V = \mathbb{R}u^*$, and suppose that $PMu^* = 0$. Then conditions (a) and (b) of Theorem 5.1 become

(a') $P\mathcal{B}[u^*, u^*] = 0$,

(b') the linear map $v \to P\mathcal{B}[v^*, v]$ from V to Z is invertible.

If (a')–(b') hold true then an application of Theorem 5.1 yields the following.

Theorem 5.3 *Suppose $F \in C^2(\mathbb{R} \times X, Y)$ is such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Let λ^* be such that $L = F_u(\lambda^*, 0)$ satisfies Assumption (I) and let $V = \mathbb{R}u*$. Moreover, we set $M = F_{u,\lambda}(\lambda, 0)$, $\mathcal{B} = F_{u,u}(\lambda^*, 0)$ and we assume that $Mu^* \in \mathbb{R}$ and that (a')–(b') hold true. Then λ^* is a bifurcation point for F.*

An application

Let us apply Theorem 5.1 to the following problem: given a continuous 2π-periodic function h, to find 2π-periodic solutions of

$$u'' + \lambda u + hu^2 = 0, \quad (5.7)$$

We set $X = C^2_{2\pi}$, $Y = C_{2\pi}$, where $C^k_{2\pi}$ (resp. $C_{2\pi}$) denotes the space of 2π-periodic C^k functions (resp. continuous functions), and let $F : \mathbb{R} \times X \to Y$,

$$F(\lambda, u) = u'' + \lambda u + hu^2. \quad (5.8)$$

Here $Lv = v'' + \lambda v$, $Mv = v$ and $\mathcal{B}[u, v] = 2huv$. For $\lambda = \lambda^* =$

k^2, $V = \mathrm{Ker}(L)$ is two-dimensional and spanned by $\{\cos kt, \sin kt\}$. Moreover, $Z = \mathrm{span}\{\cos kt, \sin kt\}$, too, and $R = R(L)$ is L^2-orthogonal to Z. As for the corresponding projection $P : Y \to Z$, one has that $Ph = (a_k \cos kt, b_k \sin kt)$ where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} h(t)\cos kt\,\mathrm{d}t,$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} h(t)\sin kt\,\mathrm{d}t.$$

If we write $v = A\cos kt + B\sin kt$ condition (a) leads us to find nontrivial solutions of the system

$$A + \frac{1}{\pi}\int_0^{2\pi} (A\cos kt + B\sin kt)^2 h(t)\cos kt = 0,$$

$$B + \frac{1}{\pi}\int_0^{2\pi} (A\cos kt + B\sin kt)^2 h(t)\sin kt = 0.$$

This system is of the form

$$\left.\begin{aligned} A + \mathcal{P}(A,B) = 0,\\ B + \mathcal{Q}(A,B) = 0, \end{aligned}\right\} \tag{5.9}$$

where $\mathcal{P}, \mathcal{Q}$ are homogeneous polynomials of degree 2, whose coefficients depend on h. From the geometrical point of view, the solutions can be thought of as the intersections of two conics crossing through the origin transversally to each other. So they intersect in another point in the projective plane. This intersection is not on the "line at infinity", that is, $A, B \in \mathbb{R}$, provided the system

$$\left.\begin{aligned} \mathcal{P}(A,B) = 0,\\ \mathcal{Q}(A,B) = 0, \end{aligned}\right\}$$

has the trivial solution $A = B = 0$ only. It is easy to see that this is the case for all $h \in Y \backslash Y_0$, for some thin set Y_0 (in the sense of Baire). Then, for a "generic" h (5.9) has a nontrivial solution $(A^*, B^*) \in \mathbb{R}^2$; as for condition (b), it also holds for all h up to a thin set. Then we can conclude that, for all $h \in Y$, up to a set of first category in Y, each $\lambda = k^2, k = 1, 2, \ldots$, is a bifurcation for F given by (5.6); each bifurcating branch gives rise to a family of 2π-periodic solutions of (5.5).

Appendix

In this short appendix we want to review some very important bifurcation results, which require tools other than the Local Inversion Theorem.

We will deal with equations of the type

$$F(\lambda, u) = \lambda u - G(u) = 0, \tag{A1}$$

where G satisfies

(G1) $G \in C(X, X)$ and is differentiable at $u = 0$, with (compact) derivative $A = G'(0)$,
(G2) G is compact.

It is always understood that $G(0) = 0$.

Theorem A1 (Krasnoselskii, [Kr1]) *Suppose that* (G1–2) *hold and let λ^* be an eigenvalue of A with odd (algebraic) multiplicity. Then λ^* is a bifurcation point for F.*

Roughly, the proof relies on the following arguments. If, supposing the contrary, λ^* is not a bifurcation point then there exist a ball D around $u = 0$ and $\varepsilon > 0$ such that

$$F(\lambda, u) \neq 0 \text{ for all } \lambda \in [\lambda - \varepsilon, \lambda + \varepsilon], \text{ for all } u \in \partial D. \tag{A2}$$

In view of (A2), it makes sense to consider the Leray–Schauder topological degree, $\mathrm{d}(F_\lambda, D, 0)$, of $F_\lambda := F(\lambda, .)$, with respect to D and $u = 0$, and, by the homotopy invariance of the degree, one has

$$\mathrm{d}(F_{\lambda^*-\varepsilon}, D, 0) = \mathrm{d}(F_{\lambda^*+\varepsilon}, D, 0). \tag{A3}$$

On the other hand, if necessary taking D smaller, the degree of F_λ can be evaluated by linearization: more precisely, if λ is not an eigenvalue of $A = G'(0)$, then one has

$$\mathrm{d}(F_\lambda, D, 0) = \mathrm{d}(\lambda I - A, D, 0) = (-1)^k, \tag{A4}$$

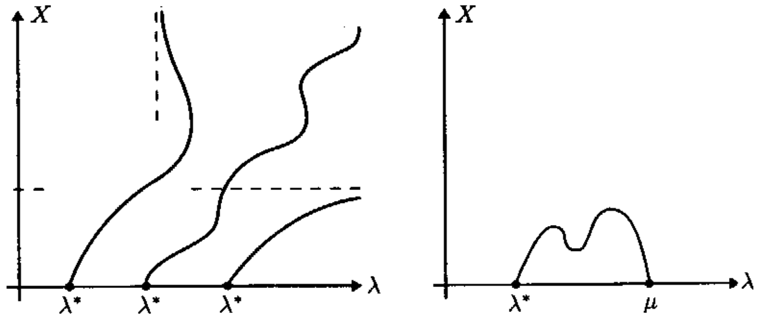
where k denotes the sum of the (algebraic) multiplicities (see subsection 0.4) of the eigenvalues μ of A, with $\mu > \lambda$.

Let m denote the sum of the (algebraic) multiplicities of the eigenvalues μ of A, with $\mu > \lambda^*$, and m^* that of λ^*. If necessary taking ε smaller, we can assume that λ^* is the only eigenvalue of A in the interval $[\lambda - \varepsilon, \lambda + \varepsilon]$. Then from (A4) we infer

$$\mathrm{d}(F_{\lambda^*+\varepsilon}, D, 0) = (-1)^m,$$

$$\mathrm{d}(F_{\lambda^*-\varepsilon}, D, 0) = (-1)^{m+m^*}.$$

Since m^* is odd, it follows that $\mathrm{d}(F_{\lambda^*-\varepsilon}, D, 0) \neq \mathrm{d}(F_{\lambda^*+\varepsilon}, D, 0)$, in contradiction with (A3). This proves that λ^* is a bifurcation point.



(*a*) Possible bifurcation diagrams in case (i)

(*b*) A possible bifurcation diagram in case (ii)

Figure 5.11

Actually, the global nature of the topological degree can be used to improve Theorem A1 as follows.

Theorem A2 (Rabinowitz, [R2]) *Suppose that* (G1–2) *hold and let* λ^* *be an eigenvalue of* A *with odd (algebraic) multiplicity. Then from* λ^* *there branches off a continuum (namely a closed connected set)* Σ *of nontrivial solutions of* $F = 0$ *such that either*

(i) Σ *is unbounded, or*

(ii) Σ *meets another eigenvalue* $\mu \neq \lambda^*$ *of* A.

Theorem A2 applies to a large variety of problems. Among others, we mention Sturm–Liouville problems [**CrR**], existence of positive solutions of nonlinear eigenvalue problems [**AH**], existence of vortex rings in an ideal fluid [**AmiT**].

A last result which is worth recalling deals with the case in which G is a *variational operator*. To be precise, let us assume that X is a Hilbert space and that there exists $g : X \to \mathbb{R}$, such that $G = \nabla g$. Note that in such a case (A1) becomes $\nabla g(u) = \lambda u$, whose solutions can be found as *critical points* of g on the Hilbert sphere $\|u\| = \rho$, the parameter λ playing the role of the Lagrange multiplier.

Theorem A3 (Krasnoselski, [Kr1]) *Suppose* $G \in C^1(X, X)$ *is a variational operator and satisfies* (G2). *Then any eigenvalue of* $A = G'(0)$ *is a bifurcation point for* $F = 0$.

For a proof using Morse Theory, see [**MP**]. Improvements can be found in [**Bö**] and [**Mar**].

6

Bifurcation problems

There is a broad variety of problems arising in applications that can be handled by the Bifurcation Theorem stated in Section 5.4. In the present chapter we will discuss some of them. We have tried to choose problems that are relevant from the physical point of view but that do not need too much technicality. Only one of them, the Bénard Problem discussed in Section 2, is not self-contained. Indeed, the analysis of linearized equations requires some delicate tools that would need much more space. Neverthless, the relevance of the problem has driven us to include it in this chapter, even if we had to be sketchy in several points.

1 The rotating heavy string

Following the formulation of Kolodner [**Ko**], we consider a string with uniform density ρ and length $= 1$, hung at the origin of the coordinates (the z-axis will be considered to be pointing downwords) in $\mathbb{R}^3$. The points on the string will be parametrized through the arclength $s \in [0, 1]$ and denoted by $\mathbf{x}(s,t) = (x(s,t), y(s,t), z(s,t))$. It is convenient to take s in such a way that $\mathbf{x}(1,t) = (0,0,0)$ is the fixed endpoint of the string.

The equations of the motion are

$$\rho \mathbf{x}_{tt} = \rho \mathbf{g} + (T\mathbf{x}_s)_s \tag{1.1}$$

together with

$$|\mathbf{x}_s|^2 = 1, \tag{1.2}$$