CHAPTER 10

Unbounded Operators and Operator Semigroups

This chapter is devoted to fundamentals of the spectral theory of unbounded selfadjoint operators and some elements of the theory of operator semigroups. Some of the principal applications of these theories are connected with partial differential equations and mathematical physics, in particular, quantum mechanics, but there are also important applications in many other areas of mathematics, for example, in the theory of random processes and geometry. The spaces considered in this chapter are complex unless if explicitly stated otherwise.

10.1. Graphs and Adjoints

We already know that on every infinite-dimensional normed space there are discontinuous linear functionals. However, the theory of unbounded selfadjoint operators deals with unboundedness of another sort: a characteristic feature of these operators is that their domain of definition does not coincide with the whole space. A typical example is a differential operator regarded as an operator on the space L^2, although its actual domain of definition is smaller.

10.1.1. Definition. *Let X and Y be Banach spaces. A linear mapping T from a dense linear subspace $\mathfrak{D}(T) \subset X$, called the domain of definition of T, to the space Y is called a densely defined linear operator on X.*

An operator T with the domain of definition $\mathfrak{D}(T)$ is called closed if its graph

$$\Gamma(T) := \bigl\{\{x, Tx\} \in X \times Y : \ x \in \mathfrak{D}(T)\bigr\}$$

is closed in $X \times Y$.

Note that the kernel $\operatorname{Ker} T$ of a closed operator T is always closed, although $\mathfrak{D}(T)$ can fail to be closed. Indeed, if $x_n \in \mathfrak{D}(T)$, $x_n \to x$ and $Tx_n = 0$, then $x \in \mathfrak{D}(T)$ and $Tx = 0$.

The set of values of a linear mapping T defined on a linear domain of definition $\mathfrak{D}$ is called the range of T and denoted by $\operatorname{Ran} T$, i.e.,

$$\operatorname{Ran} T := T(\mathfrak{D}).$$

According to the closed graph theorem, every closed operator defined on the whole Banach space is bounded. Hence every unbounded everywhere defined operator serves as an example of a densely defined nonclosed linear operator. Let us give an example of an unbounded densely defined closed operator. Until §10.4 inclusive for simplification of exposition we shall discuss only operators on Hilbert

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spaces. Only in §§10.5–10.6 in connection with semigroups we return to general Banach spaces.

10.1.2. Example. Let $H = l^2$. Set

$$\mathfrak{D}(T) = \Big\{x \in l^2\colon\ \sum_{n=1}^{\infty} n^2 x_n^2 < \infty\Big\}, \quad Tx = (x_1, 2x_2, 3x_3, \ldots).$$

If vectors $x^k = (x_n^k)$ converge to x and vectors Tx^k converge to y, then it is clear that $x \in \mathfrak{D}(T)$ and $Tx = y$. So T is closed.

10.1.3. Definition. *We shall say that an operator $\big(S, \mathfrak{D}(S)\big)$ is an extension of an operator $\big(T, \mathfrak{D}(T)\big)$ if*

$$\mathfrak{D}(T) \subset \mathfrak{D}(S) \quad \textit{and} \quad S|_{\mathfrak{D}(T)} = T.$$

Notation: $T \subset S$.

For an unbounded operator its domain of definition is a characteristic as important as the way of defining the operator on this domain. Operators defined by the same expression on different domains are different operators and, as we shall see below, can possess completely different properties.

10.1.4. Definition. *A densely defined linear operator T is called closable if it has a closed extension.*

If an operator T is closable, then the closure $\overline{\Gamma(T)}$ of its graph is the graph of the operator called the *closure* of T and denoted by the symbol $\overline{T}$.

The domain $\mathfrak{D}(T)$ of an operator T on a Hilbert space H can be equipped with the inner product

$$(x, y)_T := (x, y) + (Tx, Ty).$$

The Euclidean space obtained in this way is denoted by D_T. The norm

$$\|x\|_T := \sqrt{(x, x)_T}$$

is equivalent to the norm induced from H only if the operator T is bounded.

10.1.5. Lemma. *A densely defined operator T is closed precisely when the space D_T is Hilbert.*

PROOF. Obviously, the mapping $x \mapsto \{x, Tx\}$ is an isometry of the spaces D_T and $\Gamma(T)$. Hence the completeness of one of them is equivalent to the completeness of the other. □

Let us give an example of an operator that has no closure.

10.1.6. Example. Let $H = L^2[0,1]$, $\mathfrak{D}(T) = C[0,1]$, $Tx(t) = x(1) \cdot 1(\mathrm{t})$, where 1 is the function identically equal to 1. The operator T has no closure. Indeed, let $x_n(t) = t^n$. Then $x_n \in \mathfrak{D}(T)$, $x_n \to 0$ in H, but $Tx_n = 1$.

For consideration of graphs of operators on a Hilbert space H let us introduce on $H \times H$ two unitary operators

$$U\colon\ H \times H \to H \times H, \quad U\{x, y\} = \{y, x\},$$
$$V\colon\ H \times H \to H \times H, \quad V\{x, y\} = \{-y, x\}.$$

The operators U and V satisfy the equalities

$$U^2 = I, \quad V^2 = -I, \quad UV = -VU.$$

Let us note the following relation valid for every set $E \subset H \times H$:

$$(VE)^\perp = V(E^\perp) = V^{-1}(E^\perp). \tag{10.1.1}$$

The first equality follows from the fact that a unitary operator preserves the orthogonality and the second one follows from the equalities $V^2 = -I$ and $E^\perp = -E^\perp$.

10.1.7. Proposition. *Suppose that a densely defined operator T on a Hilbert space H has a dense range and $\operatorname{Ker} T = \{0\}$. Then the closedness of T is equivalent to the closedness of the operator T^{-1} with domain $\mathfrak{D}(T^{-1}) = \operatorname{Ran} T$.*

PROOF. It suffices to observe that $\Gamma(T^{-1}) = U\Gamma(T)$. $\square$

10.1.8. Definition. *Suppose that an operator T on a Hilbert space H has a dense domain of definition $\mathfrak{D}(T)$. Let us define the operator T^* as follows: $y \in \mathfrak{D}(T^*)$ if there exists an element $T^*y := z \in H$ such that*

$$(Tx, y) = (x, z) \quad \textit{for all } x \in \mathfrak{D}(T).$$

By the Riesz theorem the vector y belongs to $\mathfrak{D}(T^*)$ precisely when the functional $x \mapsto (Tx, y)$ on $\mathfrak{D}(T)$ is continuous with respect to the norm from H.

The graph of T^* is connected with the graph of T in the following way.

10.1.9. Proposition. *Let T be a densely defined operator on a Hilbert space H. Then*

$$\Gamma(T^*) = [V\Gamma(T)]^\perp,$$

where the orthogonal complement is taken in $H \times H$. In particular, the operator T^* is closed.

PROOF. By definition $\{y, z\} \in \Gamma(T^*)$ precisely when $(Tx, y) = (x, z)$ for all $x \in \mathfrak{D}(T)$, which can be written as the orthogonality of $\{-Tx, x\}$ and $\{y, z\}$ in $H \times H$. The latter is the orthogonality of $\{y, z\}$ to the subspace $V\Gamma(T)$. $\square$

10.1.10. Corollary. *If T is a closed densely defined operator on H, then $H \times H$ is the orthogonal sum of the closed subspaces $V\Gamma(T)$ and $\Gamma(T^*)$, i.e.,*

$$H \times H = V\Gamma(T) \oplus \Gamma(T^*).$$

In particular, for every pair of vectors $x, y \in H$, there exists a unique pair of vectors $u \in \mathfrak{D}(T)$ and $v \in \mathfrak{D}(T^*)$ satisfying the equalities

$$x = -Tu + v, \quad y = u + T^*v,$$

$$\|x\|^2 + \|y\|^2 = \|u\|^2 + \|Tu\|^2 + \|v\|^2 + \|T^*v\|^2.$$

It is clear that $0 \in \mathfrak{D}(T^*)$ for every densely defined operator T. However, it can happen that $\mathfrak{D}(T^*)$ contains no nonzero elements.

10.1.11. Example. It suffices to construct an operator on $H = l^2$ the graph of which is everywhere dense in $H \oplus H$. For this we take a linearly independent everywhere dense countable set $B_1 = \{b_n^1\}$ in H, add an everywhere dense countable set $B_2 = \{b_n^2\}$ linearly independent with B_1, and continue by induction. We obtain a countable collection of everywhere dense countable sets $B_k = \{b_n^k\}$ that are linearly independent in total. Let us complement the constructed countable linearly independent set to a Hamel basis and set $Ab_n^k = b_k^1$; on the additional vectors of the basis we define A by zero. We extend A to H by linearity. The graph of A is everywhere dense in $H \times H$: by the linearity of the graph it suffices to verify that every vector form $\{b_m^1, b_k^1\}$ belongs to the closure of the graph and for this it suffices to take a sequence $b_{n_j}^k \to b_m^1$.

10.1.12. Proposition. *Let T be an operator with a dense domain $\mathfrak{D}(T)$ in a Hilbert space H. Then*

(i) $(T+B)^* = T^* + B^*$ *for all* $B \in \mathcal{L}(H)$,

(ii) $(\lambda T)^* = \overline{\lambda} T^*$ *and* $(T+\lambda I)^* = T^* + \overline{\lambda} I$ *for all* $\lambda \in \mathbb{C}$.

PROOF. (i) The equalities $\mathfrak{D}(T+B) = \mathfrak{D}(T)$, $\mathfrak{D}(T^*+B^*) = \mathfrak{D}(T^*)$ are obvious. For any $y \in \mathfrak{D}(T^*)$, $x \in \mathfrak{D}(T)$ we have

$$(Tx + Bx, y) = (x, T^*y) + (x, B^*y) = (x, T^*y + B^*y),$$

so $T^* + B^* \subset (T+B)^*$. Applying this to $T+B$ and $-B$, we obtain the inverse inclusion. (ii) The first equality is obvious, the second one follows from (i). □

There is the same connection between the range of the operator and the kernel of its adjoint as in the case of bounded operators.

10.1.13. Proposition. *Let T be a linear operator with a dense domain of definition $\mathfrak{D}(T)$ in a Hilbert space H. Then the closed subspaces $\overline{\mathrm{Ran}\, T}$ and $\mathrm{Ker}\, T^*$ are mutually orthogonal and $H = \overline{\mathrm{Ran}\, T} \oplus \mathrm{Ker}\, T^*$.*

In addition, for every $\lambda \in \mathbb{C}$ we have

$$H = \overline{\mathrm{Ran}\,(T - \lambda I)} \oplus \mathrm{Ker}\,(T^* - \overline{\lambda} I).$$

PROOF. The subspace $\mathrm{Ker}\, T^*$ is closed by the closedness of T^*. The inclusion $y \in \mathrm{Ker}\, T^*$ is equivalent to the property that $(Tx, y) = 0$ for all $x \in \mathfrak{D}(T)$. The latter is precisely the condition $y \perp \mathrm{Ran}\, T$. Therefore, for every $u \in H$ we obtain an element v in $\overline{\mathrm{Ran}\, T}$ that is the orthogonal projection of u onto this closed subspace and also the element $w := u - v$ orthogonal to $\mathrm{Ran}\, T$, i.e., belonging to $\mathrm{Ker}\, T^*$ according to what has been said above. The last assertion of the proposition follows from the first one applied to the operator $T - \lambda I$. □

10.1.14. Proposition. (i) *If a densely defined operator T has the closure $\overline{T}$, then $(\overline{T})^* = T^*$.*

(ii) *Suppose that an operator T is densely defined. The operator T^* is densely defined precisely when T is closable. In this case $T^{**} = \overline{T}$.*

(iii) *Let T be an operator such that the sets $\mathfrak{D}(T)$ and $\mathrm{Ran}\, T$ are dense and $\mathrm{Ker}\, T = \{0\}$. Then the operator $(T^*)^{-1}$ is densely defined and $(T^*)^{-1} = (T^{-1})^*$, where $\mathfrak{D}(T^{-1}) = \mathrm{Ran}\, T$.*

PROOF. (i) We have $\Gamma(T^*) = [V\Gamma(T)]^{\perp}$. If T is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$, hence

$$[V\Gamma(T)]^{\perp} = [V\Gamma(\overline{T})]^{\perp} = \Gamma(\overline{T}^*).$$

(ii) Let T be densely defined and closable. According to assertion (i) we have $T^* = (\overline{T})^*$. Hence we can assume that T is closed. If $y \perp \mathfrak{D}(T^*)$, then $\{y, 0\} \perp \Gamma(T^*)$, which by Corollary 10.1.10 gives the inclusion $\{y, 0\} \in \Gamma(T)$. Hence $y = 0$.

Conversely, let the operator T^* be densely defined. The operator T^{**} is closed. In addition, $T \subset T^{**}$, which yields the closability of T and the inclusion $\overline{T} \subset T^{**}$. We show that actually the equality holds. It is clear that it suffices to show the equality $T = T^{**}$ for the closed operator T. This equality follows from the relations

$$\Gamma(T^{**}) = [V\Gamma(T^*)]^{\perp}, \quad \Gamma(T) = V^{-1}\big([\Gamma(T^*)]^{\perp}\big)$$

and equality (10.1.1).

(iii) The density of the range of T gives the equality $\operatorname{Ker} T^* = 0$. Hence the operator $(T^*)^{-1}$ is defined on the range of T^*. Let us compare the graphs of the operators $(T^*)^{-1}$ and $(T^{-1})^*$. We have the equalities

$$\Gamma\big((T^*)^{-1}\big) = U\Gamma(T^*) = U\big([V\Gamma(T)]^{\perp}\big),$$
$$\Gamma\big((T^{-1})^*\big) = [V\Gamma(T^{-1})]^{\perp} = [VU\Gamma(T)]^{\perp}.$$

Their right-hand sides coincide, which follows from relations (10.1.1) and the equality $UV = -VU$. □

For unbounded operators it is also useful to introduce the notion of a regular point. Let T be a closed operator. The number $\lambda \in \mathbb{C}$ is called *regular* for T if $T - \lambda I$ has a bounded inverse on the domain $\operatorname{Ran}(T - \lambda I)$, i.e., for some $c > 0$ we have

$$\|(T - \lambda I)x\| \geqslant c\|x\|, \quad x \in \mathfrak{D}(T). \tag{10.1.2}$$

Set

$$d_T(\lambda) := \dim \big(\operatorname{Ran}(T - \lambda I)\big)^{\perp}, \ \lambda \in \mathbb{C}.$$

The number $d_T(\lambda) \in [0, +\infty]$ is called the *defect* of the operator $T - \lambda I$.

10.1.15. Proposition. *Let T be a closed operator on a Hilbert space and let λ be its regular point. Then there exists an open disc centered at λ consisting of regular points and the function d_T is constant on this disc.*

PROOF. If we have (10.1.2) and $|\lambda - \mu| < c$, then

$$\|(T - \mu I)x\| \geqslant (c - |\lambda - \mu|)\|x\|, \quad x \in \mathfrak{D}(T).$$

The operator $T - \lambda I$ is closed, since if

$$x_n \in \mathfrak{D}(T), \ x_n \to x \quad \text{and} \quad Tx_n - \lambda x_n \to y,$$

then $Tx_n \to \lambda x + y$, whence $x \in \mathfrak{D}(T)$ and $Tx = \lambda x + y$, i.e., $(T - \lambda I)x = y$. Hence we can assume that $\lambda = 0$. Then (10.1.2) yields that the ranges of T and $T - \mu I$ are closed. Let us show that $d_T(0) \leqslant d_T(\mu)$. If $d_T(0) > d_T(\mu)$, then the subspace $E_\mu := \big(\operatorname{Ran}(T - \mu I)\big)^{\perp}$ is finite-dimensional and there exists a nonzero element $z \in (\operatorname{Ran} T)^{\perp}$ orthogonal to E_μ. Hence $z \in \operatorname{Ran}(T - \mu I)$, i.e., $z = (T - \mu I)y$, $y \in \mathfrak{D}(T)$, $y \neq 0$. Since $z \perp \operatorname{Ran} T$, we have $(z, Ty) = 0$. Hence

$$(Ty, Ty) = \mu(y, Ty) \leqslant |\mu| \, \|y\| \, \|Ty\|,$$

whence $\|Ty\| \leqslant |\mu|\,\|y\|$. Therefore, $y = 0$, which is a contradiction. Hence $d_T(0) \leqslant d_T(\mu)$. Similarly we verify that $d_T(0) \geqslant d_T(\mu)$, since in the opposite case there is a nonzero vector y for which $z = Ty$ is orthogonal to $Ty - \mu y$. $\square$

10.2. Symmetric and Selfadjoint Operators

A very important class for applications is constituted by unbounded selfadjoint operators on Hilbert spaces.

10.2.1. Definition. *An operator A with a dense domain in a Hilbert space is called selfadjoint if $A^* = A$.*

The equality of operators includes the equality of their domains of definition. So for unbounded operators the selfadjointness is stronger than symmetry.

10.2.2. Definition. *An operator A with a dense domain of definition $\mathfrak{D}(A)$ in a Hilbert space is called symmetric if*

$$(Ax, y) = (x, Ay) \quad \textit{for all } x, y \in \mathfrak{D}(A).$$

We have seen in Example 6.8.8 that an *everywhere* defined symmetric operator on a Hilbert space is necessarily continuous. Hence an unbounded symmetric operator cannot be extended to the whole space as a symmetric operator. The following example of the differentiation operator is classical in the theory of operators in many respects. In particular, it gives a simple example of a symmetric operator that is not selfadjoint. Let $AC[0,1]$ be the class of all absolutely continuous functions on $[0,1]$.

10.2.3. Example. (The differentiation operator) (i) Let $H = L^2[0,1]$. On the set $\mathfrak{D}(A) = C_0^\infty(0,1)$ of all infinitely differentiable complex functions with support in the open interval $(0,1)$ we define an operator A by the formula

$$Au(t) = iu'(t).$$

This operator is densely defined. For all $u, v \in \mathfrak{D}(A)$ we have

$$(Au, v) = i\int_0^1 u'(t)\overline{v(t)}\,dt = \int_0^1 u(t)\overline{iv'(t)}\,dt = (u, Av)$$

by the integration by parts formula. Thus, the operator A is symmetric. However, it is not selfadjoint, since it is not closed. Indeed, let u be a continuously differentiable, but not infinitely differentiable function on $[0,1]$ with $u(0) = u(1) = 0$. It is readily seen that there exists a sequence of functions $u_n \in C_0^\infty(0,1)$ such that $u_n \to u$ and $u_n' \to u'$ in $L^2[0,1]$.

(ii) We show that

$$\mathfrak{D}(A^*) = \bigl\{u \in AC[0,1]\colon\ u' \in L^2[0,1]\bigr\}, \quad A^*u = iu'.$$

The fact that every absolutely continuous function u with $u' \in L^2[0,1]$ belongs to $\mathfrak{D}(A^*)$ and that $A^*u = iu'$ is obvious from the integration by parts formula above with $v \in C_0^\infty(0,1)$. This also shows that $\mathfrak{D}(A^*)$ does not coincide with $\mathfrak{D}(A)$. Let $u \in \mathfrak{D}(A^*)$, which means that there exists a function $w \in L^2[0,1]$ such that for all $v \in C_0^\infty(0,1)$ we have

$$\int_0^1 w(t)\overline{v(t)}\,dt = (w, v) = (A^*u, v) = (u, Av) = \int_0^1 u(t)\overline{iv'(t)}\,dt.$$

Set $u_0(t) = -i \int_0^t w(s)\,ds$. For real $v \in C_0^\infty(0,1)$, the integration by parts formula and the equality above give

$$\int_0^1 u_0(t)v'(t)\,dt = i\int_0^1 w(t)v(t)\,dt = \int_0^1 u(t)v'(t)\,dt.$$

For every function h from $L^2[0,1]$ with zero integral we can find a sequence of functions $v_n \in C_0^\infty(0,1)$ such that $v_n' \to h$ in $L^2[0,1]$. Hence the function $u - u_0$ is orthogonal to all such elements h. Hence it equals a constant almost everywhere (to its integral). Thus, along with u_0, the function u is also absolutely continuous and has a square integrable derivative. Finally, it is clear that $A^*u = iu'$. Note that the operator A^* is not symmetric.

(iii) Let us now describe the closure of A. We show that

$$\mathfrak{D}(A^{**}) = \bigl\{u \in AC[0,1]\colon\ u' \in L^2[0,1], u(0) = u(1) = 0\bigr\},\ A^{**}u = iu'.$$

The indicated domain belongs to $\mathfrak{D}(A^{**})$ by the integration by parts formula. As above, the main thing is to prove the inverse inclusion. Let $u \in \mathfrak{D}(A^{**})$. It follows from what has been proved that u is an absolutely continuous function, $u' \in L^2[0,1]$ and $A^{**}u = iu'$. Now, however, at our disposal we have all smooth functions v on $[0,1]$, not only having support in $(0,1)$. Hence the equality

$$\int_0^1 iu'(t)v(t)\,dt = (A^{**}u, v) = (u, A^*v) = -i\int_0^1 u(t)v'(t)\,dt$$

for smooth real functions v gives a new relation $u(1)v(1) = u(0)v(0)$, whence $u(0) = u(1) = 0$. The operator A^{**} is closed as any adjoint. By assertion (ii) in Proposition 10.1.14 the operator A^{**} coincides with the closure of A. In particular, A^{**} is a symmetric closed, but not selfadjoint operator, since $(A^{**})^* = \overline{A}^* = A^*$ differs from A^{**}.

(iv) The operator A has selfadjoint extensions (which are also extensions of $\overline{A}$), moreover, all these extensions have the form A_θ, where

$$\mathfrak{D}(A_\theta) = \bigl\{u \in AC[0,1]\colon\ u' \in L^2[0,1], u(1) = \theta u(0)\bigr\},\ |\theta| = 1,$$
$$A_\theta u = iu'.$$

Indeed, if $\widetilde{A}$ is a selfadjoint extension of A, then $\widetilde{A} = \widetilde{A}^* \subset A^*$, i.e., we have $\mathfrak{D}(\widetilde{A}) \subset \mathfrak{D}(A^*)$ and $\widetilde{A}u = iu'$. The condition that $\widetilde{A}$ is symmetric leads to the requirement $u(0)\overline{v(0)} = u(1)\overline{v(1)}$ for all $u, v \in \mathfrak{D}(\widetilde{A})$. Since $\overline{A}$ is not selfadjoint, we see that $\mathfrak{D}(\widetilde{A})$ is larger than $\mathfrak{D}(\overline{A})$, which means that there exists a function $u \in \mathfrak{D}(\widetilde{A})$ for the continuous version of which we have $u(0) \neq 0$. We can assume that $u(0) = 1$. Then we set $\theta = \overline{u(1)}$ and for $v = u$ obtain $|\theta|^2 = 1$. In addition, for all $v \in \mathfrak{D}(\widetilde{A})$ we obtain $\overline{v(0)} = \overline{\theta v(1)}$, i.e., $v(0) = \theta v(1)$. Conversely, the operator A_θ on $\mathfrak{D}(A_\theta)$ is symmetric and extends A. Hence $A_\theta^* \subset A^*$, i.e., for all $u \in \mathfrak{D}(A_\theta^*)$ we have $u \in AC[0,1]$ and $u' \in L^2[0,1]$. Hence for all $u \in \mathfrak{D}(A_\theta^*)$, $v \in \mathfrak{D}(A_\theta)$ we obtain

$$\int_0^1 iu'(t)\overline{v(t)}\,dt = (A_\theta^* u, v) = (u, A_\theta v) = \int_0^1 u(t)\overline{iv'(t)}\,dt,$$

which leads to the condition $u(0)\overline{\theta v(1)} = u(1)\overline{v(1)}$. Since we can take v with $v(1) = 1$, the equality $u(0) = \theta u(1)$ holds. Thus, we obtain $\mathfrak{D}(A_\theta^*) = \mathfrak{D}(A_\theta)$ and the operator A_θ is selfadjoint.

The next example is also important for the general theory. We shall establish below that every selfadjoint operator on a separable Hilbert space is unitarily equivalent to an operator of such a form.

10.2.4. Example. Let μ be a finite nonnegative measure on a measurable space $(\Omega, \mathcal{B})$ and let φ be a real μ-measurable function (not necessarily bounded). Let us define the operator A_φ (the same notation as in case of bounded functions) on the domain

$$\mathfrak{D}(A_\varphi) = \{x \in L^2(\mu)\colon\ \varphi \cdot x \in L^2(\mu)\}$$

by the formula $A_\varphi x(\omega) = \varphi(\omega)x(\omega)$. Then the operator A_φ is selfadjoint. Indeed, the set $\mathfrak{D}(A_\varphi)$ is dense in $L^2(\mu)$, since for every $x \in L^2(\mu)$ it contains all functions xI_{Ω_n}, where $\Omega_n := \bigl\{\omega\colon\ |\varphi(\omega)| \leqslant n\bigr\}$. The operator A_φ is symmetric on this domain. Let $u \in \mathfrak{D}(A_\varphi^*)$. Then there is $w \in L^2(\mu)$ with

$$\int_\Omega u(\omega)\varphi(\omega)v(\omega)\,\mu(d\omega) = \int_\Omega w(\omega)v(\omega)\,\mu(d\omega)$$

for all real functions $v \in \mathfrak{D}(A_\varphi)$. Taking for v indicator functions of measurable subsets of the sets Ω_n, we obtain for μ-a.e. ω that $w(\omega) = \varphi(\omega)u(\omega)$. Thus, $\mathfrak{D}(A_\varphi^*) = \mathfrak{D}(A_\varphi)$ and the operator A_φ is selfadjoint.

10.2.5. Example. Let H be a Hilbert space and let Π be a projection-valued measure on $\mathcal{B}(\mathbb{R}^1)$ with values in $\mathcal{L}(H)$. Set

$$\mathfrak{D}(A) := \left\{x \in H\colon\ \int_{\mathbb{R}^1} \lambda^2\, d\Pi_{x,x}(\lambda) < \infty\right\}, \quad Ax := \int_{\mathbb{R}^1} \lambda\, d\Pi_x(\lambda),$$

where the integral is understood as the limit in H of the integrals over the intervals $[-n, n]$ (which exist, as we have seen in §7.9). The operator A is selfadjoint. Indeed, it is easy to see that $\mathfrak{D}(A)$ is a linear space. It is dense, since for every $x \in H$ the vectors $x_n := \Pi([-n, n])x$ converging to x belong to $\mathfrak{D}(A)$. In addition,

$$Ax_n = A_n x = \int_{[-n,n]} \lambda\, d\Pi_x(\lambda),$$

where the bounded selfadjoint operator A_n is defined by restricting the projection-valued measure Π to $[-n, n]$ (see §7.9). If $k > n$, we have

$$\|A_n x - A_k x\|^2 = \bigl((A_n - A_k)^2 x, x\bigr) = \int_{n<|\lambda|\leqslant k} \lambda^2\, d\Pi_{x,x}(\lambda),$$

whence it follows that $\{Ax_n\}$ is Cauchy in norm and there is $Ax = \lim\limits_{n\to\infty} A_n x$. We show that $A^* = A$. Let $y \in \mathfrak{D}(A^*)$. Hence $|(Ax, y)| \leqslant \|A^*y\|\,\|x\|$ for all $x \in \mathfrak{D}(A)$. Therefore, $y \in \mathfrak{D}(A)$, since otherwise $c_n := \|A_n y\| \to \infty$, which is impossible by the relation

$$\|A_n y\| = c_n^{-1}(A_n y, A_n y) = c_n^{-1}(A_n A_n y, y) = c_n^{-1}(AA_n y, y) \leqslant \|A^* y\|$$

following from the estimate $\|c_n^{-1}A_n y\| \leqslant 1$. Since the operator A is symmetric and $\mathfrak{D}(A^*) = \mathfrak{D}(A)$, we have $A^* = A$.

10.3. The Spectral Theorem

Here we shall see that unbounded selfadjoint operators are also unitarily isomorphic to operators of multiplication by functions, i.e., Example 10.2.4 is universal. Unitary equivalence of unbounded selfadjoint operators is defined naturally (but now it is also required that isomorphisms must interchange the domains). The Caley transform also extends to such operators.

10.3.1. Lemma. *Let A be a selfadjoint operator on a Hilbert space $H \neq 0$. Then the operators $A + iI$ and $A - iI$ are injective and their ranges coincide with H. In addition, the operator $U = (A - iI)(A + iI)^{-1}$ is unitary.*

PROOF. We have $\mathrm{Ker}(A - iI) = \mathrm{Ker}(A - iI) = 0$, since if $Ax = ix$, then $(Ax, x) = i(x, x)$ and $x = 0$, because $(Ax, x) \in \mathbb{R}^1$. There is a similar equality for $A+iI$. The range of $A+iI$ is dense, since if $y \in H$ is such that $(Ax+ix, y) = 0$ for all $x \in \mathfrak{D}(A)$, then $y \in \mathfrak{D}(A^*) = \mathfrak{D}(A)$, whence $(x, Ay-iy) = 0$, so $Ay = iy$ and $y = 0$. We observe that

$$\|Ax + ix\|^2 = \|Ax\|^2 + \|x\|^2 = \|Ax - ix\|^2 \quad \forall\, x \in \mathfrak{D}(A),$$

because $(Ax, ix) = -i(Ax, x) = -i(x, Ax)$. Hence there exists a linear isometry U between the everywhere dense linear subspaces $(A + iI)(H)$ and $(A - iI)(H)$ defined by the formula

$$U(Ax + ix) = Ax - ix, \quad x \in \mathfrak{D}(A).$$

By the injectivity of $A + iI$ the operator U on $(A + iI)(H)$ can be written in the form $U = (A - iI)(A + iI)^{-1}$. This operator uniquely extends to a unitary operator, also denoted by U. We show that $(A+iI)\bigl(\mathfrak{D}(A)\bigr) = H$. Let $y \in H$. Find $x_n \in \mathfrak{D}(H)$ for which $Ax_n + ix_n \to y$. Then $Ax_n - ix_n = U(Ax_n + ix_n) \to Uy$, which gives convergence of $\{x_n\}$ to some $x \in H$. Since A is closed, we obtain $x \in \mathfrak{D}(A)$ and $Ax = y$. Similarly, $(A - iI)\bigl(\mathfrak{D}(A)\bigr) = H$. □

The unitary operator U from this lemma is called the *Caley transform* of the operator A.

10.3.2. Lemma. *Suppose that selfadjoint operators A and B on H possess equal Caley transforms. Then $A = B$. In addition, if U the Caley transform of A, then the operator $U - I$ is injective and*

$$\mathfrak{D}(A) = (U - I)(H), \quad Ax = i(I + U)(I - U)^{-1}x.$$

PROOF. Let $y \in H$. As shown above, there is $x \in \mathfrak{D}(A)$ with $y = Ax + ix$. Then $Uy = Ax - ix$ and $y - Uy = 2ix$. Hence $\mathrm{Ker}(U - I) = 0$. In addition, $(U - I)(H) = \mathfrak{D}(A)$, since for every $x \in \mathfrak{D}(A)$ we can take $y = Ax + ix$. Therefore, if U is the common Caley transform of the operators A and B, then $\mathfrak{D}(A) = \mathfrak{D}(B)$ and for every x from this common domain we obtain

$$Ax = i(I + U)(I - U)^{-1}x = Bx,$$

which completes the proof. □

10.3.3. Theorem. *Every selfadjoint operator A on a separable Hilbert space $H \neq 0$ is unitarily equivalent to some operator A_φ of multiplication by a function φ from Example* 10.2.4 *with some probability measure μ.*

PROOF. Let U be the Caley transform of the operator A. According to Corollary 7.10.9 we can assume that U is the operator of multiplication on $L^2(\mu)$ by a μ-measurable function ψ with $|\psi| = 1$, where μ is some probability measure on a space Ω. Since U is the Caley transform of A, the operator $U - I$ is injective, i.e., $\psi(\omega) \neq 1$ for μ-a.e. ω. Set $\varphi = i(1+\psi)(1-\psi)^{-1}$. It is straightforward to verify that the Caley transform of the operator A_φ is the operator $A_\psi = U$. By the previous lemma $A = A_\varphi$. $\square$

For μ one can take some Borel probability measure on the real line and for φ some Borel function. By using Remark 7.8.7 this theorem can be extended to nonseparable spaces. There is an analog of Theorem 7.10.11 in which the commutativity of unbounded operators means the commutativity of their projection measures.

Similarly to the case of bounded operators, by means of a functional model it is easy to define Borel functions of selfadjoint operators and construct projection-valued measures. Let A be a selfadjoint operator on a separable Hilbert space and let f be a complex Borel function on the real line. Let us define the operator $f(A)$ as follows: we represent A in the form of multiplication by a μ-measurable real function φ and set

$$\mathfrak{D}\bigl(f(A)\bigr) := \{x \in L^2(\mu)\colon\ (f\circ\varphi)\cdot x \in L^2(\mu)\}, \quad f(A)x := (f\circ\varphi)\cdot x.$$

If the function f is real, then the operator $f(A)$ is selfadjoint. If the function f is bounded, then the operator $f(A)$ is bounded as well. If f is the indicator function of a Borel set B, then $\Pi(B) := I_B(A)$ is the orthogonal projection and the mapping $B \mapsto \Pi(B)$ is a projection-valued measure. This measure generates complex scalar measures $\Pi_{x,y}(B) = \bigl(\Pi(B), x, y\bigr)$. Similarly to Theorem 7.9.6 the following relations hold; they can be obtained as a corollary of the cited theorem and Example 10.2.5 if we represent A as the direct sum of the countable collection of bounded operators of multiplication by the functions $\varphi I_{\{k\leqslant\varphi<k+1\}}$.

10.3.4. Theorem. *There holds the equality*

$$A = \int_{\sigma(A)} \lambda\, d\Pi(\lambda), \quad \textit{where } \mathfrak{D}(A) = \biggl\{x\colon \int_{\sigma(A)} \lambda^2\, d\Pi_{x,x}(\lambda) < \infty\biggr\},$$

understood as the identity

$$(Ax, y) = \int_{\sigma(A)} \lambda\, \Pi_{x,y}(d\lambda), \quad x \in \mathfrak{D}(A), y \in H.$$

In addition, for every Borel function f we have the equality

$$f(A) = \int_{\sigma(A)} f(\lambda)\, d\Pi(\lambda),$$

understood similarly.

10.4. Unitary Invariants of Selfadjoint Operators

It was already noted in Chapter 7 that the representations of selfadjoint operators constructed there do not provide any way to establish the equivalence or non-equivalence of two such operators. Of course, in some cases non-equivalence can be seen directly from characteristics such as the spectrum or existence and absence of cyclic vectors. But even for operators with cyclic vectors the equality of spectra does not imply the equivalence of the operators. In the finite-dimensional case the eigenvalues and their multiplicities determine operators up to unitary equivalence. What can serve as an analog of this in the infinite-dimensional case? It is clear that we should somehow distinguish measures used in our representations. The first step in this direction is the following result.

10.4.1. Theorem. *Let μ and ν be two bounded nonnegative Borel measures on the real line and let A_μ and A_ν be the operators of multiplication by the argument on $L^2(\mu)$ and $L^2(\nu)$, respectively. These operators are unitarily equivalent precisely when the measures μ and ν are equivalent.*

PROOF. Let μ and ν be equivalent and $\varrho = d\nu/d\mu$. We recall (see Chapter 3) that equivalent measures possess equal supplies of measurable functions and a measurable function φ is integrable with respect to ν precisely when the function $\varphi\varrho$ is integrable with respect to μ. In addition,

$$\int \varphi(t)\,\nu(dt) = \int \varphi(t)\varrho(t)\,\mu(dt).$$

Let us define an operator $U\colon\ L^2(\nu) \to L^2(\mu)$ by the equality

$$Uf(t) = \sqrt{\varrho(t)}f(t).$$

Then

$$\|Uf\|^2_{L^2(\mu)} = \int |f(t)|^2\varrho(t)\,\mu(dt) = \int |f(t)|^2\,\nu(dt) = \|f\|^2_{L^2(\nu)},$$

i.e., U is an isometry. The operator U is surjective, since for every function $g \in L^2(\mu)$ the function $\varrho^{-1/2}g$ belongs to $L^2(\nu)$. Finally,

$$A_\mu Uf(t) = t\sqrt{\varrho(t)}f(t) = \sqrt{\varrho(t)}tf(t) = UA_\nu f(t).$$

Let A_μ and A_ν be unitarily equivalent. Let $U\colon L^2(\nu)\to L^2(\mu)$ be their isometry. We can assume that μ and ν have supports in some $S = [-n, n]$, since $g(A_\mu)$ and $g(A_\nu)$ for $g(t) = tI_S(t)$ are also equivalent. Set $\psi := U1$ and $p_k(t) := t^k$. Then

$$Up_k(t) = UA_\nu p_{k-1}(t) = tUp_{k-1}(t),$$

which gives the equality $Up_k(t) = t^k\psi(t)$. For every polynomial f we obtain $Uf(t) = f(t)\psi(t)$. Since U is unitary, we have

$$\int_S |f(t)|^2\,\nu(dt) = \int_S |\psi(t)|^2|f(t)|^2\,\mu(dt).$$

Let now B be a Borel set in S. Taking the measure $\mu + \nu$, we find a uniformly bounded sequence of polynomials f_n such that $f_n(t) \to I_B(t)$ for almost all t with respect to both measures. By the Lebesgue dominated convergence theorem

$$\nu(B) = \int_B |\psi(t)|^2 \, \mu(dt),$$

which means that $\nu \ll \mu$ and $d\nu/d\mu = |\psi|^2$. By the symmetry of the roles of both measures they are equivalent. □

In the case of decompositions into subspaces with cyclic vectors the situation becomes more complicated. For example, the operator of multiplication by the argument on $L^2[0,1]$ can be decomposed into the sum of operators of multiplication by the argument on the intervals $[0,1/2]$ and $(1/2,1]$. How can we avoid such redundant terms? On the other hand, one should somehow take into account the multiplicities of several copies of the operator of multiplication by the argument. In the finite-dimensional case this reduces to counting the multiplicity of every eigenvalue, but only different eigenvalues are taken (for example, an eigenvalue of multiplicity 2 is not considered as two eigenvalues of multiplicity 1). Moreover, in case of multiplicity 1 the corresponding subspace is one-dimensional and cannot be further decomposed. In the infinite-dimensional case an analog of an operator with a simple spectrum is an operator with a cyclic vector, but, as the example above shows, such an operator can fail to have a "minimal" canonical decomposition. For this reason for obtaining unitarily invariant representations in the infinite-dimensional space one has to reject the finite-dimensional picture $\mathbb{C}^n = H = H_{\lambda_1} \oplus H_{\lambda_2} \oplus \cdots \oplus H_{\lambda_k}$, where $\lambda_1, \ldots, \lambda_k$ are all distinct eigenvalues of A and H_{λ_i} are the corresponding n_i-dimensional kernel subspaces. To this picture there corresponds the Jordan form

$$\begin{pmatrix} \lambda_1 \mathrm{I}_{n_1} & & \\ & \ddots & \\ & & \lambda_k \mathrm{I}_{n_k} \end{pmatrix}.$$

Here $\lambda_j \mathrm{I}_{n_j}$ denotes the block consisting of the n_j-dimensional unit matrix multiplied by λ_j. We consider instead another picture obtained by some rearrangement of the previous one. In this new picture eigenvalues are ordered according to increasing of their multiplicities: $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$. Suppose that there are N different multiplicities $m_1 < m_2 < \cdots < m_N$. Then we create the following blocks:

$$B_1 = \begin{pmatrix} A_1 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_1 \end{pmatrix}, \quad \text{where} \quad A_1 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_l \end{pmatrix},$$

λ_l is the last number of multiplicity m_1 and A_1 is taken m_1 times, next,

$$B_2 = \begin{pmatrix} A_2 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_2 \end{pmatrix}, \quad \text{where} \quad A_2 = \begin{pmatrix} \lambda_{l+1} & & & \\ & \lambda_{l+2} & & \\ & & \ddots & \\ & & & \lambda_p \end{pmatrix},$$

here the block A_2 is taken m_2 times (it employs the eigenvalues λ_j of multiplicity m_2, i.e., λ_2 appears in it only in the case where $n_2 > n_1$), and further blocks $A_3, \ldots, A_N$ of increasing multiplicity appear. This procedure represents the operator A as the direct sum of N blocks $B_1, \ldots, B_N$, where every B_j is the direct sum of m_j copies of the operator A_j with a simple spectrum (the spectra of A_j are disjoint). Thus, the algorithm is this: we take the eigenvalue of the minimal multiplicity m_1 and separated the part of the standard Jordan form constituted by the blocks $\lambda_1 \mathrm{I}_{m_1}, \ldots, \lambda_l \mathrm{I}_{m_1}$, which excludes the eigenvalue of the minimal multiplicity (or several eigenvalues of the same minimal multiplicity). Next we repeat the same with the remaining part.

10.4.2. Definition. *A selfadjoint operator is called an operator of homogeneous multiplicity m, where $m \in \mathbb{N} \cup \{\infty\}$, if it is unitarily equivalent to the direct sum of m copies of the operator of multiplication by the argument on the space $L^2(\mu)$ for some nonnegative σ-finite Borel measure μ on the real line.*

It is clear that bounded operators of homogeneous multiplicity correspond to measures with bounded supports.

In the finite-dimensional example described above the operators B_j are operators of homogeneous multiplicity. It turns out that such a representation already has a reasonable infinite-dimensional analog. In the final classification we use the notions of equivalence and mutual singularity of measures. We recall that two measures μ and ν are equivalent ($\mu \sim \nu$) if they are given by densities with respect to each other; the measures μ and ν are mutually singular ($\mu \perp \nu$) if they are concentrated on disjoint sets. These notions extend to classes of measures: two classes of measures M and L are equivalent if $\mu \sim \lambda$ for all $\mu \in M$ and $\lambda \in L$; two classes of measures M and L are mutually singular if $\mu \perp \lambda$ for all $\mu \in M$ and $\lambda \in L$. Let $\langle \mu \rangle$ be the equivalence class of μ.

We first show that for operators of homogeneous multiplicity m a full unitary invariant is the equivalence type of the measure.

10.4.3. Theorem. (i) *Let μ be a nonnegative σ-finite Borel measure on the real line that is not identically zero. If the direct sum of m copies of the operator of multiplication by the argument on the space $L^2(\mu)$ is unitarily equivalent to the direct sum of n copies of this operator, where $m, n \in [1, \ldots, \infty]$, then $m = n$.*

(ii) *Two operators of homogeneous multiplicities m and n are unitarily equivalent precisely when $m = n$ and the measures generating these operators are equivalent.*

PROOF. (i) Let $U\colon \bigl(L^2(\mu)\bigr)^m \to \bigl(L^2(\mu)\bigr)^n$ be a unitary isomorphism of the regarded operators A_n and A_m and let $n < m \leqslant \infty$. The isomorphism of operators gives the equality

$$I_B(A_n)U = UI_B(A_m)$$

for all Borel sets B. The projections $I_B(A_n)$ and $I_B(A_m)$ are operators of multiplication by I_B. Let us take a Borel set E with $0 < \mu(E) < \infty$ and consider $n+1$ vectors $\varphi_1, \ldots, \varphi_{n+1}$ from the direct sum of m copies of $L^2(\mu)$ defined as follows: the component with the number j of the element φ_j is I_E and the remaining components are zero. Set

$$\psi_j := U\varphi_j = (\psi_{j,1}, \ldots, \psi_{j,n}).$$

For every Borel set $B \subset E$ we have

$$\big(I_B(A_m)\varphi_i, \varphi_j\big) = \big(I_B(A_n)\psi_i, \psi_j\big) = \int_B \sum_{k=1}^n \psi_{i,k}(t)\overline{\psi_{j,k}(t)}\,\mu(dt).$$

The left-hand side equals $\mu(B)\delta_{ij}$. Since this is true for all $B \subset E$, for any fixed i, j we obtain

$$\sum_{k=1}^n \psi_{i,k}(t)\overline{\psi_{j,k}(t)} = \delta_{ij} \quad \text{for } \mu\text{-a.e. } t \in E.$$

Then this equality is true μ-almost everywhere on E for all $i, j = 1, \ldots, n+1$ at once. Hence there exists a point t at which the indicated equality is fulfilled for all $i, j = 1, \ldots, n+1$. This leads to a contradiction, since gives $n+1$ linearly independent vectors in an n-dimensional space.

(ii) Suppose that the operators $A_{\mu,n}$ and $A_{\nu,m}$ of homogeneous multiplicities n and m, generated by the measures μ and ν, are unitarily equivalent. Then $\mu \sim \nu$, since the projections $I_B(A_{\mu,n})$ and $I_B(A_{\nu,m})$ are also unitarily equivalent and $I_B(A_{\mu,n}) = 0$ precisely when $\mu(B) = 0$, and similarly for the second operator. According to the previous theorem the operators $A_{\mu,n}$ and $A_{\nu,n}$ are equivalent. Assertion (i) gives the equality $n = m$. □

Let A be a selfadjoint operator on a separable Hilbert space H. In addition to the idea to consider operators of homogeneous multiplicity, the structural spectral theorem employs some analysis of the measures μ_x, $x \in H$, defined by

$$\mu_x(B) = \Pi_{x,x}(B) = \big(\Pi(B)x, x\big),$$

where Π is the projection-valued measure generated by the operator A.

We shall say that $x \in H$ is a *vector of maximal type* in H if $\mu_y \ll \mu_x$ for all $y \in H$. In a natural way we define vectors of maximal type in any subspace $H' \subset H$.

Given x, let H_x be the closure of the linear span of all vectors $\Pi(B)x$, where $B \in \mathcal{B}(\mathrm{I\!R}^1)$. For a bounded operator A, the subspace H_x equals the closure of the linear span of the sequence $\{x, Ax, A^2x, \ldots\}$; see Exercise 10.7.35.

10.4.4. Lemma. (i) *If $y \in H_x$, then $\mu_y \ll \mu_x$.*

(ii) *If $\mu_x \perp \mu_y$, then $H_x \perp H_y$ and $\mu_{x+y} = \mu_x + \mu_y$.*

(iii) *There are vectors of maximal type. Moreover, for every $v \in H$, there exists a vector x of maximal type such that $v \in H_x$.*

PROOF. (i) Let B be a Borel set such that $\mu_x(B) = 0$, i.e.,

$$\big(\Pi(B)x, \Pi(B)x\big) = \big(\Pi(B)x, x\big) = 0.$$

Then $\Pi(B)z = 0$ for every vector z of the form $z = \Pi(E)x$, where $E \in \mathcal{B}(\mathrm{I\!R}^1)$, since $\Pi(B)\Pi(E)x = \Pi(E)\Pi(B)x$. The equality $\Pi(B)z = 0$ remains valid for all vectors z from the closure of the linear span of the vectors $\Pi(E)x$, i.e., for all $z \in H_x$. Thus, $\Pi(B)y = 0$ and $\mu_y(B) = \big(\Pi(B)y, y\big) = 0$ for all $y \in H_x$. Since $\big(\Pi(B)x, y\big) = 0$, we have $\mu_{x+y} = \mu_x + \mu_y$.

(ii) There are Borel sets S_1 and S_2 such that
$$S_1 \cap S_2 = \varnothing, \quad S_1 \cup S_2 = \mathbb{R}^1, \quad \mu_x(S_2) = 0, \quad \mu_y(S_1) = 0.$$
For every set $B \in \mathcal{B}(\mathbb{R}^1)$ we have $\Pi(B \cap S_2)x = \Pi(B \cap S_1)y = 0$, since $\Pi(B \cap S_2) \leqslant \Pi(S_2)$ and $\|\Pi(S_2)x\| = 0$, and similarly for $\Pi(B \cap S_1)y$. Then for any $B_1, B_2 \in \mathcal{B}(\mathbb{R}^1)$ we obtain
$$\Pi(B_1)x = \Pi(B_1 \cap S_1)x + \Pi(B_1 \cap S_2)x = \Pi(B_1 \cap S_1)x \in \Pi(S_1)(H)$$
and $\Pi(B_2)y = \Pi(B_2 \cap S_2)y \in \Pi(S_2)(H)$. Since $S_1 \cap S_2 = \varnothing$, we have $\Pi(S_1)(H) \perp \Pi(S_2)(H)$ (by the properties of projection-valued measure).

(iii) It suffices to consider a unit vector v. Let us take the set $\mathfrak{M}$ the elements of which are all possible families of measures μ_x with $x \neq 0$ that are pairwise mutually singular (within every family) and are also mutually singular with μ_v. It is partially ordered by inclusion. Zorn's lemma gives a maximal family, which by the separability of H consists of some countable collection of measures μ_{v_n} (for every family, the corresponding subspaces H_x are mutually orthogonal by assertion (ii)). For the required vector we take $x = v_0 + \sum_{n=1}^{\infty} c_n v_n$, where $v_0 = v$, $c_n := n^{-2}\|v_n\|^{-1}$. Since the measures μ_{v_n} and μ_v are mutually singular, there exist pairwise disjoint Borel sets B_n, $n = 0, 1, \ldots$, such that each measure μ_{v_n} is concentrated on B_n. Indeed, for every pair of different numbers n, k we can find Borel sets $B_{n,k}$ with $\mu_{v_k}(B_{n,k}) = \mu_{v_n}(\mathbb{R}^1 \backslash B_{n,k}) = 0$ and then take $B_n := \bigcap_{k \neq n} B_{n,k}$. This proves the inclusion $v_0 = \Pi(B_0)v_0 = \Pi(B_0)x \in H_x$. Let us verify that x is a vector of maximal type. If this is not true, then there exists a vector $u \in H$ such that the measure μ_u is not absolutely continuous with respect to μ_x. This means that for some Borel set B we have $\mu_x(B) = 0$ and $\mu_u(B) > 0$. Set $z := \Pi(B)u$. Then $\mu_z(B) = \mu_u(B) > 0$ and
$$\mu_z(\mathbb{R}^1 \backslash B) = \bigl(\Pi(\mathbb{R}^1 \backslash B)\Pi(B)u, z\bigr) = 0,$$
since $\Pi(\mathbb{R}^1 \backslash B)\Pi(B) = 0$. The equality $\mu_x = \mu_{v_0} + \sum_{n=1}^{\infty} c_n^2 \mu_{v_n}$ yields that $\mu_z \perp \mu_{v_n}$ for all $n \geqslant 0$ contrary to the maximality of the regarded family. □

10.4.5. Lemma. *Let $x \in H$ and let the measure μ_x be written as the sum $\mu_x = \sum_{n=1}^{\infty} \mu_n$ of pairwise mutually singular nonnegative Borel measures. Then there exist pairwise orthogonal vectors $x_n \in H_x$ such that*
$$x = \sum_{n=1}^{\infty} x_n, \quad \mu_{x_n} = \mu_n, \quad H_x = \bigoplus_{n=1}^{\infty} H_{x_n}.$$
If a finite measure $\nu \geqslant 0$ satisfies the condition $\nu \ll \Pi_{x,x}$, then $\nu = \Pi_{y,y}$ for some vector $y \in H_x$.

PROOF. As in the previous lemma, we split the real line into pairwise disjoint Borel sets B_n such that μ_n is concentrated on B_n. Let $x_n := \Pi(B_n)x$. Since the measure μ is concentrated on the union of B_n, we have
$$x = \Pi\Bigl(\bigcup_{n=1}^{\infty} B_n\Bigr)x = \sum_{n=1}^{\infty} x_n.$$
The vectors x_n are pairwise orthogonal by the disjointness of the sets B_n. For the same reason for every n the measure μ_{x_n} is concentrated on the set B_n. In

addition, $\mu_{x_n} = \mu_n$, since for every set $B \in \mathcal{B}(\mathbb{R}^1)$ by the equality

$$\Pi(B \cap B_n) = \Pi(B)\Pi(B_n) = \Pi(B_n)\Pi(B)$$

we have

$$\mu_x(B) = \bigl(\Pi(B)x, x\bigr) = \sum_{n=1}^{\infty} \bigl(\Pi(B \cap B_n)x, x\bigr) = \sum_{n=1}^{\infty} \bigl(\Pi(B)\Pi(B_n)x, x\bigr)$$
$$= \sum_{n=1}^{\infty} \bigl(\Pi(B)\Pi(B_n)x, \Pi(B_n)x\bigr) = \sum_{n=1}^{\infty} \mu_{x_n}(B).$$

Further, by the previous lemma $H_{x_n} \perp H_{x_k}$ if $n \neq k$. For every $B \in \mathcal{B}(\mathbb{R}^1)$ we have $\Pi(B)x_n = \Pi(B)\Pi(B_n)x = P(B \cap B_n)x \in H_x$, i.e., $H_{x_n} \subset H_x$. On the other hand, $\Pi(B)x = \sum_{n=1}^{\infty} \Pi(B \cap B_n)x$ belongs to $\bigoplus_{n=1}^{\infty} H_n$. Thus, $\bigoplus_{n=1}^{\infty} H_n = H_x$. Finally, let $\nu \ll \Pi_{x,x}$, where $\|x\| = 1$, and let A be the selfadjoint operator on H_x generated by Π. Let us represent A as the multiplication by the argument on $L^2(\mu)$, $\mu = \Pi_{x,x}$. Then $\nu = f \cdot \mu$, $f \in L^1(\mu)$, hence we can take $y = f^{1/2} \in L^2(\mu)$. □

The following fundamental result gives a complete classification of selfadjoint operators.

10.4.6. Theorem. *For every selfadjoint operator A on a separable Hilbert space H there is a decomposition of H into the sum $H = H_1 \oplus H_2 \oplus \cdots \oplus H_\infty$ of mutually orthogonal closed subspaces H_m (some of them can be absent) with the following properties*:

(i) *$A\bigl(H_m \cap \mathfrak{D}(A)\bigr) \subset H_m$ ($A(H_m) \subset H_m$ for bounded A) and $A|_{H_m}$ is an operator of homogeneous multiplicity m for each $m \in [1, 2, \ldots, \infty]$;*

(ii) *the classes of measures $\langle \mu_m \rangle$, corresponding to the operators $A|_{H_m}$, are mutually singular for different m.*

The equivalence classes of measures μ_m give a complete collection of unitary invariants of the operator, i.e., two operators with equal collections are unitarily equivalent.

PROOF. Actually we shall find unitary invariants of the projection-valued measure of the operator and use them to obtain the desired invariants of the operator itself. We establish the existence of a decomposition of the indicated form. We first show that there exists a finite or countable set of unit vectors v_n such that

$$H = \bigoplus_n H_{v_n}, \quad \mu_{v_{n+1}} \ll \mu_{v_n}.$$

It is natural to take a unit vector v_1 of maximal type, next in the orthogonal complement $H_{v_1}^{\perp}$ of the subspace H_{v_1} take a unit vector v_2 of maximal type for this complement and so on: at the nth step we take a unit vector v_n of maximal type in the orthogonal complement to $H_{v_1} \oplus \cdots \oplus H_{v_{n-1}}$. We observe that $H_{v_2} \perp H_{v_1}$, since for any sets $B, C \in \mathcal{B}(\mathbb{R}^1)$ we have

$$\bigl(\Pi(B)v_1, \Pi(C)v_2\bigr) = \bigl(\Pi(C)\Pi(B)v_1, v_2\bigr) = \bigl(\Pi(C \cap B)v_1, v_2\bigr) = 0,$$

because $\Pi(C \cap B)v_1 \in H_{v_1}$. Hence we have mutually orthogonal subspaces H_{v_n} in a finite or countable number (for example, if v_1 is cyclic, then $H_{v_1} = H$).

We would like to obtain the whole space H as the sum of H_{v_n}, but this is not always true. For example, if A is the identity operator and $\{e_n\}$ is an orthonormal basis in H, then as the result of our construction we can obtain e_1, e_3, e_5 and so on. In order to avoid this unpleasant thing, we have to slightly modify the construction. Taking a basis $\{e_n\}$, we pick v_1 such that $e_1 \in H_{v_1}$, which is possible by the lemma. If e_2 has a nonzero projection φ_2 on $H_{v_1}^{\perp}$, then we pick v_2 such that $\varphi_2 \in H_{v_2}$. Hence $e_1, e_2 \in H_{v_1} \oplus H_{v_2}$. At the nth step we pick $v_n \in H_{v_1} \oplus \cdots \oplus H_{v_{n-1}}$ such that H_{v_n} contains the projection of e_n onto the orthogonal complement of $H_{v_1} \oplus \cdots \oplus H_{v_{n-1}}$. As a result, the direct sum of H_n contains all e_n and hence coincides with H.

We now rearrange the obtained decomposition. For every n, the measure μ_{v_n} has the form $\mu_{v_n} = \varrho_n \cdot \mu_{v_1}$, where ϱ_n is a Borel measurable Radon–Nikodym density of the measure μ_{v_n} with respect to μ_{v_1}. By the relation $\mu_{v_{n+1}} \ll \mu_{v_n}$, these densities can be chosen such that the sets $S_n := \{t\colon\, \varrho_n(t) > 0\}$ decrease. This decreasing need not be strict, because S_n can coincide with S_{n+1} if $\mu_{v_{n+1}}$ and μ_{v_n} are equivalent. The restriction of the measure μ_{v_1} to $S_{n-1}\backslash S_n$, where $S_0 := \mathbb{R}^1$, will be denoted by μ_n, and the restriction of μ_{v_1} to $S_\infty := \bigcap_{n=1}^\infty S_n$ will be denoted by μ_∞. It is clear that the obtained measures are pairwise orthogonal and their sum is μ_{v_1}. For every n, the measure μ_{v_n} is equivalent to the measure $\mu_n \oplus \mu_{n+1} \oplus \cdots \oplus \mu_\infty$ (note that such sums are finite measures dominated by μ_{v_1}).

Suppose now that the operator A is bounded. Then v_n is a cyclic vector for the restriction of A to H_{v_n} and this restriction is unitarily equivalent to multiplication by the argument on L^2 with respect to the measure $\mu_n \oplus \mu_{n+1} \oplus \cdots \oplus \mu_\infty$, which can be written as the direct sum of operators of multiplication by the argument on the spaces $L^2(\mu_n), L^2(\mu_{n+1}), \ldots, L^2(\mu_\infty)$. It is clear that we have obtained the desired decomposition.

If the operator A is not bounded, then we first represent it as the direct sum of bounded operators A_k each of which is already decomposed into the sum of operators of homogeneous multiplicity n by means of measures $\mu_{k,n}$ concentrated on some bounded Borel set B_k, where B_k are pairwise disjoint. We can assume that $\mu_{k,n}(\mathbb{R}^1) \leqslant 2^{-n-k}$. For every m we take the measure $\mu_m := \sum_{k=1}^\infty \mu_{k,m}$.

Let us proceed to the proof of unitary invariance of the obtained objects: the appearing multiplicities n and measure types μ_n. It is clear from Theorem 10.4.1 that two decompositions of the indicated form with $\mu_n \sim \mu_n'$ are unitarily equivalent. Suppose that two operators A_1 and A_2 of such a form corresponding to collections of measures $\{\mu_n\}$ and $\{\nu_n\}$ are unitarily equivalent. Suppose that for some $n \in \mathbb{N} \cup \{\infty\}$ the measures μ_n and ν_n are not equivalent. We can assume that there is a Borel set B with $\mu_n(B) = 0$ and $0 < \nu_n(B) < \infty$. Pairwise orthogonal measures ν_k are concentrated on some disjoint Borel sets B_k. Then $\nu_n(B \cap B_n) > 0$ and $\nu_k(B \cap B_n) = 0$ if $k \neq n$. Hence the operator $A_2 I_{B\cap B_n}(A_2)$, which is the direct sum of n copies of the operator of multiplication by the argument on the space $L^2(\nu_n|_{B\cap B_n})$, is nonzero and of homogeneous multiplicity n. This contradicts Theorem 10.4.3, since the operators $A_1 I_{B\cap B_n}(A_1)$ and $A_2 I_{B\cap B_n}(A_2)$ are unitarily equivalent and by our construction $A_1 I_{B\cap B_n}(A_1) = 0$. $\square$

Let us explain how this theorem excludes "false" components of the type mentioned above in the decomposition of $L^2[0,1]$ into the sum $L^2[0,1/2]\oplus L^2(1/2,1]$ for the operator of multiplication by the argument. In this sum the summands are generated by mutually singular measures, but have the same multiplicity 1, which is forbidden in the theorem. The theorem can be restated in terms of the unitary equivalence of projection-valued measures of operators (see [**67**, Chapter 7]).

Note also that even in the case of operators with simple spectra the equivalence of generating measures cannot be in general seen from the equality of spectra. For example, the operator of multiplication by the argument on $L^2(\lambda)$ with Lebesgue measure λ has the same spectrum $[0,1]$ as the operator of multiplication by the argument on $L^2(\mu)$ with any singular measure μ such that the interval $[0,1]$ is the minimal closed set of measure 1 (i.e., positive on all intervals in $(0,1)$).

10.5. Operator Semigroups

Let X be a real or complex Banach space. In this section we study families of operators that in the finite-dimensional case correspond to semigroups of the form $\exp(tA)$. The main feature of the infinite-dimensional case is that for A one cannot always take a bounded operator. For this reason, as in the previous sections, we deal with operators whose domains of definition do not coincide with the whole space. Generators of semigroups are among the most typical examples of unbounded operators.

10.5.1. Definition. *A family of bounded operators $\{T_t\}_{t\geqslant 0}$ on X is called a strongly continuous operator semigroup (or a C_0-semigroup) if*

(i) $T_0 = I$, $T_{t+s} = T_t T_s$ *for all* $t, s \geqslant 0$,

(ii) $\lim\limits_{t\to 0} \|T_t x - x\| = 0$ *for all* $x \in X$.

The definition yields the continuity of all mappings

$$t \mapsto T_t x, \quad x \in X,$$

on the half-line (not only at zero), since $\|T_{t+s}x - T_t x\| \leqslant \|T_t\|\,\|T_s x - x\|$.

Set

$$\mathfrak{D}(L) := \Bigl\{x \in X\colon\ \exists \lim_{t\to 0} \frac{T_t x - x}{t}\Bigr\},$$

$$Lx := \lim_{t\to 0} \frac{T_t x - x}{t},\ x \in \mathfrak{D}(L).$$

The operator L with the indicated domain of definition $\mathfrak{D}(L)$ is called the *generator* or the *infinitesimal operator* of the semigroup $\{T_t\}_{t\geqslant 0}$.

The Banach–Steinhaus theorem yields the uniform boundedness of operators T_t with $t \in [0,1]$, i.e., $\sup_{t\in[0,1]} \|T_t\| \leqslant C < \infty$. Since $T_n = T_1^n$ and $T_t = T_{[t]}T_r$, where $[t]$ is the integer part of t and r is the fractional part of t, we arrive at the following estimate with some constant β:

$$\|T_t\| \leqslant C\|T_1\|^{[t]} \leqslant Ce^{\beta t}. \tag{10.5.1}$$

We observe that the operators $S_t = e^{-\beta t}T_t$ also form a strongly continuous semigroup, but are uniformly bounded. The generator of the semigroup $\{S_t\}_{t\geqslant 0}$ equals $L - \beta I$, where L is the generator of the semigroup $\{T_t\}_{t\geqslant 0}$.

The estimate obtained above enables us to consider the operators R_λ defined for $\operatorname{Re}\lambda > \beta$ by the formula

$$R_\lambda x = \int_0^\infty e^{-\lambda s} T_s x\, ds. \qquad (10.5.2)$$

This integral exists as a limit in norm of the integrals over $[0, N]$ as $N \to \infty$, and the integrals over the intervals exist as Riemann integrals of continuous vector-valued functions (i.e., as limits of usual Riemann sums).

10.5.2. Example. Let L be a bounded operator and

$$T_t = \exp(tL) := \sum_{n=0}^{\infty} t^n L^n / n!.$$

Then this series converges with respect to the operator norm and defines a strongly continuous semigroup the generator of which coincides with the operator L.

PROOF. Norm convergence of the series above is obvious from the estimate $\|L^n\| \leqslant \|L\|^n$. Multiplying the series for $\exp(tL)$ and $\exp(sL)$, it is readily verified that at L^m we have the coefficient $(t+s)^m/m!$, since this is true for scalar expansions. Finally, the equality $\lim\limits_{t\to 0} t^{-1}[\exp(tL) - I] = L$ follows from the equality

$$t^{-1}[\exp(tL) - I] - L = \sum_{n=2}^{\infty} t^{n-1} L^n / n!$$

and convergence of the series with the general term $|t|^n \|L\|^n / n!$. □

10.5.3. Example. Let A be a selfadjoint (possibly, unbounded) operator on a Hilbert space H. Then the unitary operators $\exp(itA)$, $t \in \mathbb{R}^1$, form a strongly continuous group with the generator iA.

PROOF. It suffices to consider the case of a separable space H. We can assume that A is the operator of multiplication on $L^2(\mu)$ by a real measurable function φ and

$$\mathfrak{D}(A) = \{x \in L^2(\mu)\colon\ \varphi \cdot x \in L^2(\mu)\}.$$

Then $\exp(itA)$ is the operator of multiplication by $\exp(it\varphi)$. It is clear that this is a strongly continuous group. Let L be its generator. Then $\mathfrak{D}(L) \subset \mathfrak{D}(A)$ and $Lx = i\varphi \cdot x$ for all $x \in \mathfrak{D}(L)$, since convergence $t^{-1}[\exp(it\varphi)x - x] \to Lx$ in $L^2(\mu)$ yields convergence $t_n^{-1}\bigl[\exp\bigl(it_n\varphi(\omega)\bigr)x(\omega) - x(\omega)\bigr] \to Lx(\omega)$ μ-a.e. for some sequence $t_n \to 0$ and the left-hand side converges to $i\varphi(\omega)x(\omega)$. On the other hand, for every $x \in \mathfrak{D}(A)$ we have convergence $t^{-1}[\exp(it\varphi)x - x] \to i\varphi x$ in $L^2(\mu)$. This follows from the Lebesgue dominated convergence theorem, since we have the pointwise convergence and the bound

$$\bigl|t^{-1}\bigl[\exp\bigl(it\varphi(\omega)\bigr)x(\omega) - x(\omega)\bigr]\bigr| \leqslant 2|\varphi(\omega)x(\omega)|.$$

Thus, $L = iA$. In particular, $\mathfrak{D}(L) = \mathfrak{D}(A)$. □

The next example is justified similarly.

10.5.4. Example. Let A be a selfadjoint (possibly, unbounded) operator on a Hilbert space H such that $A \geqslant 0$. Then the operators $\exp(-tA)$, $t \geqslant 0$, form a strongly continuous semigroup with the generator $-A$.

We shall consider below complex spaces, but all results in this section remain valid for real spaces. Let β satisfy (10.5.1).

10.5.5. Proposition. *For every λ with $\operatorname{Re}\lambda > \beta$ we have*

$$R_\lambda(X) \subset \mathfrak{D}(L), \quad (\lambda I - L)R_\lambda = I,$$

$$\lim_{\lambda>0,\ \lambda\to\infty} \lambda R_\lambda x = x \quad \textit{for all} \ \ x \in X.$$

PROOF. We observe that

$$\begin{aligned} t^{-1}(T_t - I)R_\lambda x &= t^{-1}\int_0^\infty e^{-\lambda s} T_t T_s x\, ds - t^{-1}\int_0^\infty e^{-\lambda s} T_s x\, ds \\ &= t^{-1}\int_0^\infty e^{-\lambda s} T_{t+s} x\, ds - t^{-1}\int_0^\infty e^{-\lambda s} T_s x\, ds \\ &= t^{-1}e^{\lambda t}\int_t^\infty e^{-\lambda s} T_s x\, ds - t^{-1}\int_0^\infty e^{-\lambda s} T_s x\, ds. \end{aligned}$$

The right-hand side can be written in the form

$$\begin{aligned} &t^{-1}(e^{\lambda t} - 1)\int_t^\infty e^{-\lambda s} T_s x\, ds - t^{-1}\int_0^t e^{-\lambda s} T_s x\, ds \\ &\qquad = t^{-1}(e^{\lambda t} - 1)\Bigl(R_\lambda x - \int_0^t e^{-\lambda s} T_s x\, ds\Bigr) - t^{-1}\int_0^t e^{-\lambda s} T_s x\, ds. \end{aligned}$$

By the continuity of the function $s \mapsto e^{-\lambda s} T_s x$ we conclude that, as $t \to 0+$, the obtained expression tends to $\lambda R_\lambda x - x$. Thus, we have shown that $R_\lambda x \in \mathfrak{D}(L)$ and $LR_\lambda x = \lambda R_\lambda x - x$. For the proof of the second assertion we observe that

$$\lambda R_\lambda x - x = \lambda \int_0^\infty e^{-\lambda s}(T_s x - x)\, ds.$$

In addition, whenever $\lambda > 0$, we have

$$\lambda \int_{\beta+1}^\infty e^{-\lambda s}\|T_s x - x\|\, ds \leqslant \lambda \int_{\beta+1}^\infty e^{-\lambda s}(Ce^{\beta s} + 1)\|x\|\, ds,$$

which tends to zero as $\lambda \to \infty$. Since $\|T_s x - x\| \to 0$ as $s \to 0$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\lambda \int_0^\delta e^{-\lambda s}\|T_s x - x\|\, ds \leqslant \varepsilon\lambda \int_0^\infty e^{-\lambda s}\, ds = \varepsilon$$

for all $\lambda > \beta$. Finally, the integral of $\lambda e^{-\lambda s}\|T_s x - x\|$ over $[\delta, \beta + 1]$ tends to zero as $\lambda \to \infty$ by the Lebesgue dominated convergence theorem, since $\lambda e^{-\lambda s} \to 0$ for all $s > 0$ and $\lambda e^{-\lambda s} \leqslant \lambda e^{-\delta\lambda}$. $\square$

10.5.6. Theorem. *For every strongly continuous operator semigroup* $\{T_t\}_{t\geqslant 0}$ *the following assertions hold*:

(i) *the operator* L *— the generator of the semigroup — is densely defined and closed*;

(ii) *for every* $x \in \mathfrak{D}(L)$ *we have* $T_t x \in \mathfrak{D}(L)$ *and*

$$\frac{d}{dt}T_t x = LT_t x = T_t L x;$$

(iii) *for every* $x \in X$ *we have*

$$T_t x = \lim_{\varepsilon\to 0} \exp\Bigl(t\frac{T_\varepsilon - I}{\varepsilon}\Bigr)x, \quad t \geqslant 0,$$

moreover, convergence is uniform on compact intervals in $[0, +\infty)$.

PROOF. (i) Set $A_\varepsilon := \varepsilon^{-1}(T_\varepsilon - I)$. Then

$$\exp(tA_\varepsilon) = \exp(-t/\varepsilon)\exp(t\varepsilon^{-1}T_\varepsilon) = \exp(-t/\varepsilon)\sum_{n=0}^{\infty}\frac{t^n T_{n\varepsilon}}{\varepsilon^n n!},$$

since the operators T_t are bounded. The continuity of the mapping $t \mapsto T_t x$ yields the existence of the vector integral

$$B_t x := \frac{1}{t}\int_0^t T_s x\,ds, \quad t > 0.$$

We observe that (10.5.1) implies the estimate

$$\|B_t x\| \leqslant \frac{C}{\beta t}(e^{\beta t} - 1).$$

Hence for every t the operator B_t is bounded. We show that

$$A_\varepsilon B_t = A_t B_\varepsilon, \quad \varepsilon > 0, t > 0. \tag{10.5.3}$$

Indeed, we have

$$\begin{aligned}\varepsilon t A_\varepsilon B_t x &= \int_0^t (T_\varepsilon - I)T_s x\,ds = \int_0^t (T_{s+\varepsilon} - T_s)x\,ds\\ &= \int_t^{t+\varepsilon} T_s x\,ds - \int_0^\varepsilon T_s x\,ds,\end{aligned}$$

$$\begin{aligned}\varepsilon t A_t B_\varepsilon x &= \int_0^\varepsilon (T_t - I)T_s x\,ds = \int_0^\varepsilon (T_{s+t} - T_s)x\,ds\\ &= \int_t^{t+\varepsilon} T_s x\,ds - \int_0^\varepsilon T_s x\,ds.\end{aligned}$$

Next, the continuity of $T_t x$ in t yields that $B_t x \to x$ as $t \to 0$. By (10.5.3) this gives the equality

$$\lim_{\varepsilon\to 0} A_\varepsilon B_t x = A_t \lim_{\varepsilon\to 0} B_\varepsilon x = A_t x.$$

It follows that $B_t x \in \mathfrak{D}(L)$ for all $t > 0$ and that $LB_t x = A_t x$. The set of elements of the form $B_t x$ is everywhere dense in X, since we have $\lim\limits_{t\to 0} B_t x = x$ for every $x \in X$.

We now show that the operator L is closed. Let $\{x_n\} \subset \mathfrak{D}(L)$, $x_n \to x$ and $Lx_n \to y$ in X as $n \to \infty$. Since the operators T_t and T_s commute, the operators A_ε and B_t commute as well. Hence (10.5.3) yields the equalities

$$B_t L u = B_t \lim_{\varepsilon \to 0} A_\varepsilon u = \lim_{\varepsilon \to 0} B_\varepsilon A_t u = A_t u, \quad u \in \mathfrak{D}(L), \varepsilon > 0, t > 0.$$

Therefore, $A_t x_n = B_t L x_n$, whence $A_t x = B_t y$. Since $\lim\limits_{t \to 0} B_t y = y$, we obtain $x \in \mathfrak{D}(L)$ and $Lx = y$. Thus, the operator L is closed.

(ii) Let $x \in \mathfrak{D}(L)$. For any $t > 0$ we have

$$\lim_{\varepsilon \to 0} A_\varepsilon T_t x = \lim_{\varepsilon \to 0} T_t A_\varepsilon x = T_t L x.$$

Hence $T_t x \in \mathfrak{D}(L)$ and

$$L T_t x = T_t L x.$$

The equality $B_t L x = A_t x$ proved above can be written in the form

$$\int_0^t T_s L x \, ds = T_t x - x.$$

By the continuity of the integrand this completes the proof of (ii).

(iii) Suppose first that $x \in \mathfrak{D}(L)$. According to assertion (ii), whenever $0 < s < t$ we have

$$\frac{d}{ds}\Big[\exp\big((t-s)A_\varepsilon\big)T_s x\Big] = \exp\big((t-s)A_\varepsilon\big)T_s(Lx - A_\varepsilon x),$$

which after integration in s over the interval $[0,t]$ gives the equality

$$T_t x - \exp(tA_\varepsilon)x = \int_0^t \exp\big((t-s)A_\varepsilon\big)T_s(Lx - A_\varepsilon x)\, ds.$$

From (10.5.1), letting $\gamma := e^\beta$, we obtain the following estimates:

$$\begin{aligned}\|\exp(tA_\varepsilon)\| &\leqslant \exp(-t/\varepsilon)\sum_{n=0}^\infty \frac{t^n}{\varepsilon^n n!}\|T_{n\varepsilon}\| \\ &\leqslant C\exp(-t/\varepsilon)\sum_{n=0}^\infty \frac{t^n}{\varepsilon^n n!}\exp(\beta\varepsilon n) = C\exp\Big(\frac{t}{\varepsilon}(\gamma^\varepsilon - 1)\Big).\end{aligned}$$

Whenever $0 < \varepsilon \leqslant 1$, by the bound $\gamma^\varepsilon - 1 \leqslant \varepsilon\gamma$ we obtain

$$\|\exp(tA_\varepsilon)\| \leqslant Ce^{\gamma t}.$$

Therefore, we arrive at the estimate

$$\begin{aligned}\|T_t x - \exp(tA_\varepsilon)x\| &\leqslant C^2\|Lx - A_\varepsilon x\| \int_0^t e^{\gamma(t-s)}e^{\beta s}\, ds \\ &\leqslant C_1 e^{\gamma t}\|Lx - A_\varepsilon x\|, \end{aligned} \tag{10.5.4}$$

where C_1 is a constant. We now fix $x_0 \in X$, an interval $[0,\tau]$ and $\delta > 0$. Set $M := Ce^{\beta\tau} + Ce^{\gamma\tau}$. Pick $x \in \mathfrak{D}(L)$ with $\|x - x_0\| \leqslant \delta/M$. By using (10.5.4) for $t \in [0, \tau]$ we obtain

$$\|T_t x_0 - \exp(tA_\varepsilon)x_0\| \leqslant \|T_t x - \exp(tA_\varepsilon)x\| + \|T_t - \exp(tA_\varepsilon)\| \, \|x_0 - x\| \\ \leqslant C_1 e^{\gamma\tau} \|Lx - A_\varepsilon x\| + \delta.$$

For all sufficiently small ε the right-hand side is estimated by 2δ, since we have $\|Lx - A_\varepsilon x\| \to 0$ as $\varepsilon \to 0$. □

10.5.7. Corollary. *If* $\mathrm{Re}\,\lambda > \beta$, *where* β *satisfies inequality* (10.5.1), *then the operator* R_λ *is inverse to the operator* $\lambda I - L$ *and maps* X *one-to-one onto* $\mathfrak{D}(L)$.

PROOF. By Proposition 10.5.5 we have the equality $(\lambda I - L)R_\lambda = I$. We show that $R_\lambda(\lambda I - L)x = x$ for all $x \in \mathfrak{D}(L)$. The semigroup $S_t = e^{-\lambda t}T_t$ has generator $L - \lambda I$. This gives the equality

$$e^{-\lambda t}T_t x - x = \int_0^t e^{-\lambda s}T_s(L - \lambda I)x\, ds, \quad t \geqslant 0.$$

Letting $t \to +\infty$ we obtain $e^{-\lambda t}T_t x \to 0$, since $\|e^{-\lambda t}T_t\| \leqslant Ce^{(\beta-\lambda)t}$, where $\mathrm{Re}\,(\beta - \lambda) < 0$. Hence $R_\lambda(L - \lambda I)x = -x$. □

Thus, we have the following representation for all λ with a sufficiently large real part (if $\|T_t\| \leqslant 1$, then for all λ with $\mathrm{Re}\,\lambda > 0$):

$$(\lambda I - L)^{-1}x = \int_0^\infty e^{-\lambda s}T_s x\, ds. \tag{10.5.5}$$

This representation is a basis for many further results in the theory of operator semigroups. Let us give without proof (which can be found in [**317**, p. 622] or [**471**, p. 85]) the following theorem due to Trotter.

10.5.8. Theorem. *Let* L *and* L_n, *where* $n \in \mathbb{N}$, *be the generators of strongly continuous semigroups* $\{T_t\}_{t\geqslant 0}$ *and* $\{T_t^{(n)}\}_{t\geqslant 0}$ *on a Banach space* X *for which there exists a number* $C, \omega \in \mathbb{R}^1$ *such that* $\|T_t^{(n)}\| \leqslant Ce^{\omega t}$, $\|T_t\| \leqslant Ce^{\omega t}$ *and for some* λ *with* $\mathrm{Re}\,\lambda > \omega$ *we have* $\lim\limits_{n\to\infty} (\lambda I - L_n)^{-1}x = (\lambda I - L)^{-1}x$ *for every* $x \in X$. *Then this is true for all* λ *with* $\mathrm{Re}\,\lambda > \omega$ *and, as* $n \to \infty$, *for every* $x \in X$ *we have* $T_t^{(n)}x \to T_t x$ *uniformly in* t *from every compact interval.*

Conversely, if $T_t^{(n)}x \to T_t x$ *for all* $x \in X$ *and* $t \geqslant 0$, *then for every* λ *with* $\mathrm{Re}\,\lambda > \omega$ *we have* $(\lambda I - L_n)^{-1}x \to (\lambda I - L)^{-1}x$ *for all* $x \in X$.

There is also a condition for convergence of semigroups in terms of their generators (see [**471**, p. 88]).

10.5.9. Theorem. *The pointwise convergence of semigroups in the situation of the previous theorem holds if in place of convergence of resolvents the following condition is satisfied: there exist a dense linear subspace* D *in the intersection of the domains of definition of* L *and* L_n *and a number* λ *with* $\mathrm{Re}\,\lambda > \omega$ *such that* $L_n x \to Lx$ *for all* $x \in D$ *and* $(\lambda I - L)(D)$ *is dense in* X.

In the next section we continue our discussion of semigroups.

10.6. Generators of Semigroups

Here we prove two important results about semigroup generators: Stone's theorem about generators of unitary groups on Hilbert spaces and the Hille–Yosida theorem, which gives a description of all semigroup generators. We first observe that the generator uniquely determines the semigroup.

10.6.1. Proposition. *Suppose that two strongly continuous operator semigroups $\{T_t\}_{t\geqslant 0}$ and $\{S_t\}_{t\geqslant 0}$ on a Banach space have equal generators. Then $T_t = S_t$ for all $t \geqslant 0$.*

PROOF. Let L be the common generator of the given semigroups. Then by the equality $(\lambda - L)^{-1} = R_\lambda$ proved above for every λ with a sufficiently large real part we have

$$\int_0^\infty e^{-\lambda t} T_t x\,dt = \int_0^\infty e^{-\lambda t} S_t x\,dt.$$

Let $l \in X^*$, $\varphi(t) = e^{-\eta t} l(T_t x)$, $\psi(t) = e^{-\eta t} l(S_t x)$, where η is a large real number and $\varphi = \psi = 0$ on $(-\infty, 0)$. The previous equality means that φ and ψ have equal Fourier transforms. Hence $\varphi(t) = \psi(t)$, which gives the equality $T_t x = S_t x$, because l was arbitrary. $\square$

We now prove the following Stone theorem.

10.6.2. Theorem. *Let $\{U_t\}_{t\in\mathbb{R}}$ be a strongly continuous group of unitary operators on a Hilbert space H. Then its generator L has the form $L = iA$ with some selfadjoint operator A and $U_t = \exp(itA)$.*

PROOF. For any $f \in C_0^\infty(\mathbb{R}^1)$ and $\varphi \in H$ set

$$\varphi_f := \int_{-\infty}^{+\infty} f(t) U_t \varphi\,dt,$$

where the integral is understood in the Riemann sense. Let D be the set of finite linear combinations of elements φ_f with $f \in C_0^\infty(\mathbb{R}^1)$ and $\varphi \in H$. The linear subspace D is dense in H, since for every $\varphi \in H$ we can take elements $\varphi_{\theta_\varepsilon}$ with $\theta_\varepsilon(t) = \varepsilon^{-1}\theta(t/\varepsilon)$, where θ is a smooth probability density with support in $[0, 1]$. Then, letting $\varepsilon \to 0$, we have

$$\|\varphi_{\theta_\varepsilon} - \varphi\| = \left\| \int_{-\infty}^{+\infty} \theta_\varepsilon(t)[U_t\varphi - \varphi]\,dt \right\| \leqslant \sup_{t\in[0,\varepsilon]} \|U_t\varphi - \varphi\| \to 0$$

by the strong continuity of the semigroup. Letting $s \to 0$ we find that

$$s^{-1}(U_s - I)\varphi_f = s^{-1} \int_{-\infty}^{+\infty} f(t)[U_{t+s} - U_t]\varphi\,dt =$$

$$= \int_{-\infty}^{+\infty} \frac{f(\tau - s) - f(\tau)}{s} U_\tau \varphi\,d\tau \to -\int_{-\infty}^{+\infty} f'(\tau) U_\tau \varphi\,d\tau = \varphi_{-f'},$$

since the difference quotients $s^{-1}[f(\tau - s) - f(\tau)]$ converge to $-f'(\tau)$ uniformly on some interval outside of which they vanish. Set

$$A\varphi_f := -i\varphi_{-f'} = -i \lim_{s\to 0} s^{-1}(U_s - I)\varphi_f.$$

So the operator A is defined on D. We observe that $U_t(D) \subset D$, $A(D) \subset D$ and $U_t A\psi = AU_t\psi$ for all $\psi \in D$, which is verified on the vectors $\psi = \varphi_f$. In addition, if $f, g \in C_0^\infty(\mathrm{I\!R}^1)$ and $\varphi, \psi \in H$, then

$$(A\varphi_f, \psi_g) = \lim_{s\to 0} \frac{1}{is}(U_s\varphi_f - \varphi_f, \psi_g) = \lim_{s\to 0} \frac{1}{is}(\varphi_f, U_{-s}\psi_g - \psi_g)$$
$$= (\varphi_f, i^{-1}\psi_{-g'}) = (\varphi_f, A\psi_g).$$

Thus, A is a symmetric operator on D. We show that it is essentially selfadjoint. Suppose that $u \in \mathfrak{D}(A^*)$ and $A^*u = iu$. Then for every $\varphi \in D$ we have

$$\frac{d}{dt}(U_t\varphi, u) = (iAU_t\varphi, u) = -i(U_t\varphi, A^*u) = -i(U_t\varphi, iu) = (U_t\varphi, u).$$

Thus, the complex function $\zeta(t) = (U_t\varphi, u)$ satisfies the equation $\zeta'(t) = \zeta$, so $\zeta(t) = ce^t$. Since $|\zeta(t)| \leqslant \|\varphi\|\,\|u\|$, we have $c = 0$, hence $(\varphi, u) = 0$. Using that D is dense, we obtain that $u = 0$. Similarly we verify that if $u \in \mathfrak{D}(A^*)$ and $A^*u = -iu$, then $u = 0$. By Corollary 10.7.11 proved below the closure $\overline{A}$ of the operator A is selfadjoint. Let $V_t = \exp(it\overline{A})$. It remains to show that $U_t = V_t$. It suffices to verify that $U_t\varphi = V_t\varphi$ for all $\varphi \in D$, since D is dense. Thus, let $\varphi \in D$. Since $D \subset \mathfrak{D}(\overline{A})$, according to Example 10.5.3 and Theorem 10.5.6 we have $V_t\varphi \in \mathfrak{D}(\overline{A})$ and $(V_t\varphi)' = i\overline{A}V_t\varphi$. On the other hand, $U_t\varphi \in D$ and $(U_t\varphi)' = iAU_t\varphi$, since $(U_t\varphi)' = U_t(U_s\varphi)'|_{s=0} = iU_tA\varphi$. Set $w(t) := U_t\varphi - V_t\varphi$. Then

$$w'(t) = iAU_t\varphi - i\overline{A}V_t\varphi = i\overline{A}w(t).$$

Therefore,

$$\frac{d}{dt}\big(w(t), w(t)\big) = -i\big(\overline{A}w(t), w(t)\big) + i\big(w(t), \overline{A}w(t)\big) = 0.$$

Since $w(0) = 0$, we obtain $w(t) = 0$ for all t, as required. □

10.6.3. Theorem. *Let $\{T_t\}_{t\geqslant 0}$ be a strongly continuous semigroup of selfadjoint operators on a Hilbert space H such that $\|T_t\| \leqslant 1$. Then its generator L is selfadjoint and $L \leqslant 0$.*

Conversely, if an operator L is selfadjoint and $L \leqslant 0$, then the operators $T_t = \exp(tL)$ form a strongly continuous semigroup of selfadjoint operators with generator L and $\|T_t\| \leqslant 1$.

PROOF. The operator L is closed and symmetric. For every $x \in \mathfrak{D}(L)$ we have

$$\frac{d}{dt}(T_tx, T_tx) = 2(T_tLx, T_tx).$$

In addition, $(T_tx, T_tx) \leqslant (x, x)$ and $(T_0x, T_0x) = (x, x)$. Therefore, $(Lx, x) \leqslant 0$, since otherwise for sufficiently small $t \geqslant 0$ we would have $2(T_tLx, T_tx) > 0$ and then $(T_tx, T_tx) > (x, x)$. As shown in §10.7(ii), the operator L possesses a nonpositive selfadjoint extension G. This extension actually coincides with L, since both operators $L - I$ and $G - I$ have bounded inverse operators.

The converse assertion is obvious from the fact that in the separable case L can be represented as multiplication by a nonpositive function, which enables us to verify our assertion directly. The nonseparable case reduces to the separable one by decomposing H into a direct sum of separable subspaces invariant with respect to L. □

We now prove the general Hille–Yosida theorem about generators of contracting semigroups, i.e., semigroups $\{T_t\}_{t\geqslant 0}$ with $\|T_t\| \leqslant 1$.

10.6.4. Theorem. (i) *Let L be the generator of a strongly continuous contracting semigroup $\{T_t\}_{t\geqslant 0}$ on a Banach space X. Then every real number $\lambda > 0$ belongs to the resolvent set of L and*

$$\|R_\lambda\| = \|(\lambda I - L)^{-1}\| \leqslant \frac{1}{\lambda}. \tag{10.6.1}$$

(ii) *Conversely, let L be a linear operator with a dense domain of definition $\mathfrak{D}(L)$ such that for every real number $\lambda > 0$ the operator $\lambda I - L$ has a bounded inverse $R_\lambda \colon X \to \mathfrak{D}(X)$ satisfying condition* (10.6.1). *Then L is the generator of a strongly continuous contracting semigroup $\{T_t\}_{t\geqslant 0}$ on X.*

PROOF. Corollary 10.5.7 and equality (10.5.5) give (i), where (10.6.1) follows from the fact that the integral of $e^{-\lambda t}$ over $(0, +\infty)$ is λ^{-1}. For the proof of (ii) we set

$$L_n := n^2 R_n - nI, \quad n \in \mathbb{N}.$$

Since $LR_n = nR_n - I$, we have $L_n = nLR_n$. The idea of the proof is to approximate L by the operators L_n. Since the operator L_n is bounded, with the aid of the usual exponent it generates the semigroup $T_t^{(n)} := \exp(tL_n)$. We show that

$$\lim_{n\to\infty} nR_n x = x, \quad x \in X. \tag{10.6.2}$$

If $x \in \mathfrak{D}(L)$, then this equality is true by the relations $nR_n x - x = R_n Lx$ and $\|R_n Lx\| \leqslant n^{-1}\|Lx\|$. Since $\mathfrak{D}(L)$ is dense in X and $\|nR_n\| \leqslant 1$, relation (10.6.2) is true for all $x \in X$. Using that for all $x \in \mathfrak{D}(L)$ we have $L_n x = nLR_n x = nR_n Lx$, from (10.6.2) we obtain the equality

$$\lim_{n\to\infty} L_n x = Lx, \quad x \in \mathfrak{D}(L). \tag{10.6.3}$$

By the definition of L_n we obtain

$$\|T_t^{(n)}\| = \|\exp(tL_n)\| = e^{-nt}\|\exp(n^2 tR_n)\| \leqslant e^{-nt}\sum_{k=0}^{\infty}\frac{1}{k!}(n^2t)^k\|R_n^k\|$$
$$\leqslant e^{-nt}\sum_{k=0}^{\infty}\frac{1}{k!}(n^2t)^k n^{-k} = e^{-nt}e^{nt} = 1.$$

Thus, the constructed semigroups are contracting.

We now show that these semigroups converge. To this end we estimate the quantity $\|T_t^{(n)}x - T_t^{(k)}x\|$ for $x \in \mathfrak{D}(L)$. It is readily seen that the operators L_n commute with $T_t^{(k)}$. This yields the relation

$$\frac{d}{dt}T_{s-t}^{(n)}T_t^{(k)}x = T_{s-t}^{(n)}T_t^{(k)}[L_k - L_n]x.$$

By the estimate $\|T_t^{(n)}\| \leqslant 1$ the norm of the right-hand side does not exceed $\|L_k x - L_n x\|$. Therefore,

$$\|T_t^{(n)}x - T_t^{(k)}x\| \leqslant t\|L_n x - L_k x\|.$$

Along with (10.6.3) this shows the existence of the limit

$$T_t x := \lim_{n\to\infty} T_t^{(n)}x, \quad x \in \mathfrak{D}(L),$$

which is uniform on every compact interval. The uniform boundedness of $T_t^{(n)}$ yields the existence of the limit for all $x \in X$. It is clear that $\{T_t\}$ is a continuous semigroup and $\|T_t\| \leqslant 1$.

It remains to show that L coincides with the generator of $\{T_t\}$. Passing to the limit in the equality

$$T_t^{(n)}x - x = \int_0^t T_s^{(n)} L_n x\, ds$$

as $n \to \infty$, for every $x \in \mathfrak{D}(L)$, on account of (10.6.3) we find

$$T_t x - x = \int_0^t T_s L x\, ds.$$

Let us denote the generator of $\{T_t\}$ by G. The previous equality shows (dividing by t and letting t go to zero) that $\mathfrak{D}(L)$ belongs to $\mathfrak{D}(G)$ and $Lx = Gx$ for all $x \in \mathfrak{D}(L)$. Thus, G extends L. However, this extension cannot be proper, since the operators $L - I$ and $G - I$ have bounded inverses (the first one by our assumption and the second one as the generator of a contracting semigroup). Hence $L = G$. $\square$

10.6.5. Corollary. *A closed densely defined operator L on a Banach space X is the generator of a strongly continuous operator semigroup precisely when there exist numbers $C \geqslant 0$ and $\beta \in \mathbb{R}^1$ such that every real number $\lambda > \beta$ belongs to the resolvent set of L and for all $n \in \mathbb{N}$ we have*

$$\|(\lambda I - L)^{-n}\| \leqslant \frac{C}{(\lambda - \beta)^n}. \tag{10.6.4}$$

PROOF. If L is the generator of a strongly continuous semigroup, then one has estimate (10.5.1). Suppose first that $\beta = 0$. Then we can introduce a new norm

$$\|x\|_0 := \sup_{t\geqslant 0} \|T_t x\|,$$

which is equivalent to the old one: $\|x\| \leqslant \|x\|_0 \leqslant C\|x\|$. The norms of operators on X corresponding to the new norm will be also denoted by $\|\cdot\|_0$. With respect to the new norm the operators T_t satisfy the estimate $\|T_t\|_0 \leqslant 1$. Hence all points λ with a positive real part are regular and $\|(\lambda I - L)^{-1}\|_0 \leqslant |\lambda|^{-1}$. Hence $\|(\lambda I - L)^{-n}\|_0 \leqslant |\lambda|^{-n}$, whence the desired estimate follows, since$\|A\|_0 \leqslant C\|A\|$ for all $A \in \mathcal{L}(X)$. In the general case we consider the semigroup of operators $S_t := e^{-t\beta}T_t$ with the generator $L - \beta I$. We have $\|S_t\| \leqslant C$, so by the previous step we obtain the regularity of all numbers λ with $\mathrm{Re}\,\lambda > \beta$ and estimate (10.6.4).

Conversely, suppose that L satisfies the indicated conditions. Again we start with the case $\beta = 0$. For any $\mu > 0$, we set

$$\|x\|_\mu := \sup\bigl\{\|\mu^n(\mu I - L)^{-n}\|\colon\ n = 0, 1, 2, \ldots\bigr\}.$$

This norm is equivalent to the original one, since $\|x\| \leqslant \|x\|_\mu \leqslant C\|x\|$. For the corresponding operator norm we have $\|\mu(\mu I - L)^{-1}\|_\mu \leqslant 1$. Moreover, the following inequality holds:

$$\|(\lambda I - L)^{-1}\|_\mu \leqslant \frac{1}{\lambda}, \quad 0 < \lambda \leqslant \mu. \tag{10.6.5}$$

For the proof we use the identity $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$ for resolvents to obtain

$$\|R_\lambda\|_\mu \leqslant \|R_\mu\|_\mu + \|(\mu - \lambda)R_\lambda R_\mu\|_\mu \leqslant \frac{1}{\mu} + \frac{\mu - \lambda}{\mu}\|R_\lambda\|_\mu,$$

which gives $\lambda\|R_\lambda\|_\mu \leqslant 1$, as required. Now, whenever $n \in \mathbb{N}$ and $0 < \lambda \leqslant \mu$, by (10.6.5) we obtain

$$\|\lambda^n(\lambda I - L)^{-n}x\| \leqslant \|\lambda^n(\lambda I - L)^{-n}x\|_\mu \leqslant \|\lambda(\lambda I - L)^{-1}\|_\mu^n\|x\|_\mu \leqslant \|x\|_\mu.$$

Therefore, $\|x\|_\lambda \leqslant \|x\|_\mu$ if $0 < \lambda \leqslant \mu$. Let us introduce the norm

$$\|x\|_0 := \sup_{\mu > 0}\|x\|_\mu = \lim_{\mu\to\infty}\|x\|_\mu.$$

It is clear that $\|x\| \leqslant \|x\|_0 \leqslant C\|x\|$. Since $\|(\lambda I - L)^{-1}x\|_\mu \leqslant \lambda^{-1}\|x\|_\mu$ whenever $0 < \lambda \leqslant \mu$ (by (10.6.5)), letting $\mu \to \infty$ we obtain $\|(\lambda I - L)^{-1}x\|_0 \leqslant \lambda^{-1}\|x\|_0$, hence $\|(\lambda I - L)^{-1}\|_0 \leqslant \lambda^{-1}$. By the Hille–Yosida theorem the operator L is the generator of some continuous semigroup $\{T_t\}$ with $\|T_t\|_0 \leqslant 1$. Then we have the inequalities $\|T_t x\| \leqslant \|T_t x\|_0 \leqslant \|x\|_0 \leqslant C\|x\|$, i.e., $\|T_t\| \leqslant C$.

In the case of arbitrary β we apply the previous step to the shifted operator $L_0 = L - \beta I$. It is the generator of some continuous semigroup $\{S_t\}_{t\geqslant 0}$. The semigroup of operators $T_t := e^{\beta t}S_t$ has generator L. □

There is yet another useful characterization of generators of contracting semigroups. A densely defined operator $\bigl(L, \mathfrak{D}(L)\bigr)$ on a Banach space X is called *dissipative* if, for every $u \in \mathfrak{D}(L)$ with $\|u\| = 1$, there exists $l \in X^*$ with $\|l\| = 1$ such that $l(u) = 1$ and $\operatorname{Re} l(Lu) \leqslant 0$. If X is a Hilbert space, then this means that $\operatorname{Re}(u, Lu) \leqslant 0$. Dissipativity is equivalent to the property that for all $\lambda > 0$ and $u \in \mathfrak{D}(L)$ we have $\|\lambda u - Lu\| \geqslant \lambda\|u\|$ (see Exercise 10.7.41). Hence the generator of a contracting semigroup is dissipative. On account of the Hille–Yosida theorem this gives the following result due to Lumer and Phillips.

10.6.6. Theorem. *Let $\bigl(L, \mathfrak{D}(L)\bigr)$ be a dissipative operator on a Banach space. Its closure is the generator of a strongly continuous contracting semigroup precisely when the set $(\lambda I - L)\bigl(\mathfrak{D}(L)\bigr)$ is dense for some (and then for every) number $\lambda > 0$.*

10.7. Complements and Exercises

(i) Extensions of symmetric operators (461). (ii) Semibounded forms and operators (466). (iii) The Chernoff and Trotter theorems (470). (iv) The mathematical model of quantum mechanics (472). (v) Sturm–Liouville operators (478). Exercises (480).

10.7(i). Extensions of symmetric operators

In this subsection we discuss in greater detail the question about symmetric and selfadjoint extensions of symmetric operators.

We have seen in Example 10.2.3 that a closed symmetric operator can fail to be selfadjoint, but at the same time possess selfadjoint extensions. Let us consider an example of a closed symmetric operator that has no selfadjoint extensions.

10.7.1. Example. Let $H = L^2[0,+\infty)$. Set

$$\mathfrak{D}(A_0) = C_0^\infty(0,+\infty), \quad A_0 u = iu'.$$

The operator A_0 is symmetric, its closure A is a closed and symmetric operator, but has no selfadjoint extensions. Indeed, similarly to Example 10.2.3 we can prove that the set $\mathfrak{D}(A_0^*) = \mathfrak{D}(A^*)$ consists of functions $u \in L^2[0,+\infty)$ such that u is absolutely continuous on bounded intervals and $u' \in L^2[0,+\infty)$. In addition, $A_0^* u = iu'$. Note that the continuous version of any function $u \in \mathfrak{D}(A_0^*)$ must have the zero limit at infinity. Indeed,

$$|u(t)|^2 = |u(0)|^2 + \int_0^t [u'(s)\overline{u(s)} + u(s)\overline{u'(s)}]\, ds.$$

The right-hand side has a limit at infinity by the quadratic integrability of u and u'. It is clear that this limit must be zero.

The inclusion $A_0 \subset A_0^*$ yields that $A_0^{**} \subset A_0^*$. Then for all $u \in \mathfrak{D}(A_0^{**})$ and $v \in \mathfrak{D}(A_0^*)$ we have

$$\int_0^\infty iu'(t)\overline{v(t)}\, dt = \int_0^\infty u(t)\overline{iv'(t)}\, dt,$$

which by the integration by parts formula and the fact that u and v tend to zero at infinity gives the relation $u(0)\overline{v(0)} = 0$. Therefore, $u(0) = 0$. Conversely, if $u \in \mathfrak{D}(A_0^*)$ and $u(0) = 0$, then we have $u \in \mathfrak{D}(A_0^{**})$ and $A_0^{**}u = iu$. Thus, $\mathfrak{D}(A_0^{**})$ consists of functions $u \in \mathfrak{D}(A_0^*)$ such that $u(0) = 0$. We observe that $A_0^{**} = A = \overline{A_0}$, since the operator A_0^{**} is closed and its graph contains the closure of the graph of A_0.

Let B be a selfadjoint operator with $A \subset B$. Then $A \subset B = B^* \subset A^*$. However, $\mathfrak{D}(A)$ is of codimension 1 in $\mathfrak{D}(A^*)$, hence the set $\mathfrak{D}(B)$ must coincide with $\mathfrak{D}(A)$ or with $\mathfrak{D}(A^*)$, which is impossible, since A is not selfadjoint.

What is the reason that symmetric operators very similar by appearance are so different with respect to existence of selfadjoint extensions? It turns out that this is due to different defect numbers, which arise in case of bounded and unbounded intervals.

10.7.2. Definition. *Let T be a closed symmetric operator on a Hilbert space H. The dimensions $n_-(T)$ and $n_+(T)$ (possibly, infinite) of the orthogonal complements to the subspaces*

$$\mathrm{Ran}\,(T+iI) = \mathrm{Ker}\,(T^*-iI) \quad \textit{and} \quad \mathrm{Ran}\,(T-iI) = \mathrm{Ker}\,(T^*+iI)$$

are called the defect numbers (defect indices) of the operator T.

In the terminology introduced at the end of §10.1 the defect $d_T(\lambda)$ can be written as $n_-(T) = d_T(-i)$, $n_+(T) = d_T(i)$.

Let us introduce the Caley transform of a densely defined symmetric operator T. As in the case of a selfadjoint operator, the operators $T+iI$ and $T-iI$ are injective. However, now their ranges need not be everywhere dense.

10.7.3. Proposition. *Let T be a densely defined symmetric operator on a Hilbert space H. Then the following assertions are true.*

(i) *One has the identity*

$$\|Tx+ix\|^2 = \|Tx\|^2 + \|x\|^2 = \|Tx-ix\|^2, \quad x \in \mathfrak{D}(T).$$

(ii) *The operator T is closed precisely when the subspaces* $\mathrm{Ran}\,(T+iI)$ *and* $\mathrm{Ran}\,(T-iI)$ *are closed.*

(iii) *There is a linear isometry U between the linear subspaces* $\mathrm{Ran}\,(T+iI)$ *and* $\mathrm{Ran}\,(T-iI)$ *defined by the formula*

$$U(Tx+ix) = Tx-ix, \quad x \in \mathfrak{D}(T).$$

The mapping U is called the Caley transform of the operator T.

PROOF. Assertion (i) is verified directly.

(ii) Let T be closed. Let $x_n \in \mathfrak{D}(T)$ and $Tx_n + ix_n \to y$. It follows from (i) that $\{x_n\}$ is a Cauchy sequence. Hence $x_n \to x \in H$. Then sequence $\{Tx_n\}$ converges to some vector $z \in X$. Since T is closed, we have $x \in \mathfrak{D}(T)$ and $z = Tx$, whence $y = Tx + ix$. Similarly we obtain that $\mathrm{Ran}\,(T-iI)$ is closed.

Conversely, suppose that the subspace $\mathrm{Ran}\,(T+iI)$ is closed. Let $x_n \in \mathfrak{D}(T)$, $x_n \to x$ and $Tx_n \to y$. Then $Tx_n + ix_n \to y + ix$. Since $\mathrm{Ran}\,(T+iI)$ is closed, we have $u \in \mathfrak{D}(T)$ and $y + ix = Tu + iu$. Hence $T(x_n - u) + i(x_n - u) \to 0$, which by (i) gives convergence $x_n - u \to 0$. Thus, $x = u \in \mathfrak{D}(T)$ and $y = Tx$. Assertion (iii) follows from (i). □

10.7.4. Corollary. *Any densely defined symmetric operator T possesses a closed symmetric extension.*

PROOF. By Proposition 10.1.9 the operator T^* is closed. Since it extends T by the symmetry of T, the operator T is closable. Its closure $\overline{T}$ is symmetric, because for all $x, y \in \mathfrak{D}(\overline{T})$ by definition there exist two sequences of vectors $x_n, y_n \in \mathfrak{D}(T)$ such that $Tx_n \to \overline{T}x$ and $Ty_n \to \overline{T}y$. Hence

$$(\overline{T}x, y) = \lim_{n\to\infty} (Tx_n, y_n) = \lim_{n\to\infty} (x_n, Ty_n) = (x, \overline{T}y).$$

By construction the operator $\overline{T}$ is closed (see Proposition 10.1.9). □

10.7.5. Corollary. *Let T be a closed symmetric operator. For every $\lambda \in \mathbb{C}$ with* $\operatorname{Im}\lambda \neq 0$ *one has the orthogonal decomposition*

$$H = \operatorname{Ran}(T - \lambda I) \oplus \operatorname{Ker}(T^* - \overline{\lambda} I).$$

PROOF. Since $\operatorname{Ran}(T-\lambda I)$ is closed, we can apply Proposition 10.1.13. □

10.7.6. Theorem. *The Caley transform establishes a one-to-one correspondence between symmetric operators T on H and isometric operators V such that $\mathfrak{D}(V)$ is a linear subspace in H and $\overline{\operatorname{Ran}(V-I)} = H$. Here T is closed if and only if $\mathfrak{D}(V)$ is a closed subspace.*

PROOF. We already know that if the operator T is symmetric, then its Caley transform V maps isometrically the linear subspace $\operatorname{Ran}(T+iI)$ onto the subspace $\operatorname{Ran}(T-iI)$. Moreover, T is closed if and only if this subspace is closed. Let us show that the range of $V-I$ is dense in H. Indeed, let $y \in H$ be an element such that

$$(Vz - z, y) = 0 \quad \forall\, z = (T+iI)x, \quad x \in \mathfrak{D}(T).$$

Since $V(T+iI)x = (T-iI)x$, this means that $(x,y) = 0$ for all $x \in \mathfrak{D}(T)$, i.e., $y = 0$. Conversely, suppose that the operator V possesses the indicated properties. We observe that the mapping $V-I$ is injective. Indeed, if $Vx - x = 0$ for some $x \in \mathfrak{D}(V)$, then for all $y \in \mathfrak{D}(V)$ we have

$$\bigl(Vx, (I-V)y\bigr) = (Vx, y) - (Vx, Vy) = (Vx, y) - (x, y) = 0,$$

which gives the equality $x = 0$, since the range of $V-I$ is dense. Now we set

$$\mathfrak{D}(T) := \operatorname{Ran}(V-I), \quad Ty = -i(V+I)x \quad \text{if } y = (V-I)x.$$

The operator T is densely defined. In addition,

$$\begin{aligned}(Ty, y) &= -i(Vx + x, Vx - x)\\ &= -i[(Vx, Vx) - (Vx, x) + (x, Vx) - (x, x)] = -2\operatorname{Re}(Vx, x),\end{aligned}$$

which shows that the form (Tx, x) is real. Thus, the operator T is symmetric. Since

$$Ty + iy = -iVx - ix + iVx - ix = -2ix,$$

we have $\operatorname{Ran}(T+iI) = \mathfrak{D}(V)$. If this subspace is closed, then the operator T is closed. Finally, the Caley transform U of the operator T coincides with V, since $\operatorname{Ran}(T+iI) = \mathfrak{D}(V)$, i.e., $\mathfrak{D}(U) = \mathfrak{D}(V)$, moreover,

$$U(Ty + iy) = Ty - iy = -2iVx, \quad U(Ty + iy) = U(-2ix) = -2iUx$$

for all $x \in \mathfrak{D}(V)$. □

The Caley transform reduces the problem of extending symmetric operators to the problem of extending linear isometries, which is easily solved.

10.7.7. Proposition. *If a symmetric operator $\widetilde{T}$ is an extension of the symmetric operator T, then the Caley transform of the operator $\widetilde{T}$ is an extension of the Caley transform of the operator T.*

Conversely, if T is a symmetric operator with the Caley transform V and $\widetilde{V}$ is an isometry operator extending V, then $\widetilde{V}$ is the Caley transform of some symmetric operator $\widetilde{T}$ that extends T.

PROOF. The inclusion $\mathrm{Ran}\,(T+iI) \subset \mathrm{Ran}\,(\widetilde{T}+iI)$ yields the first assertion. The second assertion follows from the previous theorem, which gives a symmetric operator $\widetilde{T}$ with the Caley transform $\widetilde{V}$. Moreover,

$$\mathfrak{D}(T) = \mathrm{Ran}\,(V - I) \subset \mathrm{Ran}\,(\widetilde{V} - I) = \mathfrak{D}(\widetilde{T})$$

and on $\mathfrak{D}(T)$ the operators T and $\widetilde{T}$ coincide, which is clear from the formula defining T by V (see the previous proof). □

10.7.8. Lemma. *Let H_1 and H_2 be closed linear subspaces in a Hilbert space $H \neq 0$ and let $U\colon\ H_1 \to H_2$ be a linear isometry with $U(H_1) = H_2$. The mapping U can be extended to an isometry on all of H with values in H precisely when $\dim H_1^{\perp} \leqslant \dim H_2^{\perp}$. In the case of the equality of these two dimensions U extends to a unitary operator.*

PROOF. If the indicated condition is fulfilled, we can take a linear isometric embedding $V\colon\ H_1^{\perp} \to H_2^{\perp}$ (which in case of equal dimensions of these subspaces can be made surjective) and set

$$\widetilde{U}(x+u) = Ux + Vu, \quad x \in H_1, u \in H_1^{\perp}.$$

Conversely, if U has an isometric extension $\widetilde{U}$ to all of H, then we obtain $\widetilde{U}(H_1^{\perp}) \subset H_2^{\perp}$, since $\widetilde{U}$ preserves the orthogonality. □

10.7.9. Theorem. *Let T be a closed symmetric operator and let $n_-(T)$, $n_+(T)$ be its defect numbers. Then*

(i) *the operator T is selfadjoint if and only if $n_-(T) = n_+(T) = 0$;*

(ii) *the existence of selfadjoint extensions of T is equivalent to the equality $n_-(T) = n_+(T)$;*

(iii) *the absence of proper symmetric extensions of T is equivalent to the equality to zero of at least one of its defect numbers*;

(iv) *if $n_-(T) = n_+(T) = \infty$, then T has symmetric extensions with an a priori given pair of defect numbers.*

PROOF. All assertions of this theorem follow from the established correspondence between symmetric operators T and their Caley transforms V taking into account the fact that the defect numbers $n_-(T)$ and $n_+(T)$ coincide with the dimensions $n_e(V)$ and $n_i(V)$ of the orthogonal complements to the subspaces $H_0 = \mathfrak{D}(V)$ and $H_1 = \mathrm{Ran}\,V$. If $n_e(V) \neq n_i(V)$, then V cannot be extended to a unitary operator. If $n_e(V) = n_i(V)$, then this is possible. If $n_e(V) = n_i(V) = \infty$, then for any pair of numbers n, m from $\{0, 1, \ldots, \infty\}$ we can take in $H_0^{\perp}$ and $H_1^{\perp}$ closed infinite-dimensional subspaces E_0 and E_1 having in $H_0^{\perp}$ and $H_1^{\perp}$ the codimensions n and m, respectively. Now we can extend the operator V with the aid of an isometry mapping from E_1 onto E_2. □

10.7.10. Corollary. *All symmetric extensions of a closed symmetric operator T with defect numbers $n_+ = n_- = 1$ are selfadjoint.*

These extensions are described by a complex parameter θ with $|\theta| = 1$ in the following way: a unit vector u of the one-dimensional space $\mathfrak{D}(V)^{\perp}$ is mapped by the isometric extension V to the vector θv, where v is a unit vector of the one-dimensional space $(\operatorname{Ran} V)^{\perp}$.

The established facts yield the following criterion for the closure of a symmetric operator A to be a selfadjoint operator. Such an operator A is called *essentially selfadjoint*.

10.7.11. Corollary. *A densely defined symmetric operator A is essentially selfadjoint, i.e., its closure is selfadjoint, precisely when the equations $A^*u = iu$ and $A^*u = -iu$ have only zero solutions in $\mathfrak{D}(A^*)$.*

Let us give a more precise description of symmetric extensions.

10.7.12. Theorem. *Let T be a closed symmetric operator.*

(i) *For every λ with $\operatorname{Im}\lambda \neq 0$ one has the equality*

$$\mathfrak{D}(T^*) = \mathfrak{D}(T) + \operatorname{Ker}(T^* - \lambda I) + \operatorname{Ker}(T^* - \overline{\lambda} I),$$

where we have a direct algebraic sum.

(ii) *Let $D_0 \subset \operatorname{Ker}(T^* - \lambda I)$ and $R_0 \subset \operatorname{Ker}(T^* - \overline{\lambda} I)$ be closed linear subspaces of the same dimension n_0 (possibly, both infinite-dimensional), and let V_0 be some isometry from D_0 onto R_0. Then the formula*

$$\mathfrak{D}(\widetilde{T}) := \mathfrak{D}(T) + (V_0 - I)(D_0) \tag{10.7.1}$$

defines the domain of definition of some closed symmetric extension $\widetilde{T}$ of the operator T, where $\widetilde{T}(x + V_0x_0 - x_0) = Tx - ix_0 - iV_0x_0$ for all $x \in \mathfrak{D}(T)$, $x_0 \in D_0$. Conversely, every closed symmetric extension of T has the indicated form.

PROOF. (i) By the symmetry of the operator T the right-hand side of the equality to be proved is contained in the left-hand side. Let $y \in \mathfrak{D}(T^*)$. Since

$$H = \operatorname{Ran}(T - \lambda I) \oplus \operatorname{Ker}(T^* - \overline{\lambda} I)$$

according to Corollary 10.7.5, we obtain the representation

$$T^*y - \lambda y = (T - \lambda I)x + (\overline{\lambda} - \lambda)u, \quad \text{where } x \in \mathfrak{D}(T),\ u \in \operatorname{Ker}(T^* - \overline{\lambda} I).$$

On account of the equality $T^*x = Tx$ this gives $(T^* - \lambda I)(y - x - u) = 0$, i.e., $y - x - u \in \operatorname{Ker}(T^* - \lambda I)$. Let us verify that the sum of the three indicated spaces is direct. Let $x + u + v = 0$, where $u \in \mathfrak{D}(T)$, $(T^* - \lambda I)u = 0$ and $(T^* - \overline{\lambda} I)v = 0$. Applying $T^* - \lambda I$ and taking into account that $T^*x = Tx$, we obtain $(T^* - \lambda I)x + (\overline{\lambda} - \lambda)v = 0$. Using the orthogonality of $\operatorname{Ran}(T - \lambda I)$ and $\operatorname{Ker}(T^* - \overline{\lambda} I)$ once again, we obtain $x = 0$ and $v = 0$.

(ii) Assertion (i) yields that $(V_0 - I)(D_0) \cap \mathfrak{D}(T) = \{0\}$, since

$$V_0(D_0) \subset R_0 \subset \operatorname{Ker}(T^* - \overline{\lambda} I), \ quad D_0 \subset \operatorname{Ker}(T^* - \lambda I).$$

Without loss of generality we can assume that $\lambda = i$. Let V be the Caley transform of T; H is the orthogonal sum of the closed subspaces $\operatorname{Ran}(T + iI) = \mathfrak{D}(V)$ and $\operatorname{Ker}(T^* - iI)$. Hence D_0 is contained in the orthogonal complement of $\mathfrak{D}(V)$.

Similarly, R_0 belongs to the orthogonal complement of $\mathrm{Ran}\,(T - iI) = \mathrm{Ran}\,V$. Thus, by means of V_0 we obtain an isometric extension $\widetilde{V}$ of the operator V. It has been shown that $\mathfrak{D}(T) = \mathrm{Ran}\,(V - I)$. Hence $\widetilde{V}$ corresponds to a symmetric extension $\widetilde{T}$ of the operator T satisfying (10.7.1). Moreover, $\widetilde{T} \subset \widetilde{T}^* \subset T^*$, hence $\widetilde{T}x = ix$ and $\widetilde{T}V_0x = -iV_0x$ for all $x \in D_0$. Since the symmetric extensions of T are obtained in the indicated way from the isometric extensions of V, we have described all closed symmetric extensions of T. $\square$

Considering extensions it is useful to include defect numbers in the parametric family $d_T(\lambda)$ (see the end of §10.1). As we know from Proposition 10.1.15, the function d_T is locally constant on the set of regular points. When applied to symmetric operators this leads to the following conclusions.

10.7.13. Proposition. *Let T be a closed symmetric operator. If* $\mathrm{Im}\,\lambda \neq 0$, *then the operator $T - \lambda I$ has a bounded inverse on the domain* $\mathrm{Ran}\,(T - \lambda I)$, *i.e., λ is a regular point. In addition, the number $d_T(\lambda)$ is constant in every half-plane* $\mathrm{Im}\,\lambda > 0$ *and* $\mathrm{Im}\,\lambda < 0$.

PROOF. Set $\lambda = \alpha + i\beta$. Since (Tx, x) is real for all $x \in \mathfrak{D}(T)$, we obtain

$$\begin{aligned}\|(T - \lambda I)x\|^2 &= \|(T - \alpha I)x\|^2 + \beta^2\|x\|^2 + 2\mathrm{Re}\,\bigl((A - \alpha I)x, i\beta x\bigr)\\ &= \|(T - \alpha I)x\|^2 + \beta\|x\|^2 \geqslant \beta^2\|x\|^2,\end{aligned}$$

which proves the first assertion. The second assertion is obvious from Proposition 10.1.15. $\square$

10.7.14. Corollary. *If a closed symmetric operator T has a real regular point, then its defect numbers are equal. Hence T has selfadjoint extensions.*

10.7.15. Corollary. *Let T be a closed symmetric operator such that for some $m \in \mathbb{R}^1$ we have*

$$(Tx, x) \geqslant m(x, x) \quad \forall\, x \in \mathfrak{D}(T).$$

Then the ray $(-\infty, m)$ is contained in the set of regular points and the defect numbers of T are equal. Hence T has selfadjoint extensions.

In the next section we discuss in greater detail the situation of this corollary and see that there is an extension with the same bound.

10.7(ii). Semibounded forms and operators

In this subsection we study symmetric operators the quadratic forms of which are estimated from below by $\mathrm{const}\|x\|^2$. Such operators arise frequently in applications (of particular importance are operators with nonnegative forms). They possess certain special selfadjoint extensions bounded from below: the so-called Friedrichs extensions constructed through closures of forms.

Suppose that in a complex Hilbert space H we are given a dense linear subspace $\mathfrak{D}(Q)$ and a function

$$Q\colon\ \mathfrak{D}(Q) \times \mathfrak{D}(Q) \to \mathbb{C}$$

which is linear in the first argument and conjugate-linear in the second argument. It generates the quadratic form $x \mapsto Q(x,x)$ on $\mathfrak{D}(Q)$. This form is called *semibounded from below* if there exists a real number m such that

$$Q(x,x) \geqslant m(x,x), \quad x \in \mathfrak{D}(Q).$$

If m can be taken nonnegative, then the form is called *nonnegative*, and if m can be taken strictly positive, then the form is called *positive definite* or *positive*. The largest possible m is called the *precise lower bound of the form Q*.

We shall consider forms generated by symmetric operators by the formula

$$Q(x,x) := (Ax,x), \quad x \in \mathfrak{D}(A).$$

The terminology introduced above extends from forms to operators. In particular, a symmetric operator is called *semibounded from below* if so is its quadratic form; if the form is nonnegative or positive, then the operator is called nonnegative (respectively, positive). It is quite often in applications that the primary object is the quadratic form, which is used to construct the generating operator; next, its domains of definition and selfadjointness is studied. Here is a typical example: given a nonnegative Borel measure μ on $\mathbb{R}^n$, the so-called Dirichlet form on the space $L^2(\mu)$ is introduced by the formula

$$\mathcal{E}(\varphi,\varphi) = \int_{\mathbb{R}^n} \big(\nabla\varphi(x), \nabla\varphi(x)\big)\, \mu(dx), \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

The question arises: can we define this form by a selfadjoint operator? Say, if μ is Lebesgue measure, then the form $\mathcal{E}$ is generated by the Laplace operator Δ. Already in this example it is clear that the form generated by an operator can have a natural domain of definition larger than the domain of definition of the generating operator. For example, the natural domain of definition of the Laplace operator Δ on $L^2(\mathbb{R}^n)$ is the Sobolev class $W^{2,2}(\mathbb{R}^n)$, but the generated gradient Dirichlet form is naturally defined on the larger Sobolev class $W^{2,1}(\mathbb{R}^n)$. We shall see below that this phenomenon is very typical.

Suppose first that the form Q is positive-definite. Then $\mathfrak{D}(Q)$ can be equipped with the inner product

$$(x,y)_Q := Q(x,y), \quad x,y \in \mathfrak{D}(Q).$$

If $\mathfrak{D}(Q)$ is Hilbert with respect to $(\,\cdot\,,\,\cdot\,)_Q$, then the form Q is called *closed*.

10.7.16. Theorem. (i) *Let A be a positive selfadjoint operator. Then it generates a closed form*

$$Q_A(x,x) := (\sqrt{A}x, \sqrt{A}x), \quad \mathfrak{D}(Q_A) := \mathfrak{D}(\sqrt{A}).$$

Moreover, Q_A is the unique closed positive form Q satisfying the conditions

$$\mathfrak{D}(A) \subset \mathfrak{D}(Q),\ (Ax,y) = Q(x,y) \text{ for all } x \in \mathfrak{D}(A), y \in \mathfrak{D}(Q). \qquad (10.7.2)$$

(ii) *Conversely, every densely defined closed positive form Q can be obtained from some positive selfadjoint operator in this way and the corresponding operator is unique.*

PROOF. (i) Clearly, the form Q_A is closed, since the selfadjoint operator $\sqrt{A}$ is closed. Next, suppose that a positive form Q satisfies conditions (10.7.2). Then $Q_A(x,y) = Q(x,y)$ for all $x, y \in \mathfrak{D}(A)$. For the proof of the equality of the forms Q_A and Q we shall verify that $\mathfrak{D}(A)$ is dense in the Hilbert spaces $\mathfrak{D}(Q_A)$ and $\mathfrak{D}(Q)$ with their norms $\|\cdot\|_{Q_A}$ and $\|\cdot\|_Q$. If $y \in \mathfrak{D}(Q)$ and $Q(x,y) = 0$ for all $x \in \mathfrak{D}(A)$, then by (10.7.2) the vector y is orthogonal in H to the set $\operatorname{Ran} A$, which is dense, since A is positive. Hence $y = 0$. This proves that $\mathfrak{D}(A)$ is dense in $\mathfrak{D}(Q)$. The same reasoning applies to $\mathfrak{D}(Q_A)$.

(ii) For any $u \in H$, let us consider the functional $l_u(y) := (u,y)$ on the Hilbert space $\mathfrak{D}(Q)$ equipped with the inner product $(\,\cdot\,,\,\cdot\,)_Q$. Then

$$|l_u(y)| \leqslant \|u\|\,\|y\| \leqslant m^{-1/2}\|u\|\,\|y\|_Q.$$

By the Riesz representation theorem there exists a unique element $v \in \mathfrak{D}(Q)$ such that $l_u(y) = Q(v,y)$ for all $y \in \mathfrak{D}(Q)$. It is clear that the element v depends linearly on u. Set $Bu := v$. Then

$$Q(Bu,y) = (u,y), \quad y \in \mathfrak{D}(Q).$$

Since $\|Bu\|_Q^2 = (u,Bu) \leqslant \|u\|\,\|Bu\| \leqslant m^{-1/2}\|u\|\,\|Bu\|_Q$, we obtain the bound $\|Bu\|_Q \leqslant m^{-1/2}\|u\|$, i.e., B is a continuous operator from H to $\mathfrak{D}(Q)$ with the norm $\|\cdot\|_Q$. Hence B is continuous from H to H. Since for $y = Bu$ we have $(u,Bu) = Q(Bu,Bu)$, the operator B is selfadjoint. This operator is injective. Indeed, if $Bu = 0$, then $(y,u) = 0$ for all $y \in \mathfrak{D}(Q)$, whence $u = 0$. Hence the operator $A := B^{-1}$ on the domain $\mathfrak{D}(A) := \operatorname{Ran} B \subset \mathfrak{D}(Q)$ is selfadjoint. We have obtained the required operator. Indeed, for $u = Ax$ with $x \in \mathfrak{D}(A)$ we have

$$(Ax,y) = Q(x,y), \quad x \in \mathfrak{D}(A),\, y \in \mathfrak{D}(Q).$$

According to (i) the form Q_A coincides with Q. Let us verify that A is unique. If A_1 is one more selfadjoint operator generating the form Q, then

$$\mathfrak{D}(Q) = \mathfrak{D}\bigl(\sqrt{A}\bigr) = \mathfrak{D}\bigl(\sqrt{A_1}\bigr).$$

Hence for every $y \in \mathfrak{D}(A_1)$ we have

$$(x, A_1 y) = Q(x,y) = (\sqrt{A}x, \sqrt{A}y), \quad x \in \mathfrak{D}(Q).$$

This equality means that

$$\sqrt{A}y \in \mathfrak{D}\Bigl(\bigl(\sqrt{A}\bigr)^*\Bigr) = \mathfrak{D}\bigl(\sqrt{A}\bigr) \quad \text{and} \quad \bigl(\sqrt{A}\bigr)^*\sqrt{A}y = A_1 y,$$

i.e., $y \in \mathfrak{D}(A)$ and $Ay = A_1 y$. Thus, $A_1 \subset A$. Similarly, $A \subset A_1$. □

Suppose now that T is a symmetric densely defined operator semibounded from below. By Corollary 10.7.4 it has the closure that is a closed symmetric operator. This closure is also semibounded from below, because for every $x \in \mathfrak{D}(\overline{T})$ there is a sequence $\{x_n\} \subset \mathfrak{D}(T)$ such that $x_n \to x$ and $Tx_n \to \overline{T}x$. Hence the estimate $(Tx_n, x_n) \geqslant m(x_n, x_n)$ yields the estimate $(\overline{T}x, x) \geqslant m(x,x)$. Hence the operator T has selfadjoint extensions. The connection with quadratic forms described above enables us to construct a selfadjoint extension semibounded from below. To this end we shall deal with the closure of T and replace T by the positive operator $T_0 := T + \lambda I$ for some $\lambda > m$. Let us consider the positive definite form $Q(x) = (T_0 x, x)$.

10.7.17. Lemma. *Let Q be a positive definite quadratic form on a dense linear subspace $\mathfrak{D}(Q)$ in H. Suppose that the following condition is fulfilled*:

(C) *if a sequence $\{x_n\} \subset \mathfrak{D}(Q)$ is Cauchy with respect to the norm $\|\cdot\|_Q$ and $\|x_n\| \to 0$, then $Q(x_n, y) \to 0$ for all $y \in \mathfrak{D}(Q)$.*

Then the form Q is closable in the following sense: *there is a closed quadratic form $\overline{Q}$ with a domain of definition $\mathfrak{D}(\overline{Q})$ in H such that $\mathfrak{D}(\overline{Q})$ is the completion of $\mathfrak{D}(Q)$ with respect to the norm $\|\cdot\|_Q$ and $\overline{Q}|_{\mathfrak{D}(Q)} = Q$.*

PROOF. Let $\{x_n\}$ be a sequence in $\mathfrak{D}(Q)$ fundamental with respect to the norm $\|\cdot\|_Q$. Then it is fundamental in H and hence converges in H to some element $x \in H$. Let $\mathfrak{D}(\overline{Q})$ consist of all elements in H obtained in this way. Set $\overline{Q}(x,x) = \lim\limits_{n\to\infty} Q(x_n, x_n)$. We show that $\mathfrak{D}(\overline{Q})$ can be identified with the completion of $\mathfrak{D}(Q)$. For this we have to verify the following: if two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in $\mathfrak{D}(Q)$ converge in the completion to different elements, then their limits in H are also different. If they have a common limit x in H, then we obtain the sequence $z_n := x_n - y_n$ that is Cauchy with respect to the norm $\|\cdot\|_Q$ and converges to zero in H. By condition (C) we have $Q(z_n, y) \to 0$ for all $y \in \mathfrak{D}(Q)$. Since $\mathfrak{D}(Q)$ is dense in the completion, this means that $\{z_n\}$ converges in the completion to zero, contrary to our supposition that $\{x_n\}$ and $\{y_n\}$ converge to different elements of the completion. It is easy to see that we have obtained the desired closure. □

Thus, the form $Q(x) = (Tx, x)$ has the closure, moreover, the closure is also positive definite. According to the established facts there exists a positive selfadjoint operator T_0 generating the closure of our form. This operator is called the *Friedrichs extension* of the operator $T + \lambda I$, and the operator $T_0 - \lambda I$ is called the Friedrichs extension of the operator T. It is not always the closure of T.

10.7.18. Example. Let the operator $A = -d^2/dt^2$ be defined on the domain $\mathfrak{D}(A) = C_0^\infty(0,1)$. Then

$$(A\varphi, \varphi) = \int_0^1 |\varphi'(t)|^2\, dt \geqslant 0.$$

The completion of $\mathfrak{D}(A)$ with respect to the norm $\big((A\varphi,\varphi) + (\varphi,\varphi)\big)^{1/2}$ coincides with the Sobolev space $W_0^{2,1}[0,1]$ of all absolutely continuous functions φ such that $\varphi(0) = \varphi(1) = 0$ and $\varphi' \in L^2[0,1]$. The Friedrichs extension of the operator A is the nonnegative selfadjoint operator $\widetilde{A} = -d^2/dt^2$ on the domain $\mathfrak{D}_0 = \{\varphi \in W^{2,2}[0,1]\colon \varphi(0) = \varphi(1) = 0\}$. However, A has other nonnegative selfadjoint extensions: for example, $A_1 = -d^2/dt^2$ on the domain $\mathfrak{D}_1 = \{\varphi \in W^{2,2}[0,1]\colon \varphi'(0) = \varphi'(1) = 0\}$. In addition, there are selfadjoint extensions of the operator A that are not nonnegative: for example, $A_2 = -d^2/dt^2$ on the domain $\mathfrak{D}_1 = \{\varphi \in W^{2,2}[0,1]\colon \varphi'(0) = \varphi(0), \varphi'(1) = \varphi(1)\}$. Note that the operator A is not essentially selfadjoint (its closure is defined on functions $\varphi \in W^{2,2}[0,1]$ with $\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0$), which yields different selfadjoint extensions bounded from below (Exercise 10.7.33).

The proof of the next assertion can be found in [**67**, Chapter 10, §3].

10.7.19. Theorem. *Suppose that A is a densely defined symmetric operator bounded from below. Let $\widetilde{A}$ be its Friedrichs extension and let $\mathfrak{D}_Q$ be the domain of definition of the closure of the form generated by the operator A.*

(i) *Let A_1 be a selfadjoint extension of A such that $\mathfrak{D}(A_1) \subset \mathfrak{D}_Q$. Then we have $A_1 = \widetilde{A}$.*

(ii) *Let A_1 be a bounded from below selfadjoint extension of A and Q_1 the corresponding form. Then $\mathfrak{D}_Q \subset \mathfrak{D}_{Q_1}$ and $Q_1 = Q$ on $\mathfrak{D}_Q$.*

The previous example shows that the Friedrichs extension need not be a unique selfadjoint extension bounded from below. A description of all extensions bounded from below can be found in [**6**, §109].

10.7(iii). The Chernoff and Trotter theorems

Here we prove two interesting results connected with approximation of operator semigroups. The first result — Chernoff's theorem — was actually found later than Trotter's theorem, which we obtain below as a corollary. First we establish an auxiliary result.

10.7.20. Lemma. *Let L and L_n for every $n \in \mathbb{N}$ be the generators of strongly continuous contracting semigroups on a complex Banach space X and let $D \subset \mathfrak{D}(L) \bigcap \bigl(\bigcap_{n\geqslant 1} \mathfrak{D}(L_n)\bigr)$ be a linear subspace such that $(\lambda I - L)(D)$ is dense in X. Suppose that $L_n x \to Lx$ as $n \to \infty$ for every $x \in D$. Then, for every element $x \in X$, whenever $\operatorname{Re}\lambda > 0$ we have $(\lambda I - L_n)^{-1}x \to (\lambda I - L)^{-1}x$.*

PROOF. Let $\psi = (\lambda I - L)\varphi$, $\varphi \in D$. We recall that $\|(\lambda I - L_n)^{-1}\| \leqslant |\lambda|^{-1}$. Hence

$$\begin{aligned}&\|(\lambda I - L_n)^{-1}\psi - (\lambda I - L)^{-1}\psi\|\\ &\quad = \|(\lambda I - L_n)^{-1}(\lambda I - L_n)\varphi - (\lambda I - L_n)^{-1}(L_n - L)\varphi - \varphi\|\\ &\quad = \|(\lambda I - L_n)^{-1}(L_n - L)\varphi\| \leqslant |\lambda|^{-1}\|(L_n - L)\varphi\|.\end{aligned}$$

Thus, the lemma is true for all ψ from a dense set. On account of the bounds $\|(\lambda I - L_n)^{-1}\| \leqslant 1$ and $\|(\lambda I - L)^{-1}\| \leqslant 1$ this gives the desired assertion. □

10.7.21. Theorem. *Let X be a Banach space and let $F\colon\ [0,+\infty) \to \mathcal{L}(X)$ be a mapping continuous on every vector and satisfying the following conditions: $F(0)=I$, $\|F(t)\|\leqslant e^{at}$ with some constant a and there is a dense linear subspace $D \subset X$ such that for all $x \in D$ there exists a limit $F'(0)x := \lim\limits_{t\to 0} t^{-1}[F(t)x - x]$. Suppose that $F'(0)$ on D has the closure C that is the generator of a strongly continuous semigroup $\{T_t\}_{t\geqslant 0}$. Then, for every $x \in X$, as $n \to \infty$ we have $F(t/n)^n x \to T_t x$ uniformly in t from every compact interval.*

PROOF. It suffices to consider the case where $a = 0$ and $\{T_t\}_{t\geqslant 0}$ is a contracting semigroup (multiplying $F(t)$ by e^{-tC} with a sufficiently large $C > 0$). Let us note the following fact: if $T \in \mathcal{L}(X)$ and $\|T\| \leqslant 1$, then the operators $\exp\bigl(t(T-I)\bigr)$ form a contracting semigroup and

$$\Bigl\|\bigl[\exp\bigl(n(T-I)\bigr) - T^n\bigr]x\Bigr\| \leqslant n^{1/2}\|(T-I)x\|.$$

Indeed, $\big\|\exp\big(t(T-I)\big)\big\| \leqslant e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}\|T\|^k \leqslant 1$. In addition,

$$\Big\|\big[\exp\big(n(T-I)\big)-T^n\big]x\Big\| = \Big\|e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}(T^k-T^n)x\Big\|$$
$$\leqslant e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}\|(T^k-T^n)x\|$$
$$\leqslant e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}\|(T^{|k-n|}-I)x\| \leqslant e^{-n}\sum_{k=0}^{\infty}\frac{|k-n|n^k}{k!}\|(T-I)x\|,$$

which is estimated by $n^{1/2}\|(T-I)x\|$, since

$$\sum_{k=0}^{\infty}|k-n|n^k/k! \leqslant \Big(\sum_{k=0}^{\infty}|k-n|^2n^k/k!\Big)^{1/2}\Big(\sum_{k=0}^{\infty}n^k/k!\Big)^{1/2} \leqslant (ne^n)^{1/2}e^{n/2}$$

by the easily verified equality $\sum_{k=0}^{\infty}|k-n|^2n^k/k! = ne^n$. Let us fix $\tau>0$ and set $T := F(\tau/n)$, $L_n := n\tau^{-1}\big(F(\tau/n)-I\big)$. According to the assertion proved above, the operator $T-I$ generates a contracting semigroup $\exp[t(T-I)]$. Hence the operator L_n also generates a contracting semigroup $\exp(tL_n)$. By our conditions and Theorem 10.5.8 one has the pointwise convergence of the sequence of operators $\exp\big[n\big(F(\tau/n)-I\big)\big] = \exp(\tau L_n)$ to T_τ. Let $x\in D$. Applying the aforementioned assertion once again, as $n\to\infty$ we obtain

$$\|\exp(\tau L_n) - F(\tau/n)^n x\| \leqslant n^{1/2}\big\|\big(F(\tau/n)-I\big)x\big\|$$
$$= \frac{\tau}{n^{1/2}}\big\|n\tau^{-1}\big(F(\tau/n)-I\big)x\big\| \to 0.$$

Thus, for any $x\in D$ we have $F(\tau/n)^n x \to T_\tau x$. Then the same remains in force for all $x\in X$, since D is dense and $\|F(\tau/n)\|\leqslant 1$ in the considered case. A closer look at the proof shows that convergence is uniform in τ from every compact interval. □

10.7.22. Corollary. *Let A and B be the generators of two strongly continuous contracting semigroups $\exp(tA)$ and $\exp(tB)$ on X. Suppose that $\mathfrak{D}(A)\cap\mathfrak{D}(B)$ is dense and the operator $A+B$ on this domain possesses the closure C that is the generator of a strongly continuous contracting semigroup $\exp(tC)$. Then $\exp(tC)x = \lim\limits_{n\to\infty}\big(\exp(tA/n)\exp(tB/n)\big)^n x$ for every $x\in X$ uniformly in t from every compact interval.*

PROOF. It suffices to set $F(t)=\exp(tA)\exp(tB)$. Then

$$\frac{F(t)-I}{t}x = t^{-1}[\exp(tA)(\exp(tBx-x)+(\exp(tA)x-x)] \to Bx+Ax$$

for all $x\in\mathfrak{D}(A)\cap\mathfrak{D}(B)$, hence we can apply the theorem. □

10.7.23. Example. Let $\{T_t\}_{t\geqslant 0}$ be a strongly continuous contracting semigroup with generator L on a Banach space X. Then

$$\lim_{n\to\infty}(I-tn^{-1}L)^{-n}x = T_t x, \quad x\in X.$$

Indeed, let us set

$$F(t) := (I - tL)^{-1} = \int_0^\infty e^{-s} T_{st}\, ds$$

and observe that $F'(0) = L$ on $\mathfrak{D}(L)$.

Note that in Chernoff's theorem the operator $F'(0)$ is automatically closable if it is densely defined. To see this it suffices to consider the case $a = 0$ in which the operator $F'(0)$ is obviously dissipative (since $l(F(t)x) \leqslant 1$ if $\|l\| = \|x\| = l$), hence is closable, see Exercise 10.7.42. Hence the main condition is that the set $\bigl(I - F'(0)\bigr)(D)$ is dense. It does not follow from the dissipativity: it suffices to take a nonpositive symmetric operator that is not essentially selfadjoint.

10.7(iv). The mathematical model of quantum mechanics

The mathematical model of quantum mechanics used at present is described by a set of axioms formulated in the language of the theory of operators on a Hilbert space and introduced in the book by von Neumann [**455**] published in German in 1932. Actually, this model of von Neumann is a formalization of the model introduced in Dirac's book [**153**], the first edition of which was published in 1930. It is worth noting that one year after the publication of von Neumann's book, Kolmogorov's monograph "Foundations of probability theory" was published (also first in German). These two books by von Neumann and Kolmogorov give together a solution to the 6th Hilbert problem (one of the 23 problems posed in his lecture at the 2nd International Congress of Mathematicians in Paris in 1900) concerned with an axiomatic construction of probability theory and mechanics. The principal mathematical objects used for constructing the described model are complex separable Hilbert spaces and selfadjoint operators on such spaces. The principal physical objects are a quantum system (the investigated physical object, for example, an elementary particle, a family of such particles or a quantum computer), its observables (physical quantities that can be measured) and states. States can be pure and mixed. A state of the quantum system is called pure if it is not a probability mixture of other states. Every mixed state coincides with a probability mixture of pure states.

We now list the axioms with certain comments.

Axioms. 1. To every quantum system we associate a complex separable Hilbert space H, called its state space. To observables we associate selfadjoint operators on H (which in the framework of the considered model are often called observables). To pure states we associate orthogonal projections onto one-dimensional subspaces in H. Since the projection operator can be identified with its range, we can say that pure states are defined by one-dimensional subspaces in H (i.e., by elements of the projective Hilbert spaces). In addition, pure states can be represented by vectors generating the corresponding one-dimensional subspaces (they can be taken normalized). These vectors are called state vectors of the quantum system; for this reason H is called the state space. If every vector from the space H defines some (pure) state, then we say that the quantum system has no superselection rules (superselection rules are discussed in Comment 8).

Axioms. 2. A time evolution of a state of the quantum system is described by a one-parameter strongly continuous group of unitary operators $U(t)$. This means that if $\psi(t_1)$ is a state vector of the quantum system at time t_1, then its state vector at time t_2 is $\psi(t_2) = U(t_2 - t_1)\psi(t_1)$. According to Stone's theorem, we have $U(t) = \exp\bigl(-i\frac{t}{\hbar}\widehat{H}\bigr)$, where $\hbar$ is the Planck constant and $\widehat{H}$ is a selfadjoint operator, called the Hamiltonian of the system; it corresponds to (= is) an observable, called the energy. It is usually assumed that the operator $\widehat{H}$ is nonnegative definite or semibounded (from below), because it is believed that the energy of a physical system cannot decrease unboundedly.

Comments. 1. We have the equality $i\hbar\psi'(t) = \widehat{H}\psi(t)$, called the Schrödinger equation.

Axioms. 3. If $A_1, A_2, \ldots, A_n$ are pairwise commuting observables, then there exists an experiment that enables us to measure these observables simultaneously for any state of the system. Moreover, if ψ is the vector defining the state of the quantum system and $\Pi_j, j = 1, 2, \ldots, n$, are the projection-valued measures for A_j (see §7.9, §7.10(ii)), then the probability that the measured values of the observables A_j are contained in Borel sets $B_j \subset \mathbb{R}$ with $j = 1, \ldots, n$, equals the quantity $\|\Pi_1(B_1) \cdots \Pi_n(B_n)\psi\|^2/\|\psi\|^2$.

Comments. 2. The expectation of the result of the measurement of the observable A in the pure state given by a vector ψ equals $(A\psi, \psi)/\|\psi\|^2$ (if the right-hand side is defined). Axiom 3 in turn follows (under certain assumptions) from this assertion.

Axioms. 4. An experiment of measurement of an observable A with a discrete spectrum can be performed in such a way that if the original state was pure and represented by a vector φ and the result of measurement gave a number λ, then immediately after measurement the system will be in a pure state that is the projection of the vector φ onto the subspace corresponding to the eigenvalue λ. Passage from the vector φ to its projection is called reduction of a state vector.

Comments. 3. This axiom is a reinforcement of the original axiom of von Neumann (according to which the dimension of the corresponding subspace equals one), called sometimes the Lüders postulate; it is still arguable in the literature whether this postulate can be accepted. Note that in the well-known textbook by Landau and Lifshits the original axiom of von Neumann is also rejected.

Comments. 4. In order to extend the axiom about reduction of state vectors to the case of observables with continuous spectra it is necessary to enlarge the considered model by using the theory of distributions. Actually, this was already done by Dirac himself (before von Neumann's axiomatic), but at a heuristic level.

Comments. 5. There are two types of evolution of quantum systems: the one described by Axiom 2 (unitary or Hamiltonian) and the one described by Axiom 4 concerning reduction of state vectors. In the first case the future of the state of the system is uniquely determined by its initial state; in the second case the future of the state depends on the initial state only in a probabilistic way. The question in which degree the second type of evolution can be reduced to the first type is still under active discussion. Evolution of the first type takes place in an isolated system (such systems are called closed); evolution of the second type is the result of interaction of the system with another system — the measurement device.

Axioms. 5. If a quantum system with a Hilbert space H consists of two subsystems with Hilbert spaces H_1 and H_2, respectively, then it is assumed that H can be identified with the Hilbert tensor product of the spaces H_1 and H_2 (see §7.10(vi); here the Hilbert tensor product will be denoted by $H_1 \otimes H_2$) or with one of its subspaces: the closure of the subspace of symmetric tensors or the closure of the space of antisymmetric tensors (the first option is used for describing particles called bosons and the second one for describing particles called fermions). It is supposed that if the quantum systems H_1 and H_2 are in pure states represented by vectors x_1 and x_2, respectively, then the state of the joint system is also pure and represented by the vector $x_1 \otimes x_2$. In addition, if A_1 and A_2 are observables of the corresponding subsystems, then their measurement is the same as the measurement of the observables $A_1 \otimes I_2$ and $I_1 \otimes A_2$ in the joint system; here and below $I_1,\ I_2$ are the identity mappings of the corresponding spaces. Quantum systems with Hilbert spaces H_1 and H_2 regarded as subsystems of their union are called open.

Comments. 6. The following theorem holds. *Suppose that the joint system is in the pure state represented by a vector $\varphi \in H = H_1 \otimes H_2$. Then there exists a nonnegative nuclear operator T^1_φ on H_1 with $\operatorname{tr} T^1_\varphi = 1$ such that for every bounded selfadjoint operator A on H_1 we have $\operatorname{tr}(AT^1_\varphi) = (A \otimes I_2 \varphi, \varphi)_H$.*

Indeed, let $\{e^2_n\}$ be an orthonormal basis in H_2; then the relation

$$(T^1_\varphi x, z)_{H_1} = \|\varphi\|^{-2} \sum_{n=1}^{\infty} (x \otimes e^2_n, \varphi)_H \overline{(z \otimes e^2_n, \varphi)_H}, \quad x, z \in H_1,$$

uniquely defines T^1_φ. The operator T^1_φ is called a density operator (generated by the state φ of the joint system); we shall say that it defines a mixed state of the system with the Hilbert space H_1 (generated by the pure state of the joint system). Every nonnegative nuclear operator with unit trace can be obtained in this way.

Comments. 7. In the framework of the considered model a density operator on a Hilbert space K is defined as an arbitrary nonnegative nuclear operator with trace 1. Let $L^+_1(K)$ be the set of all such operators. We shall say that an operator $T \in L^+_1(K)$ defines a mixed state; if the system is in this state, then the mean value of results of measurement of the observable A equals $\operatorname{tr} AT$ (if the expression on the right is defined; for example, this is true for bounded A). Pure states are a particular case of mixed states; namely, a pure state given by a normalized vector φ is also defined by the density operator $\varphi \otimes \varphi$ (which acts by the formula $\varphi \otimes \varphi \colon x \mapsto \varphi(x, \varphi)$). If a state of a joint system with a Hilbert space $H = H_1 \otimes H_2$ is itself mixed and given by a density operator S, then there exists a density operator T^1_S on H_1 and a density operator T^2_S on H_2 such that for all bounded observables A_1 on H_1 and A_2 on H_2 we have $\operatorname{tr}(A_1 T^1_S) = \operatorname{tr}[(A_1 \otimes I_2)S]$ and $\operatorname{tr}(A_2 T^2_S) = \operatorname{tr}[(I_1 \otimes A_2)S]$. The states T^1_S and T^2_S are called reductions of the state S; it is also customary to say that T^1_S is the partial trace of the operator S with respect to H_2 and T^2_S is the partial trace of the operator S with respect to H_1;

these partial traces are denoted by the symbols $\mathrm{tr}_{H_2} S$ and $\mathrm{tr}_{H_1} S$, respectively, the same symbols are used to denote the usual traces in the same spaces. Thus, $\mathrm{tr}_{H_1}(\mathrm{tr}_{H_2} S) = \mathrm{tr}\, S = \mathrm{tr}_{H_2}(\mathrm{tr}_{H_1} S)$.

We emphasize that by means of the states T_S^1 and T_S^2 of the subsystems that are reductions of the state S of the joint system this state S can be reconstructed only in the case when both states T_S^1 and T_S^2 are pure (then $S = T_S^1 \otimes T_S^2$). However, in the general case this equality can fail. This will be the case if the state S is pure and T_S^1 and T_S^2 are mixed. Thus, in quantum mechanics, unlike the classical mechanics, the states of subsystems can fail to determine the states of the joint system (but, as we have seen, the state of the joint system determines the states of the subsystems).

The one-parameter group $U(t)$ of unitary operators from Axiom 2 defines one-parameter strongly continuous groups (also called dynamical) $F(t)$ and $G(t)$ of continuous mappings of the Banach space of nuclear operators into itself and of the Banach space of bounded observables into itself in following way:

$$F(t)T = U(t)TU(t)^*, \quad G(t)A = U(t)^* AU(t).$$

For every t, the mapping $F(t)$ is the extension by continuity of the mapping on finite linear combinations of pure states that is in turn the extension by linearity of the mapping $U(t)$. Every mapping $G(t)$ is adjoint to the corresponding mapping $F(t)$; in the definition of adjoint mappings we use Theorem 7.10.40, according to which the space of all continuous linear operators on a Hilbert space can be identified with the dual to the space of all nuclear operators (any continuous linear operator A is identified with the functional $T \mapsto \mathrm{tr}\, AT$). In this case the functions $t \mapsto F(t)T$ and $t \mapsto G(t)A$ are solutions to the Cauchy problems for the equations

$$f'(t) = i[\widehat{H}, f(t)]/\hbar, \quad g'(t) = i[g(t), \widehat{H}]/\hbar$$

with initial conditions T and A, respectively. Usually these equations are called the Heisenberg equations, although the first of them is called sometimes the Schrödinger equation and the second one is called the Liouville equation. It is customary to say that the groups $U(t)$, $F(t)$ and $G(t)$, and also the Heisenberg and Schrödinger equations, define the Hamiltonian dynamics of the quantum system. If $U_H(t)$ is a one-parameter group of unitary transformations of H defining the dynamics of the quantum system, then, for any state T of the joint system, the function $\mathcal{F}$ given by the equality $\mathcal{F}(t) = \mathrm{tr}_{H_2} U(t)TU(t)^*$ defines the dynamics of the first subsystem generated by the dynamics of this joint system. Of course, the function $\mathcal{F}$ need not be a one-parameter semigroup. The equation it satisfies is called the master equation. Nevertheless, for describing the so-called Markov approximation of the dynamics of quantum systems interacting with quantum fields it becomes useful to employ limits of some families of functions like the function $\mathcal{F}$, depending on a parameter characterizing interaction (see Accardi, Lu, Volovich [**3**]). These limit functions are one-parameter semigroups and satisfy certain equations analogous to the backward Kolmogorov equation in the theory of diffusion processes.

Comments. 8. Superselection rules (if they are introduced) are defined by means of some projection-valued measure Π. It is supposed that physically admissible are only observables commuting with all operators $\Pi(B)$. If the measure Π is concentrated on a finite or countable set, then in a similar way physically admissible are only states representable by density operators commuting with all operators $\Pi(B)$. The set of vectors corresponding to physically admissible pure states coincides with the union of ranges of all projections $\Pi(\{q_j\})$, $j = 1, 2, \ldots$ If there are no superselection rules in a quantum system, then every linear combination of vectors representing (pure) states is again a vector representing some pure state; this fact, called the principle of superposition, reflects the so-called wave properties of quantum systems. The principle of superposition yields a fundamental difference between quantum and classical systems.

Comments. 9. Every mixed state can be obtained as a probability mixture of pure states. Namely, let ν be a probability measure on K such that $\nu(\{0\}) = 0$. Then the expectation values of the results of measurement of a bounded observable A in supposition that this measure determines the probability distribution of pure states equals $\displaystyle\int (Az, z)_K \|z\|^{-2} \, \nu(dz)$. It follows from Theorem 7.10.40 that there exists a nuclear nonnegative operator S on K such that for every bounded observable A this integral equals $\mathrm{tr}\,(AS)$ (the theorem gives the equality first on compact operators, but then it extends to all operators by pointwise approximation). This implies that for a measure ν (defined by the equality not uniquely) one can take the measure that is the product of the function $z \mapsto \|z\|^2$ and another probability measure μ. This means that $\mathrm{tr}\,(AS)$ is the integral of $(Az, z)_K$ against the measure μ, so that for μ we can take any probability measure with zero mean and the correlation operator S (among such measures there exists precisely one Gaussian measure). Thus, every mixed state can be obtained by two principally different procedures: as a probability mixture of pure states and as a state generated by a pure state of some larger system.

Comments. 10. The density operator characterizes the membership of the quantum system in some statistical ensemble (by the way, it is still under discussion whether the element of the Hilbert space defining a pure state of the quantum system is a characteristic of the individual system or only of the statistical ensemble to which it belongs). It should be emphasized that by means of an experiment it is impossible to distinguish mixed states obtained by the two procedures described at the end of the previous comment. However, if it is known a priori that the mixed state of the quantum system is a probability mixture of pure states, then this means that each copy of the corresponding statistical ensemble is in some pure state (if the system is in a mixed state generated by a pure state of some larger system, then the assumption that actually every copy of the corresponding statistical ensemble of copies of this system is in some pure state and the assumption about finite speed of propagation of interactions lead together to a contradiction with Bell's inequality from the classical probability theory). One can raise the question about estimating this state by means of a result of an individual experiment.

If, for example, the probability measure characterizing a mixed state is concentrated on the set of two orthogonal vectors $\{h, k\}$ (say, of unit norm), then for determining in which of the two pure states the considered system is, it suffices to accomplish a measurement of every observable that is a selfadjoint operator with eigenvectors h and k.

If some mixed state is given by a probability measure concentrated on the set of three vectors h, $\cos\frac{\pi}{3}h + \sin\frac{\pi}{3}k$, $\cos\frac{\pi}{3}h - \sin\frac{\pi}{3}k$ and its values do not vanish on singletons in this set, then there is no experiment enabling one to determine precisely in which of the three pure states the considered system is.

Nevertheless, there exists an experiment enabling one to give the best (in some sense) estimate of this state. It consists of a measurement of some observable A belonging to an enlarged quantum system obtained by adding to the investigated system (with a Hilbert space H) an auxiliary quantum system (with a Hilbert space K) that is in an especially selected pure state S_0. If Π_A is the projection-valued measure generated by the operator A, then the operator-valued set function $E(\,\cdot\,)$ defined by the equality $E(B) = \mathrm{tr}_K\big[(I_H \otimes S_0)\Pi_A(B)\big]$ is a measure with values in the space of bounded nonnegative operators on H (called the partition of unity). In this case, if the state of the original quantum system is represented by a density operator S, then the probability that the measurement just described gives a value from a Borel set B equals $\mathrm{tr}\,[SE(B)]$. More details can be found in the book Holevo [**279**]. The corresponding proofs are based on Naimark's theorem, according to which every symmetric operator on a Hilbert space possesses a selfadjoint extension in some larger Hilbert space.

Comments. 11. The presented system of axioms of quantum mechanics is not complete, because there exist non-isomorphic quantum systems. A large class of systems is obtained by the procedure of quantization of Hamiltonian systems of classical mechanics.

A symplectic locally convex space is a pair (E, I), where E is a locally convex space over $\mathbb{R}$ and $I\colon E \to E^*$ is a linear mapping (E^* is equipped with a topology that agrees with duality between E^* and E) such that $I^* = -I$. A Poisson bracket of complex or real functions f and g on E is the function $\{f, g\}$ on E defined by $\{f, g\}(x) = f'(x)(I(g'(x)))$. Now let $E = Q \times P (= Q \oplus P)$, where Q is a locally convex space and $P = Q^*$ is equipped with a topology consistent with duality between P and Q. The space E is identified with a space of linear functionals on it by means of the mapping associating to the element $(q, p) \in E$ the linear functional on E denoted by the symbol (p, q) and given by the formula $\langle (p, q), (q', p')\rangle = q(p') + q'(p)$. The mapping I is defined as follows: $I(p, q) = (q, -p)$. The linear space of smooth functions on E with the operation of multiplication defined by the Poisson bracket forms a Lie algebra, which is denoted by P_E; its subalgebra P_C generated by linear functions is a central extension of the commutative Lie algebra $E^* = E$ with the trivial multiplication (P_C is called the Heisenberg algebra generated by E). A mapping from the Lie algebra P_C to the set of selfadjoint operators on a Hilbert space H such that for all $k, g \in P_C$, $a, b \in \mathbb{R}$ we have $F(ak + bg) = aF(k) + bF(g)$ and $iF(\{k, g\}) = [F(k), F(g)] = F(k)F(g) - F(g)F(k)$ is called a representation in H of the canonical commutation relations (CCR) in the Heisenberg form; a unitary representation of the central extension E_C of the commutative group that is the additive group of $E = E^*$ (this extension is called the Heisenberg group or the Weyl group) is called a representation of the CCR in Weyl's form.

A Hamiltonian system is a collection $(E, I, \mathcal{H})$, where (E, I) is a symplectic locally convex space, called the phase space, and $\mathcal{H}$ is a real function on E, called the Hamiltonian. The Hamilton equation is the following equation with respect to the function f of a real argument, interpreted as time, taking values in E:

$$f'(t) = I\big(\mathcal{H}'(f(t))\big).$$

If the space Q is finite-dimensional, then this equation coincides with the standard system of Hamilton equations. An example of an infinite-dimensional Hamilton equation is the Schrödinger equation (in case of an infinite-dimensional space H of the quantum system). Indeed, let E be the realification of H, I the operator on E generated by multiplication by the imaginary unit on H and $\mathcal{H}(x) = -\frac{i}{2\hbar}(\widehat{H}x, x)$. Then $(E, I, \mathcal{H})$ is the Hamiltonian system and the corresponding Hamilton equation coincides with the Schrödinger equation.

A quantization of a Hamiltonian system is a procedure of defining an operator $\mathcal{H}(F(p), F(q))$ on H, where F is the representation of the CCR in the Heisenberg form. Since the operators $F(q)$ and $F(p)$ do not commute, there is no unique "natural" definition of the operator $\mathcal{H}(F(q), F(p))$; moreover, in the infinite-dimensional case also a representation of the CCR is not unique. If the space Q is finite-dimensional, then a privileged role is played by the so-called Schrödinger representation. For every $p \in P$, let $F(p)$ denote the operator of multiplication by the linear functional $q \mapsto p(q)$ acting on the space $L_2(Q)$; for every $q \in Q$, let $F(q)$ denote the operator $g \mapsto -i\hbar g'q$ on the same space; $F(q)$ is the impulse operator in the direction q. If $\mathcal{H}(q,p) = V(q) + \frac{1}{2}(Bp, p)$, where B is a linear operator on Q, then as the result of application of the described method of quantization we obtain the standard Schrödinger equation (with a potential V and an "anisotropic mass", characterized by the operator B). This is the form of the Hamiltonian of the Hamiltonian system introduced above, for which for the Hamiltonian equation we obtain the Schrödinger equation. The quantization of this Hamiltonian system is called the second quantization; the name is explained by the fact that usually the original Schrödinger equation is also obtained by means of a quantization. If E is infinite-dimensional, then it has no reasonable analog of Lebesgue measure (there is no nonzero translation-invariant σ-finite locally finite countably additive Borel measure); however, in place of it one can use a suitable Gaussian measure (the space of functions on Q quadratically integrable with respect to this measure is called the Wiener–Segal–Fock space).

10.7(v). Sturm–Liouville operators

We know that the operator $D = id/dt$ is selfadjoint on $L^2[0,1]$ on the domain consisting of absolutely continuous functions x with a derivative in $L^2[0,1]$ satisfying the boundary condition $x(0) = x(1)$. Theorem 10.3.3 yields the selfadjointness of the operator $D^2x = -x''$ of taking the second derivative on the domain consisting of functions x for which the derivative x' is absolutely continuous, $x'' \in L^2[0,1]$, and $x(0) = x(1)$, $x'(0) = x'(1)$.

We now study a more general Sturm–Liouville operator

$$Lx(t) = -\big(p(t)x'(t)\big)' + q(t)x(t),$$

where the function p is continuously differentiable and strictly positive on $[0,1]$ and the function q is continuous and real. Let us take for the domain of definition $\mathfrak{D}(L)$ the class of all absolutely continuous functions x on $[0,1]$ for which the function x' is absolutely continuous, $x'' \in L^2[0,1]$, and the boundary conditions $x(0) = x(1) = 0$ hold. On this domain the operator L is symmetric, since

$$\int_0^1 Lx(t)\overline{y(t)}\,dt = \int_0^1 \big[p(t)x'(t)\overline{y'(t)} + q(t)x(t)\overline{y(t)}\big]\,dt$$

and the same holds for $x(t)\overline{Ly(t)}$. But we have not yet proved that L is selfadjoint! It follows from the symmetry that if L has eigenvectors, then the corresponding eigenvalues are real and the eigenvectors corresponding to different eigenvalues are orthogonal. Hence the set of all eigenvalues is at most countable.

Suppose that $\operatorname{Ker} L = 0$; to this case we can pass by replacing L with the operator of the same form $L - \lambda I$, where $\lambda \in \mathbb{R}$ is not an eigenvalue (which means that we replace q with $q - \lambda$). The selfadjointness of L can be seen from the following reasoning. The operator $L_0 x = -x''$ is also injective on $\mathfrak{D}(L)$ and maps $\mathfrak{D}(L)$ onto $L^2[0,1]$. The space $\mathfrak{D}(L)$ is a closed subspace in the Sobolev space $H = W^{2,2}\big((0,1)\big)$, which is a Hilbert space with the inner product $(u,v)_H := (u,v)_{L^2} + (u',v')_{L^2} + (u'',v'')_{L^2}$. Since the operator $L_0\colon H \to L^2[0,1]$ is continuous, the operator $L_0\colon \mathfrak{D}(L) \to L^2[0,1]$ is invertible. Then the operator $L_1 x = -px''$ from $\mathfrak{D}(L)$ to $L^2[0,1]$ is also invertible, since multiplication by the positive continuous function p is an invertible operator on $L^2[0,1]$. Our operator $L\colon \mathfrak{D}(L) \to L^2[0,1]$ has the form $L = L_1 + S$, where the operator $Sx = -p'x' + qx$, as is easily verified, is compact from H with the indicated Hilbert norm to $L^2[0,1]$; hence it is also compact on $\mathfrak{D}(L)$. The injectivity of L gives the surjectivity, since $L = L_1(I + L_1^{-1}S)$, where $L_1^{-1}S$ is compact on $\mathfrak{D}(L)$.

The operator $K = L^{-1}\colon L^2[0,1] \to \mathfrak{D}(L)$ is symmetric. Our discussion shows that it takes the unit ball from $L^2[0,1]$ to a set bounded in H. The ball of H is compact in $L^2[0,1]$ (even in $C[0,1]$). Hence L is the inverse to the selfadjoint compact operator K and is also selfadjoint on $\mathfrak{D}(L) = K(L^2[0,1])$. By the Hilbert–Schmidt theorem K has an orthonormal eigenbasis $\{e_n\}$ with eigenvalues $k_n \to 0$ of finite multiplicity. Hence L has the same eigenbasis and eigenvalues $\lambda_n = k_n^{-1}$, where $|\lambda_n| \to \infty$. We recall that all this has been done assuming the injectivity of L. In the general case, as noted above, we have the injective operator $L - \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$, so it has an orthonormal eigenbasis $\{e_n\}$ with real eigenvalues μ_n; in this basis L is also diagonal and has eigenvalues $\lambda_n = \mu_n + \lambda_0$.

Conclusion: *the equation $Lx = y$ with our boundary conditions is solvable for those and only those functions $y \in L^2[0,1]$ which are orthogonal in $L^2[0,1]$ to the subspace* $\operatorname{Ker} L$*, consisting of all solutions to the equation $Lu = 0$ with given boundary conditions.*

In case $\operatorname{Ker} L = 0$, the operator $K = L^{-1}$ is defined by a continuous real integral kernel $\mathcal{G}(t,s) = \mathcal{G}(s,t)$, called the Green's function of the operator L. To see this, we take two solutions u_1 and u_2 to the equation $Lu = 0$ with the boundary conditions $u_1(0) = 0$, $u_1'(0) = 1$ and $u_2(1) = 0$, $u_2'(1) = 1$. Since L is injective, the functions u_1 and u_2 are linearly independent. Then the Wronskian $\Delta := p(u_1'u_2 - u_1u_2')$ is a nonzero constant (its derivative equals zero). Finally, set $\mathcal{G}(t,s) = \Delta^{-1}u_1(s)u_2(t)$ if $0 \leqslant s \leqslant t \leqslant 1$, $\mathcal{G}(t,s) = \Delta^{-1}u_1(t)u_2(s)$ if $0 \leqslant t \leqslant s \leqslant 1$. It is straightforward to verify that K coincides with the integral operator $K_{\mathcal{G}}$ defined by $\mathcal{G}$, i.e., $LK_{\mathcal{G}}f = f$.

One can take more general selfadjoint boundary conditions; for example, for $p = 1$ and $q = 0$ one can obtain the aforementioned operator $(id/dt)^2$, and the boundary conditions we considered give a different selfadjoint extension of $-d^2/dt^2$ on $C_0^\infty(0,1)$.

In a similar way one can investigate multidimensional boundary value problems for second order elliptic operators (for example, for the Laplace operator Δ), although the Green's function cannot be constructed in such a simple manner.

Exercises

10.7.24.° Let T be a densely defined operator on a Hilbert space H such that T^* is densely defined and continuous. Prove that T is also continuous and extends to a bounded operator on H.

HINT: show that the adjoint operator for the extension of T^* by continuity extends T.

10.7.25.° Let S and T be densely defined operators on a Hilbert space H such that $S \subset T$. Prove that $T^* \subset S^*$.

10.7.26.° Let S and T be densely defined operators on a Hilbert space H such that $S \subset T$. Prove that if the operator T is closable, then S is closable as well and $\overline{S} \subset \overline{T}$.

10.7.27. Let S and T be densely defined closed operators on a Hilbert space H and suppose that the set $D := \mathfrak{D}(S) \cap \mathfrak{D}(T)$ is dense.

(i) Is it true that the operator $S + T$ on D is closable?

(ii) If $S + T$ is closable on D, then is it true that $(S+T)^* = S^* + T^*$ on the domain $\mathfrak{D}(S^*) \cap \mathfrak{D}(T^*)$? Is it true that $\mathfrak{D}(S^* + T^*) = \mathfrak{D}(S^*) \cap \mathfrak{D}(T^*)$, where on the left we take the domain of the closure?

HINT: (i) Take $H = L^2[0,1] \oplus \mathbb{C}$, $S(u,z) = \bigl(u', u(1/2)\bigr)$, $Tu = (-u', 0)$ for absolutely continuous functions u on $[0,1]$ with $u' \in L^2[0,1]$. Show that the operator $u \mapsto \bigl(0, u(1/2)\bigr)$ is not closable. (iii) Consider the case where $T = -S + I$.

10.7.28. Let T_0 be a closed operator on a Hilbert space H and let T be an operator such that $T_0 \subset T$ and $\mathfrak{D}(T_0)$ has a finite codimension in $\mathfrak{D}(T)$. Prove that T is closed.

HINT: observe that the Hilbert space D_{T_0} is a closed subspace of finite codimension in D_T. Hence D_T is also complete.

10.7.29. Investigate selfadjoint extensions of the operator $\varphi \mapsto i\varphi'$ on the domain of definition consisting of smooth compactly supported functions in the spaces $L^2(\mathbb{R}^1)$ and $L^2[0,+\infty)$.

10.7.30. Investigate selfadjoint extensions of the operator $\varphi \mapsto \varphi''$ on the domain of definition consisting of smooth compactly supported functions in the spaces $L^2(\mathbb{R}^1)$ and $L^2[0,+\infty)$.

10.7.31.° Let A be a bounded selfadjoint injective operator on a Hilbert space H. Show that the set $A(H)$ is dense in H, and the inverse operator $T = A^{-1}$ on the domain $\mathfrak{D}(T) := A(H)$ is selfadjoint (verify this directly by definition).

10.7.32. (i) Let A be a symmetric operator bounded from below and having a finite defect number. Prove that every selfadjoint extension of A is bounded from below (possibly, by a number less than the original bound). (ii) Construct an example of a symmetric operator bounded from below that has a selfadjoint extension not bounded from below.

HINT: see [**503**, Chapter X, Section 3].

10.7.33. Suppose that a symmetric operator A bounded from below is not essentially selfadjoint. Prove that A has selfadjoint extensions bounded from below that are different from its Friedrichs extension.

HINT: we can assume that A is closed. Let B be the Friedrichs extension of A. Then $A \subset B = B^* \subset A^*$ and $\mathfrak{D}(B)$ is the direct sum of $\mathfrak{D}(A)$, $\mathrm{Ker}\,(A^* - iI)$, $\mathrm{Ker}\,(A^* + iI)$. Let $D_0 \subset \mathrm{Ker}\,(A^* - iI)$, $R_0 \subset \mathrm{Ker}\,(A^* + iI)$ and $V_0 \colon D_0 \to R_0$ be the subspaces and the isometry corresponding to the Friedrichs extension, i.e., $\mathfrak{D}(B) = \mathfrak{D}(A) \oplus (V_0 - I)(D_0)$. Let us write D_0 is a direct sum of the linear span of a nonzero vector e and a subspace D_1. Let $R_1 = V_0(D_1)$, $e_1 = V_0(e)$. Define V_1 as follows: $V_1|_{D_1} = V_0$, $V_1(e) = -e_1$. Then $V_1 \neq V_0$. The operator V_1 corresponds to some selfadjoint extension A_1 of A different from B. Observe that $(V - I)(D_0)$ and $(V_1 - I)(D_0)$ differ by a finite-dimensional subspace and on the intersection A_1 acts by the formula $A_1(x + V_0 x_0 - x_0) = V_0 x - i x_0 - i V_0 x_0$. This yields that A_1 is bounded from below.

10.7.34.* Let A be a closed densely defined operator and

$$\mathfrak{D}(A^*A) := \{x \in \mathfrak{D}(A)\colon\ Ax \in \mathfrak{D}(A^*)\}, \quad A^*Ax := A^*(Ax).$$

Prove that A^*A is selfadjoint.

HINT: consider the form $Q(x,y) = (Ax, Ay) + (x,y)$, which is positive and closed (since A is closed). Hence there is a selfadjoint operator B such that $Q(x,y) = (Bx,y)$ for all $x \in \mathfrak{D}(B)$ and $y \in \mathfrak{D}(Q)$. It follows from the proof of Theorem 10.7.16 that $\mathfrak{D}(B)$ is closed in the Hilbert space $\mathfrak{D}(Q)$ (with the corresponding norm). We show that $\mathfrak{D}(B) \subset \mathfrak{D}(A^*A)$. For all $v \in \mathfrak{D}(B)$ and $x \in \mathfrak{D}(A)$ we have $(Sv, x) = (Av, Ax)$, where $S := B - I$. Hence $x \mapsto (Ax, Av)$ is a continuous functional and $Av \in \mathfrak{D}(A^*)$, $A^*Av = Sv$. Thus, S is selfadjoint and $S \subset A^*A$. It is obvious that A^*A is symmetric, whence it is easy to derive that $A^*A = S$.

10.7.35. Let A be a bounded selfadjoint operator on a separable Hilbert space H, $x \in H$ and H_x the closure of the linear span of the vectors $\Pi(B)x$, where $B \in \mathcal{B}(\mathrm{I\!R}^1)$. Prove that H_x equals the closure of the linear span of the sequence $\{x, Ax, A^2x, \ldots\}$.

HINT: for $A = A_\varphi$ on $L^2(\mu)$, the closed subspaces generated by the functions $I_B \circ \varphi$ and by the functions $f \circ \varphi$, where f is a polynomial, coincide.

10.7.36.° Construct an example of a selfadjoint operator on a separable Hilbert space that cannot be the generator of a strongly continuous semigroup.

HINT: consider the operator of multiplication by the argument on $L^2(\mathrm{I\!R})$.

10.7.37.° Construct an example of two different strongly continuous semigroups on a separable Hilbert space the generators of which coincide on a dense subspace.

10.7.38. Let A be a closed densely defined symmetric operator on a Hilbert space and let B be a bounded selfadjoint operator. Prove that the defect numbers of the operators A and $A + B$ coincide.

HINT: see [**6**, p. 352].

10.7.39. Suppose that A is the generator of a strongly continuous contracting semigroup $\{T_t\}_{t\geqslant 0}$ on a Hilbert space H and $D \subset \mathfrak{D}(A)$ is a linear subspace dense in H such that $T_t(D) \subset D$ for all t. Prove that the operator A on D is essentially selfadjoint, that is, A is the closure of $A|_D$.

HINT: see [**503**, Theorem X.49].

10.7.40. Let $T_t\varphi(x) = \varphi(x+t)$, $\varphi \in L^p(\mathbb{R}^1)$, $1 \leqslant p < \infty$. (i) Prove that we have obtained a strongly continuous semigroup. (ii) Prove that for $p = 1$ the semigroup of operators T_t^* on $L^\infty(\mathbb{R}^1) = \bigl(L^1(\mathbb{R}^1)\bigr)^*$ is not strongly continuous.

HINT: (i) first verify the continuity on compactly supported continuous functions. (ii) Consider T_t^* on $I_{[0,1]}$.

10.7.41. (i) Prove that an operator A on a domain $\mathfrak{D}(A)$ is dissipative precisely when $\|\lambda x - Ax\| \geqslant \lambda\|x\|$ for all $x \in \mathfrak{D}(A)$ and $\lambda > 0$.

(ii) Let A be a dissipative operator on a Banach space X. Prove that $\mathrm{Re}\, l(Ax) \leqslant 0$ for all $x \in \mathfrak{D}(A)$ and all $l \in X^*$ such that $l(x) = \|x\|^2 = \|l\|^2$ (not only for some l with this property as required by the definition).

HINT: (i) if A is dissipative, $x \in \mathfrak{D}(A)$, $\|x\| = 1$ and l is the corresponding functional, then $\|\lambda x - Ax\| \geqslant \mathrm{Re}\, l(\lambda x - Ax) \geqslant \lambda$ for all $\lambda > 0$. Conversely, let (ii) be fulfilled, $\|x\| = 1$, $l_\lambda \in X^*$, $l_\lambda(\lambda x - Ax) = \|l_\lambda\|^2 = \|\lambda x - Ax\|^2$. Let $f_\lambda = l_\lambda/\|l_\lambda\|$. Then $\mathrm{Re}\, f_\lambda(Ax) \leqslant 0$, since

$$\lambda\|x\| \leqslant \|\lambda x - Ax\| = f_\lambda(\lambda x - Ax) = \lambda \mathrm{Re}\, f_\lambda(x) - \mathrm{Re}\, f_\lambda(Ax) \leqslant \lambda\|x\| - \mathrm{Re}\, f_\lambda(Ax).$$

In addition, $\mathrm{Re}\, f_\lambda(x) \geqslant \|x\| + \lambda^{-1}\mathrm{Re}\, f_n(Ax)$. By the the weak-$*$ compactness of the balls in X^* the sequence $\{f_n\}$ has a limit point f. Clearly, $\mathrm{Re}\, f(Ax) \leqslant 0$ and $\mathrm{Re}\, f(x) \geqslant \|x\|$, whence $f(x) = \|x\| = 1$. (ii) Using the contracting semigroup $\{T_t\}_{t\geqslant 0}$ generated by A we fund that $|l(T_t x)| \leqslant \|l\|\,\|x\| = \|x\|^2$ and $\mathrm{Re}\, l(T_t x - x) = \mathrm{Re}\, l(T_t x) - \|x\|^2 \leqslant 0$; next, divide by t and let t go to zero.

10.7.42. Let A be a densely defined dissipative operator on a Banach space. Prove that A is closable and its closure is also dissipative.

HINT: let $x_n \in \mathfrak{D}(A)$, $x_n \to 0$ and $Ax_n \to y$. For every vector $z \in \mathfrak{D}(A)$ and all $\lambda > 0$ we have $\|x_n + \lambda z\| \leqslant \|x_n + \lambda z - \lambda A(x_n + \lambda z)\|$, which gives the estimate $\|\lambda z\| \leqslant \|\lambda(z - y) - \lambda^2 Az\|$, whence $\|z\| \leqslant \|z - y - \lambda Az\|$. As $\lambda \to 0$ we obtain $\|z\| \leqslant \|z - y\|$. Taking $z_n \in \mathfrak{D}(A)$ with $z_n \to 2y$, we obtain $y = 0$, so the operator A is closable. The dissipativity of $\overline{A}$ is clear from Exercise 10.7.41.

10.7.43. Let A be a dissipative operator on a Banach space X. (i) Prove that the closedness of A is equivalent to the closedness of the range of $\lambda I - A$ for some $\lambda > 0$ (then for all $\lambda > 0$). (ii) Prove that if the range of $\lambda I - A$ is X for some $\lambda > 0$, then this is true for all $\lambda > 0$.

HINT: see [**123**, §3.3].

10.7.44. Let A be a densely defined dissipative operator on a Banach space X. Prove the equivalence of the following properties. (i) The operator A^* on X^* is dissipative; (ii) the operator A^* on X^* is m-dissipative (i.e., is dissipative and for all $\lambda > 0$ the operators $A^* - \lambda I$ are surjective); (iii) the operator $\overline{A}$ is m-dissipative; (iv) the range of $\lambda I - A$ is dense for some $\lambda > 0$ (then for all $\lambda > 0$).

HINT: see [**123**, §3.3].

10.7.45. Let A be a densely defined dissipative operator on a Banach space X. Prove that $\overline{A}$ is the generator of a continuous contracting operator semigroup precisely when $\mathrm{Ker}\,(\lambda I - A^*) = 0$ for some $\lambda > 0$ (then for all $\lambda > 0$).