1
Elementary operator theory

1.1 Banach spaces

In this chapter we collect together material which should be covered in an introductory course of functional analysis and operator theory. We do not always include proofs, since there are many excellent textbooks on the subject.[1] The theorems provide a list of results which we use throughout the book.

We start at the obvious point. A normed space is a vector space $\mathcal{B}$ (assumed to be over the complex number field $\mathbf{C}$) provided with a norm $\|\cdot\|$ satisfying

$$\begin{aligned} \|f\| &\geq 0, \\ \|f\| &= 0 \text{ implies } f = 0, \\ \|\alpha f\| &= |\alpha|\,\|f\|, \\ \|f+g\| &\leq \|f\| + \|g\|, \end{aligned}$$

for all $\alpha \in \mathbf{C}$ and all $f, g \in \mathcal{B}$. Many of our definitions and theorems also apply to real normed spaces, but we will not keep pointing this out. We say that $\|\cdot\|$ is a seminorm if it satisfies all of the axioms except the second.

A Banach space is defined to be a normed space $\mathcal{B}$ which is complete in the sense that every Cauchy sequence in $\mathcal{B}$ converges to a limit in $\mathcal{B}$. Every normed space $\mathcal{B}$ has a completion $\overline{\mathcal{B}}$, which is a Banach space in which $\mathcal{B}$ is embedded isometrically and densely. (An isometric embedding is a linear, norm-preserving (and hence one-one) map of one normed space into another in which every element of the first space is identified with its image in the second.)

[1] One of the most systematic is [Dunford and Schwartz 1966].

Problem 1.1.1 Prove that the following conditions on a normed space $\mathcal{B}$ are equivalent:

(i) $\mathcal{B}$ is complete.
(ii) Every series $\sum_{n=1}^{\infty} f_n$ in $\mathcal{B}$ such that $\sum_{n=1}^{\infty} \|f_n\| < \infty$ is norm convergent.
(iii) Every series $\sum_{n=1}^{\infty} f_n$ in $\mathcal{B}$ such that $\|f_n\| \le 2^{-n}$ for all n is norm convergent.

Prove also that any two completions of a normed space $\mathcal{B}$ are isometrically isomorphic. □

The following results from point set topology are rarely used below, but they provide worthwhile background knowledge. We say that a topological space X is normal if given any pair of disjoint closed subsets A, B of X there exists a pair of disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$. All metric spaces and all compact Hausdorff spaces are normal. The size of the space of continuous functions on a normal space is revealed by Urysohn's lemma.

Lemma 1.1.2 *(Urysohn)*[2] *If A, B are disjoint closed sets in the normal topological space X, then there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

Problem 1.1.3 Use the continuity of the distance function $x \to \mathrm{dist}(x, A)$ to provide a direct proof of Urysohn's lemma when X is a metric space. □

Theorem 1.1.4 *(Tietze) Let S be a closed subset of the normal topological space X and let $f : S \to [0, 1]$ be a continuous function. Then there exists a continuous extension of f to X, i.e. a continuous function $g : X \to [0, 1]$ which coincides with f on S.*[3]

Problem 1.1.5 Prove the Tietze extension theorem by using Urysohn's lemma to construct a sequence of functions $g_n : X \to [0, 1]$ which converge uniformly on X and also uniformly on S to f. □

If K is a compact Hausdorff space then $C(K)$ stands for the space of all continuous complex-valued functions on K with the supremum norm

$$\|f\|_\infty := \sup\{|f(x)| : x \in K\}.$$

$C(K)$ is a Banach space with this norm, and the supremum is actually a maximum. We also use the notation $C_{\mathbf{R}}(K)$ to stand for the real Banach space of all continuous, real-valued functions on K.

[2] See [Bollobas 1999], [Simmons 1963, p. 135] or [Kelley 1955, p. 115].
[3] See [Bollobas 1999].

The following theorem is of interest in spite of the fact that it is rarely useful: in most applications it is equally evident that all four statements are true (or false).

Theorem 1.1.6 *(Urysohn) If K is a compact Hausdorff space then the following statements are equivalent.*

(i) K is metrizable;
(ii) the topology of K has a countable base;
(iii) K can be homeomorphically embedded in the unit cube $\Omega := \prod_{n=1}^{\infty}[0, 1]$ of countable dimension;
(iv) the space $C_{\mathbf{R}}(K)$ is separable in the sense that it contains a countable norm dense subset.

The equivalence of the first three statements uses methods of point-set topology, for which we refer to [Kelley 1955, p. 125]. The equivalence of the fourth statement uses the Stone-Weierstrass theorem 2.3.17.

Problem 1.1.7 Without using Theorem 1.1.6, prove that the topological product of a countable number of compact metrizable spaces is also compact metrizable. □

We say that $\mathcal{H}$ is a Hilbert space if it is a Banach space with respect to a norm associated with an inner product $f, g \to \langle f, g\rangle$ according to the formula

$$\|f\| := \sqrt{\langle f, f\rangle}.$$

We always assume that an inner product is linear in the first variable and conjugate linear in the second variable. We assume familiarity with the basic theory of Hilbert spaces. Although we do not restrict the statements of many theorems in the book to separable Hilbert spaces, we frequently only give the proof in that case. The proof in the non-separable context can usually be obtained by either of two devices: one may replace the word sequence by generalized sequence, or one may show that if the result is true on every separable subspace then it is true in general.

Example 1.1.8 If X is a finite or countable set then $l^2(X)$ is defined to be the space of all functions $f : X \to \mathbf{C}$ such that

$$\|f\|_2 := \sqrt{\sum_{x\in X} |f(x)|^2} < \infty.$$

This is the norm associated with the inner product

$$\langle f, g\rangle := \sum_{x\in X} f(x)\overline{g(x)},$$

the sum being absolutely convergent for all $f, g \in l^2(X)$. $\square$

A sequence $\{\phi_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is said to be an orthonormal sequence if

$$\langle \phi_m, \phi_n\rangle = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

It is said to be a complete orthonormal sequence or an orthonormal basis, if it satisfies the conditions of the following theorem.

Theorem 1.1.9 *The following conditions on an orthonormal sequence $\{\phi_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ are equivalent.*

(i) The linear span of $\{\phi_n\}_{n=1}^{\infty}$ is a dense linear subspace of $\mathcal{H}$.
(ii) The identity

$$f = \sum_{n=1}^{\infty} \langle f, \phi_n\rangle \phi_n \tag{1.1}$$

holds for all $f \in \mathcal{H}$.
(iii) The identity

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \phi_n\rangle|^2$$

holds for all $f \in \mathcal{H}$.
(iv) The identity

$$\langle f, g\rangle = \sum_{n=1}^{\infty} \langle f, \phi_n\rangle \langle \phi_n, g\rangle$$

holds for all $f, g \in \mathcal{H}$, the series being absolutely convergent.

The formula (1.1) is sometimes called a generalized Fourier expansion and $\langle f, \phi_n\rangle$ are then called the Fourier coefficients of f. The rate of convergence in (1.1) depends on f, and is discussed further in Theorem 5.4.12.

Problem 1.1.10 (Haar) Let $\{v_n\}_{n=0}^{\infty}$ be a dense sequence of distinct numbers in $[0, 1]$ such that $v_0 = 0$ and $v_1 = 1$. Put $e_1(x) := 1$ for all $x \in (0, 1)$ and define $e_n \in L^2(0, 1)$ for $n = 2, 3, \ldots$ by

$$e_n(x) := \begin{cases} 0 & \text{if } x < u_n, \\ \alpha_n & \text{if } u_n < x < v_n, \\ -\beta_n & \text{if } v_n < x < w_n, \\ 0 & \text{if } x > w_n, \end{cases}$$

where

$$u_n := \max\{v_r : r < n \text{ and } v_r < v_n\},$$

$$w_n := \min\{v_r : r < n \text{ and } v_r > v_n\},$$

and $\alpha_n > 0$, $\beta_n > 0$ are the solutions of

$$\alpha_n(v_n - u_n) - \beta_n(w_n - v_n) = 0,$$

$$(v_n - u_n)\alpha_n^2 + (w_n - v_n)\beta_n^2 = 1.$$

Prove that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis in $L^2(0, 1)$. If $\{v_n\}_{n=0}^{\infty}$ is the sequence $\{0, 1, 1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, \ldots\}$ one obtains the standard Haar basis of $L^2(0, 1)$, discussed in all texts on wavelets and of importance in image processing. If $\{m_r\}_{r=1}^{\infty}$ is a sequence of integers such that $m_1 \geq 2$ and m_r is a proper factor of m_{r+1} for all r, then one may define a generalized Haar basis of $L^2(0, 1)$ by concatenating 0, 1, $\{r/m_1\}_{r=1}^{m_1}$, $\{r/m_2\}_{r=1}^{m_2}$, $\{r/m_3\}_{r=1}^{m_3}, \ldots$ and removing duplicated numbers as they arise. □

If X is a set with a σ-algebra Σ of subsets, and $\mathrm{d}x$ is a countably additive σ-finite measure on Σ, then the formula

$$\|f\|_2 := \sqrt{\int_X |f(x)|^2 \, \mathrm{d}x}$$

defines a norm on the space $L^2(X, \mathrm{d}x)$ of all functions f for which the integral is finite. The norm is associated with the inner product

$$\langle f, g \rangle := \int_X f(x)\overline{g(x)} \, \mathrm{d}x.$$

Strictly speaking one only gets a norm by identifying two functions which are equal almost everywhere. If the integral used is that of Lebesgue, then $L^2(X, \mathrm{d}x)$ is complete.[4]

Notation If $\mathcal{B}$ is a Banach space of functions on a locally compact, Hausdorff space X, then we will always use the notation $\mathcal{B}_c$ to stand for all those

[4] See [Lieb and Loss 1997] for one among many more complete accounts of Lebesgue integration. See also Section 2.1.

functions in $\mathcal{B}$ which have compact support, and $\mathcal{B}_0$ to stand for the closure of $\mathcal{B}_c$ in $\mathcal{B}$. Also $C_0(X)$ stands for the closure of $C_c(X)$ with respect to the supremum norm; equivalently $C_0(X)$ is the space of continuous functions on X that vanish at infinity. If X is a region in $\mathbf{R}^N$ then $C^n(X)$ will stand for the space of n times continuously differentiable functions on X.

Problem 1.1.11 The space $L^1(a,b)$ may be defined as the abstract completion of the space $\mathcal{P}$ of piecewise continuous functions on $[a,b]$, with respect to the norm

$$\|f\|_1 := \int_a^b |f(x)|\,\mathrm{d}x.$$

Without using any properties of Lebesgue integration prove that $C^k[a,b]$ is dense in $L^1(a,b)$ for every $k \geq 0$. $\square$

Lemma 1.1.12 *A finite-dimensional normed space V is necessarily complete. Any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent in the sense that there exist positive constants a and b such that*

$$a\|f\|_1 \leq \|f\|_2 \leq b\|f\|_1 \tag{1.2}$$

for all $f \in V$.

Problem 1.1.13 Find the optimal values of the constants a and b in (1.2) for the norms on $\mathbf{C}^n$ given by

$$\|f\|_1 := \sum_{r=1}^n |f_r|, \qquad \|f\|_2 := \left\{\sum_{r=1}^n |f_r|^2\right\}^{1/2}. \qquad \square$$

A bounded linear functional $\phi : \mathcal{B} \to \mathbf{C}$ is a linear map for which

$$\|\phi\| := \sup\{|\phi(f)| : \|f\| \leq 1\}$$

is finite. The dual space $\mathcal{B}^*$ of $\mathcal{B}$ is defined to be the space of all bounded linear functionals on $\mathcal{B}$, and is itself a Banach space for the norm given above. The Hahn-Banach theorem states that if L is any linear subspace of $\mathcal{B}$, then any bounded linear functional ϕ on L has a linear extension ψ to $\mathcal{B}$ which has the same norm:

$$\sup\{|\phi(f)|/\|f\| : 0 \neq f \in L\} = \sup\{|\psi(f)|/\|f\| : 0 \neq f \in \mathcal{B}\}.$$

It is not always easy to find a useful representation of the dual space of a Banach space, but the Hilbert space is particularly simple.

Theorem 1.1.14 *(Fréchet-Riesz)*[5] *If $\mathcal{H}$ is a Hilbert space then the formula*

$$\phi(f) := \langle f, g\rangle$$

defines a one-one correspondence between all $g \in \mathcal{H}$ and all $\phi \in \mathcal{H}^*$. Moreover $\|\phi\| = \|g\|$.

Note The correspondence $\phi \leftrightarrow g$ is conjugate linear rather than linear, and this can cause some confusion if forgotten.

Problem 1.1.15 Prove that if ϕ is a bounded linear functional on the closed linear subspace $\mathcal{L}$ of a Hilbert space $\mathcal{H}$, then there is only one linear extension of ϕ to $\mathcal{H}$ with the same norm. $\square$

The following theorem is not elementary, and we will not use it until Chapter 13.1. The notation $C_{\mathbf{R}}(K)$ refers to the real Banach space of continuous functions $f : K \to \mathbf{R}$ with the supremum norm.[6]

Theorem 1.1.16 *(Riesz-Kakutani) Let K be a compact Hausdorff space and let $\phi \in C_{\mathbf{R}}(K)^*$. If ϕ is non-negative in the sense that $\phi(f) \geq 0$ for all non-negative $f \in C_{\mathbf{R}}(K)$ then there exists a non-negative countably additive measure μ on K such that*

$$\phi(f) = \int_X f(x)\,\mu(\mathrm{d}x)$$

for all $f \in C_{\mathbf{R}}(K)$. Moreover $\|\phi\| = \phi(1) = \mu(K)$.

One may reduce the representation of more general bounded linear functionals to the above special case by means of the following theorem. Given $\phi, \psi \in C_{\mathbf{R}}(K)^*$, we write $\phi \geq \psi$ if $\phi(f) \geq \psi(f)$ for all non-negative $f \in C_{\mathbf{R}}(K)$.

Theorem 1.1.17 *If K is a compact Hausdorff space and $\phi \in C_{\mathbf{R}}(K)^*$ then one may write $\phi := \phi_+ - \phi_-$ where $\phi_\pm$ are canonically defined, non-negative, bounded linear functionals. If $|\phi| := \phi_+ + \phi_-$ then $|\phi| \geq \pm\phi$. If $\psi \geq \pm\phi \in C_{\mathbf{R}}(K)^*$ then $\psi \geq |\phi|$. Finally $\|\,|\phi|\,\| = \|\phi\|$.*

[5] See [Dunford and Schwartz 1966, Theorem IV.4.5] for the proof.
[6] A combination of the next two theorems is usually called the Riesz representation theorem. According to [Dunford and Schwartz 1966, p. 380] Riesz provided an explicit representation of $C[0, 1]^*$. The corresponding theorem for $C_{\mathbf{R}}(K)^*$, where K is a general compact Hausdorff space, was obtained some years later by Kakutani. The formula $\phi := \phi_+ - \phi_-$ is called the Jordan decomposition. For the proof of the theorem see [Dunford and Schwartz 1966, Theorem IV.6.3]. A more abstract formulation, in terms of Banach lattices and AM-spaces, is given in [Schaefer 1974, Proposition II.5.5 and Section II.7].

Proof. The proof is straightforward but lengthy. Let $\mathcal{B} := C_{\mathbf{R}}(K)$, let $\mathcal{B}_+$ denote the convex cone of all non-negative continuous functions on K, and let $\mathcal{B}_+^*$ denote the convex cone of all non-negative functionals $\psi \in \mathcal{B}^*$.

Given $\phi \in \mathcal{B}^*$, we define $\phi_+ : \mathcal{B}_+ \to \mathbf{R}_+$ by

$$\phi_+(f) := \sup\{\phi(f_0) : 0 \le f_0 \le f\}.$$

If $0 \le f_0 \le f$ and $0 \le g_0 \le g$ then

$$\phi(f_0) + \phi(g_0) = \phi(f_0 + g_0) \le \phi_+(f + g).$$

Letting f_0 and g_0 vary subject to the stated constraints, we deduce that

$$\phi_+(f) + \phi_+(g) \le \phi_+(f + g)$$

for all $f, g \in \mathcal{B}_+$.

The reverse inequality is harder to prove. If $f, g \in \mathcal{B}_+$ and $0 \le h \le f + g$ then one puts $f_0 := \min\{h, f\}$ and $g_0 := h - f_0$. By considering each point $x \in K$ separately one sees that $0 \le f_0 \le f$ and $0 \le g_0 \le g$. hence

$$\phi(h) = \phi(f_0) + \phi(g_0) \le \phi_+(f) + \phi_+(g).$$

Since h is arbitrary subject to the stated constraints one obtains

$$\phi_+(f + g) \le \phi_+(f) + \phi_+(g)$$

for all $f, g \in \mathcal{B}_+$.

We are now in a position to extend ϕ_+ to the whole of $\mathcal{B}$. If $f \in \mathcal{B}$ we put

$$\phi_+(f) := \phi_+(f + \alpha 1) - \alpha\phi_+(1)$$

where $\alpha \in \mathbf{R}$ is chosen so that $f + \alpha 1 \ge 0$. The linearity of ϕ_+ on $\mathcal{B}_+$ implies that the particular choice of α does not matter subject to the stated constraint.

Our next task is to prove that the extended ϕ_+ is a linear functional on $\mathcal{B}_+$. If $f, g \in \mathcal{B}$, $f + \alpha 1 \ge 0$ and $g + \beta 1 \ge 0$, then

$$\begin{aligned}
\phi_+(f + g) &= \phi_+(f + g + \alpha 1 + \beta 1) - (\alpha + \beta)\phi_+(1) \\
&= \phi_+(f + \alpha 1) + \phi_+(g + \beta 1) - (\alpha + \beta)\phi_+(1) \\
&= \phi_+(f) + \phi_+(g).
\end{aligned}$$

It follows immediately from the definition that $\phi_+(\lambda h) = \lambda\phi_+(h)$ for all $\lambda \ge 0$ and $h \in \mathcal{B}_+$. Hence $f \in \mathcal{B}$ implies

$$\phi_+(\lambda f) = \phi(\lambda f + \lambda\alpha 1) - \lambda\alpha\phi_+(1) = \lambda\phi(f + \alpha 1) - \lambda\alpha\phi_+(1) = \lambda\phi_+(f).$$

If $\lambda < 0$ then

$$0 = \phi_+(\lambda f + |\lambda| f) = \phi_+(\lambda f) + \phi_+(|\lambda| f) = \phi_+(\lambda f) + |\lambda| \phi_+(f).$$

Therefore

$$\phi_+(\lambda f) = -|\lambda| \phi_+(f) = \lambda \phi_+(f).$$

Therefore ϕ_+ is a linear functional on $\mathcal{B}$. It is non-negative in the sense defined above.

We define ϕ_- by $\phi_- := \phi_+ - \phi$, and deduce immediately that it is linear. Since $f \in \mathcal{B}_+$ implies that $\phi_+(f) \geq \phi(f)$, we see that ϕ_- is non-negative. The boundedness of $\phi_\pm$ will be a consequence of the boundedness of $|\phi|$ and the formulae

$$\phi_+ = \tfrac{1}{2}(|\phi| + \phi), \qquad \phi_- = \tfrac{1}{2}(|\phi| - \phi).$$

We will need the following formula for $|\phi|$. If $f \in \mathcal{B}_+$ then the identity $|\phi| = 2\phi_+ - \phi$ implies

$$\begin{aligned} |\phi|(f) &= 2 \sup\{\phi(f_0) : 0 \leq f_0 \leq f\} - \phi(f) \\ &= \sup\{\phi(2f_0 - f) : 0 \leq f_0 \leq f\} \\ &= \sup\{\phi(f_1) : -f \leq f_1 \leq f\}. \end{aligned} \tag{1.3}$$

The inequality $|\phi| \geq \pm\phi$ of the theorem follows from

$$\begin{aligned} |\phi| = \phi + 2\phi_- &\geq \phi \\ |\phi| = 2\phi_+ - \phi &\geq -\phi. \end{aligned}$$

If $\psi \geq \pm\phi$, $f \geq 0$ and $-f \leq f_1 \leq f$ then adding the two inequalities $(\psi + \phi)(f - f_1) \geq 0$ and $(\psi - \phi)(f + f_1) \geq 0$ yields $\psi(f) \geq \phi(f_1)$. Letting f_1 vary subject to the stated constraint we obtain $\psi(f) \geq |\phi|(f)$ by using (1.3). Therefore $\psi \geq |\phi|$.

We finally have to evaluate $\| \, |\phi| \, \|$. If $f \in \mathcal{B}$ and $\phi \in \mathcal{B}^*$ then

$$\begin{aligned} |\phi(f)| &= |\phi_+(f_+) - \phi_+(f_-) - \phi_-(f_+) + \phi_-(f_-)| \\ &\leq \phi_+(f_+) + \phi_+(f_-) + \phi_-(f_+) + \phi_-(f_-) \\ &= |\phi|(|f|) \\ &\leq \| \, |\phi| \, \| \, \| \, |f| \, \| \\ &= \| \, |\phi| \, \| \, \|f\|. \end{aligned}$$

Since f is arbitrary we deduce that $\|\phi\| \leq \| \, |\phi| \, \|$.

Conversely suppose that $f \in \mathcal{B}$. The inequality $-|f| \le f \le |f|$ implies

$$-|\phi|(|f|) \le |\phi|(f) \le |\phi|(|f|).$$

Therefore

$$\begin{aligned} ||\phi|(f)| &\le |\phi|(|f|) \\ &= \sup\{\phi(f_1) : -|f| \le f_1 \le |f|\} \\ &\le \|\phi\| \sup\{\|f_1\| : -|f| \le f_1 \le |f|\} \\ &= \|\phi\| \, \|f\|. \end{aligned}$$

Hence $\| \, |\phi| \, \| \le \|\phi\|$. □

If L is a closed linear subspace of the normed space $\mathcal{B}$, then the quotient space $\mathcal{B}/L$ is defined to be the algebraic quotient, provided with the quotient norm

$$\|f + L\| := \inf\{\|f + g\| : g \in L\}.$$

It is known that if $\mathcal{B}$ is a Banach space then so is $\mathcal{B}/L$.

Problem 1.1.18 If $\mathcal{B} = C[a, b]$ and L is the subspace of all functions in $\mathcal{B}$ which vanish on the closed subset K of $[a, b]$, find an explicit representation of $\mathcal{B}/L$ and of its norm. □

The Hahn-Banach theorem implies immediately that there is a canonical and isometric embedding j from $\mathcal{B}$ into the second dual space $\mathcal{B}^{**} = (\mathcal{B}^*)^*$, given by

$$(jx)(\phi) := \phi(x)$$

for all $x \in \mathcal{B}$ and all $\phi \in \mathcal{B}^*$. The space $\mathcal{B}$ is said to be reflexive if j maps $\mathcal{B}$ one-one onto $\mathcal{B}^{**}$.

We will often use the more symmetrical notation $\langle x, \phi \rangle$ in place of $\phi(x)$, and regard $\mathcal{B}$ as a subset of $\mathcal{B}^{**}$, suppressing mention of its natural embedding.

Problem 1.1.19 Prove that $\mathcal{B}$ is reflexive if and only if $\mathcal{B}^*$ is reflexive. □

Example 1.1.20 The dual $\mathcal{B}^*$ of a Banach space $\mathcal{B}$ is usually not isometrically isomorphic to $\mathcal{B}$ even if $\mathcal{B}$ is reflexive. The following provides a large number of spaces for which they are isometrically isomorphic. We simply choose any reflexive Banach space $\mathcal{C}$ and consider $\mathcal{B} := \mathcal{C} \oplus \mathcal{C}^*$ with the norm

$$\|(x, \phi)\| := (\|x\|^2 + \|\phi\|^2)^{1/2}. \qquad \square$$

If X is an infinite set, $c_0(X)$ is defined to be the vector space of functions f which converge to 0 at infinity; more precisely we assume that for all $\varepsilon > 0$ there exists a finite set $F \subset X$ depending upon f and ε such that $x \notin F$ implies $|f(x)| < \varepsilon$.

Problem 1.1.21 Prove that $c_0(X)$ is a Banach space with respect to the supremum norm. $\square$

Problem 1.1.22 Prove that $c_0(X)$ is separable if and only if X is countable. $\square$

Problem 1.1.23 Prove that the dual space of $c_0(X)$ may be identified naturally with $l^1(X)$, the pairing being given by

$$\langle f, g \rangle := \sum_{x \in X} f(x) g(x)$$

where $f \in c_0(X)$ and $g \in l^1(X)$. $\square$

Problem 1.1.24 Prove that the dual space of $l^1(X)$ may be identified with the space $l^\infty(X)$ of all bounded functions $f : X \to \mathbf{C}$ with the supremum norm. Prove also that if X is infinite, $l^1(X)$ is not reflexive. $\square$

Problem 1.1.25 Use the Hahn-Banach theorem to prove that if $\mathcal{L}$ is a finite-dimensional subspace of the Banach space $\mathcal{B}$ then there exists a closed linear subspace $\mathcal{M}$ of $\mathcal{B}$ such that $\mathcal{L} \cap \mathcal{M} = \{0\}$ and $\mathcal{L} + \mathcal{M} = \mathcal{B}$. Moreover there exists a constant $c > 0$ such that

$$c^{-1}(\|l\| + \|m\|) \le \|l + m\| \le c(\|l\| + \|m\|)$$

for all $l \in \mathcal{L}$ and $m \in \mathcal{M}$. $\square$

We will frequently use the concept of integration[7] for functions which take their values in a Banach space $\mathcal{B}$. If $f:[a,b]\to\mathcal{B}$ is a piecewise continuous function, there is an element of $\mathcal{B}$, denoted by

$$\int_a^b f(x)\,\mathrm{d}x$$

which is defined by approximating f by piecewise constant functions, for which the definition of the integral is evident. It is easy to show that the integral depends linearly on f and that

$$\|\int_a^b f(x)\,\mathrm{d}x\| \le \int_a^b \|f(x)\|\,\mathrm{d}x.$$

Moreover

$$\left\langle \int_a^b f(x)\,\mathrm{d}x, \phi \right\rangle = \int_a^b \langle f(x), \phi\rangle\,\mathrm{d}x$$

for all $\phi \in \mathcal{B}^*$, where $\langle f, \phi\rangle$ denotes $\phi(f)$ as explained above. Both of these relations are proved first for piecewise constant functions. The integral may also be defined for functions $f:\mathbf{R}\to\mathcal{B}$ which decay rapidly enough at infinity. Many other familiar results, such as the fundamental theorem of calculus, and the possibility of taking a bounded linear operator under the integral sign, may be proved by the same method as is used for complex-valued functions.

1.2 Bounded linear operators

A bounded linear operator $A:\mathcal{B}\to\mathcal{C}$ between two Banach spaces is defined to be a linear map for which the norm

$$\|A\| := \sup\{\|Af\| : \|f\| \le 1\}$$

is finite. In this chapter we will use the term 'operator' to stand for 'bounded linear operator' unless the context makes this inappropriate. The set $\mathcal{L}(\mathcal{B},\mathcal{C})$ of all such operators itself forms a Banach space under the obvious operations and the above norm.

The set $\mathcal{L}(\mathcal{B})$ of all operators from $\mathcal{B}$ to itself is an algebra, the multiplication being defined by

$$(AB)(f) := A(B(f))$$

[7] We treat this at a very elementary level. A more sophisticated treatment is given in [Dunford and Schwartz 1966, Chap. 3], but we will not need to use this.

for all $f \in \mathcal{B}$. In fact $\mathcal{L}(\mathcal{B})$ is called a Banach algebra by virtue of being a Banach space and an algebra satisfying

$$\|AB\| \le \|A\|\,\|B\|$$

for all $A,\ B \in \mathcal{L}(\mathcal{B})$. The identity operator I satisfies $\|I\| = 1$ and $AI = IA = A$ for all $A \in \mathcal{L}(\mathcal{B})$, so $\mathcal{L}(\mathcal{B})$ is a Banach algebra with identity.

Problem 1.2.1 Prove that $\mathcal{L}(\mathcal{B})$ is only commutative as a Banach algebra if $\mathcal{B} = \mathbf{C}$, and that $\mathcal{L}(\mathcal{B})$ is only finite-dimensional if $\mathcal{B}$ is finite -dimensional. $\square$

Every operator A on $\mathcal{B}$ has a dual operator A^* acting on $\mathcal{B}^*$, satisfying the identity

$$\langle Af, \phi\rangle = \langle f, A^*\phi\rangle$$

for all $f \in \mathcal{B}$ and all $\phi \in \mathcal{B}^*$. The map $A \to A^*$ from $\mathcal{L}(\mathcal{B})$ to $\mathcal{L}(\mathcal{B}^*)$ is linear and isometric, but reverses the order of multiplication.

For every bounded operator A on a Hilbert space $\mathcal{H}$ there is a unique bounded operator A^*, also acting on $\mathcal{H}$, called its adjoint, such that

$$\langle Af, g\rangle = \langle f, A^* g\rangle,$$

for all $f,\ g \in \mathcal{H}$. This is not totally compatible with the notion of dual operator in the Banach space context, because the adjoint map is conjugate linear in the sense that

$$(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$$

for all operators A, B and all complex numbers α, β. However, almost every other result is the same for the two concepts. In particular $A^{**} = A$. The concept of self-adjointness, $A = A^*$, is peculiar to Hilbert spaces, and is of great importance. We say that an operator U is unitary if it satisfies the conditions of the problem below.

Problem 1.2.2 Let U be a bounded operator on a Hilbert space $\mathcal{H}$. Use the polarization identity

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

to prove that the following three conditions are equivalent.

(i) $U^*U = UU^* = I$;
(ii) U is one-one onto and isometric in the sense that $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$;
(iii) U is one-one onto and $\langle Uf, Ug\rangle = \langle f, g\rangle$ for all $f,\ g \in \mathcal{H}$. $\square$

The inverse mapping theorem below establishes that algebraic invertibility of a bounded linear operator between Banach spaces is equivalent to invertibility in the category of bounded operators.[8]

Theorem 1.2.3 *(Banach) If the bounded linear operator A from the Banach space $\mathcal{B}_1$ to the Banach space $\mathcal{B}_2$ is one-one and onto, then the inverse operator is also bounded.*

Let $\mathcal{A}$ be an associative algebra over the complex field with identity element e. The number $\lambda \in \mathbf{C}$ is said to lie in the resolvent set of $a \in \mathcal{A}$ if $\lambda e - a$ has an inverse in $\mathcal{A}$. We call $R(\lambda, a) := (\lambda e - a)^{-1}$ the resolvent operators of a. The $\mathrm{Spec}(a)$ of a is by definition the complement of the resolvent set. If A is a bounded linear operator on a Banach space $\mathcal{B}$ we assume that the spectrum and resolvent are calculated with respect to $\mathcal{A} = \mathcal{L}(\mathcal{B})$, unless stated otherwise.

The appearance of the spectrum and resolvent at such an early stage in the book is no accident. They are the key concepts on which everything else is based. An enormous amount of effort has been devoted to their study for over a hundred years, and sophisticated software exists for computing both in a wide variety of fields. No book could aspire to treating all of this in a comprehensive manner, but we can describe the foundations on which this vast subject has been built. One of these is the resolvent identity.

Problem 1.2.4 Prove the resolvent identity

$$R(z, a) - R(w, a) = (w - z)R(z, a)R(w, a)$$

for all $z, w \notin \mathrm{Spec}(a)$. □

Problem 1.2.5 Let a, b lie in the associative algebra $\mathcal{A}$ with identity e and let $0 \neq z \in \mathbf{C}$. Prove that $ab - ze$ is invertible if and only if $ba - ze$ is invertible. □

Problem 1.2.6 Let A, B be linear maps on the vector space $\mathcal{V}$ and let $0 \neq z \in \mathbf{C}$. Prove that the eigenspaces

$$\mathcal{M} := \{f \in \mathcal{V} : ABf = zf\}, \qquad \mathcal{N} := \{g \in \mathcal{V} : BAg = zg\}$$

have the same dimension. □

[8] See [Dunford and Schwartz 1966, Theorem II.2.2].

Problem 1.2.7 Let a be an element of the associative algebra $\mathcal{A}$ with identity e. Prove that

$$\mathrm{Spec}(a) = \mathrm{Spec}(L_a)$$

where $L_a : \mathcal{A} \to \mathcal{A}$ is defined by $L_a(x) := ax$. □

Problem 1.2.8 Let A be an operator on the Banach space $\mathcal{B}$ satisfying $\|A\| < 1$. Prove that $(I - A)$ is invertible and that

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n, \tag{1.4}$$

the sum being norm convergent. □

Theorem 1.2.9 *The set $\mathcal{G}$ of all bounded invertible operators on a Banach space $\mathcal{B}$ is open. More precisely, if $A \in \mathcal{G}$ and $\|B - A\| < \|A^{-1}\|^{-1}$ then $B \in \mathcal{G}$.*

Proof. If $C := I - BA^{-1}$ then under the stated conditions

$$\|C\| = \|(A - B)A^{-1}\| \le \|A^{-1}\|^{-1} \|A^{-1}\| < 1.$$

Therefore $(I - C)$ is invertible by Problem 1.2.8. But $B = (BA^{-1})A = (I - C)A$, so B is invertible with

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} C^n. \tag{1.5}$$

□

Theorem 1.2.10 *The resolvent operator $R(z, A)$ satisfies*

$$\|R(z, A)\| \ge \mathrm{dist}(z, \mathrm{Spec}(A))^{-1} \tag{1.6}$$

for all $z \notin \mathrm{Spec}(A)$, where $\mathrm{dist}(z, S)$ denotes the distance of z from the set S.

Proof. If $z \notin \mathrm{Spec}(A)$ and $|w - z| < \|R(z, A)\|^{-1}$ then

$$\begin{aligned} D &:= R(z, A)\{I - (z - w)R(z, A)\}^{-1} \\ &= \sum_{n=0}^{\infty} (z - w)^n R(z, A)^{n+1} \end{aligned}$$

is a bounded invertible operator on $\mathcal{B}$; the inverse involved exists by Problem 1.2.8. It satisfies

$$D\{I - (z - w)R(z, A)\} = R(z, A)$$

and hence

$$D\{zI - A - (z - w)I\} = I.$$

We deduce that $D(wI - A) = I$, and similarly that $(wI - A)D = I$. Hence $w \notin \mathrm{Spec}(A)$. The statement of the theorem follows immediately. $\square$

Our next theorem uses the concept of an analytic operator-valued function. This is developed in more detail in Section 1.4.

Theorem 1.2.11 *Every bounded linear operator A on a Banach space has a non-empty, closed, bounded spectrum, which satisfies*

$$\mathrm{Spec}(A) \subseteq \{z \in \mathbf{C} : |z| \le \|A\|\}. \tag{1.7}$$

If $|z| > \|A\|$ *then*

$$\|(zI - A)^{-1}\| \le (|z| - \|A\|)^{-1}. \tag{1.8}$$

The resolvent operator $R(z, A)$ *is a norm analytic function of* z *on* $\mathbf{C}\backslash\mathrm{Spec}(A)$.

Proof. If $|z| > \|A\|$ then $zI - A = z(I - z^{-1}A)$ and this is invertible, with inverse

$$(zI - A)^{-1} = z^{-1}\sum_{n=1}^{\infty}(z^{-1}A)^n.$$

The bound (1.8) follows by estimating each of the terms in the geometric series. This implies (1.7). Theorem 1.2.10 implies that $\mathrm{Spec}(A)$ is closed. An examination of the proof of Theorem 1.2.10 leads to the conclusion that $R(z, A)$ is a norm analytic function of z in some neighbourhood of every $z \notin \mathrm{Spec}(A)$. It remains only to prove that $\mathrm{Spec}(A)$ is non-empty.

Since

$$(zI - A)^{-1} = \sum_{n=0}^{\infty} z^{-n-1}A^n$$

if $|z| > \|A\|$, we see that $\|(zI - A)^{-1}\| \to 0$ as $|z| \to \infty$. The Banach space version of Liouville's theorem given in Problem 1.4.9 now implies that if $R(z, A)$ is entire, it vanishes identically. The contradiction establishes that $\mathrm{Spec}(A)$ must be non-empty. $\square$

We note that this proof is highly non-constructive: it does not give any clues about how to find even a single point in $\mathrm{Spec}(A)$. We will show in Section 9.1 that computing the spectrum may pose fundamental difficulties.

Problem 1.2.12 Let a be an element of the Banach algebra $\mathcal{A}$, whose multiplicative identity 1 satisfies $\|1\| = 1$. Prove that a has a non-empty, closed, bounded spectrum, which satisfies

$$\mathrm{Spec}(a) \subseteq \{z \in \mathbf{C} : |z| \leq \|a\|\}. \qquad \square$$

Our definition of the spectrum of an operator A was algebraic in that it only refers to properties of A as an element of the algebra $\mathcal{L}(\mathcal{B})$. One can also give a characterization that is geometric in the sense that it refers to vectors in the Banach space.

Lemma 1.2.13 *The number λ lies in the spectrum of the bounded operator A on the Banach space $\mathcal{B}$ if and only if at least one of the following occurs:*

(i) λ is an eigenvalue of A. That is $Af = \lambda f$ for some non-zero $f \in \mathcal{B}$.
(ii) λ is an eigenvalue of A^*. Equivalently the range of the operator $\lambda I - A$ is not dense in $\mathcal{B}$.
(iii) There exists a sequence $f_n \in \mathcal{B}$ such that $\|f_n\| = 1$ for all n and

$$\lim_{n\to\infty} \|Af_n - \lambda f_n\| = 0.$$

Proof. The operator $B := \lambda I - A$ may fail to be invertible because it is not one-one or because it is not onto. In the second case it may have closed range not equal to $\mathcal{B}$ or it may have range which is not closed. If it has closed range L not equal to $\mathcal{B}$, then there exists a non-zero $\phi \in \mathcal{B}^*$ which vanishes on L by the Hahn-Banach theorem. Therefore 0 is an eigenvalue of $B^* = \lambda I - A^*$, with eigenvector ϕ. If B is one-one with range which is not closed, then B^{-1} is unbounded; equivalently there exists a sequence f_n such that $\|f_n\| = 1$ for all n and $\lim_{n\to\infty} \|Bf_n\| = 0$. $\square$

In case (iii) we say that λ lies in the approximate point spectrum of A.

Note In the Hilbert space context we must replace (ii) by the statement that $\overline{\lambda}$ is an eigenvalue of A^*.

Problem 1.2.14 Prove that

$$\mathrm{Spec}(A) = \mathrm{Spec}(A^*)$$

for every bounded operator $A : \mathcal{B} \to \mathcal{B}$. $\square$

Problem 1.2.15 Prove that if λ lies on the topological boundary of the spectrum of A, then it is also in its approximate point spectrum. $\square$

Problem 1.2.16 Find the spectrum and the approximate point spectrum of the shift operator

$$Af(x) := f(x+1)$$

acting on $L^2(0,\infty)$, and of its adjoint operator. $\square$

Problem 1.2.17 Let $a_1, \ldots, a_n$ be elements of an associative algebra $\mathcal{A}$ with identity. Prove that if the elements commute then the product $a_1 \ldots a_n$ is invertible if and only if every a_i is invertible. Prove also that this statement is not always true if the a_i do not commute. Finally prove that if $a_1 \ldots a_n$ and $a_n \ldots a_1$ are both invertible then a_r is invertible for all $r \in \{1, \ldots, n\}$. $\square$

The following is the most elementary of a series of spectral mapping theorems in this book.

Theorem 1.2.18 *If p is a polynomial and a is an element of the associative algebra $\mathcal{A}$ with identity e then*

$$\mathrm{Spec}(p(a)) = p(\mathrm{Spec}(a)).$$

Proof. We assume that p is monic and of degree n. Given $w \in \mathbf{C}$ we have to prove that $w \in \mathrm{Spec}(p(a))$ if and only if there exists $z \in \mathrm{Spec}(a)$ such that $w = p(z)$. Putting $q(z) := p(z) - w$ this is equivalent to the statement that $0 \in \mathrm{Spec}(q(a))$ if and only if there exists $z \in \mathrm{Spec}(a)$ such that $q(z) = 0$. We now write

$$q(z) = \prod_{r=1}^{n} (z - z_r)$$

where z_r are the zeros of q, so that

$$q(a) = \prod_{r=1}^{n} (a - z_r e).$$

The theorem reduces to the statement that $q(a)$ is invertible if and only if $(a - z_r e)$ is invertible for all r. This follows from Problem 1.2.17. $\square$

Problem 1.2.19 Let $A : \mathcal{B} \to \mathcal{B}$ be a bounded operator. We say that the closed linear subspace $\mathcal{L}$ of $\mathcal{B}$ is invariant under A if $A(\mathcal{L}) \subseteq \mathcal{L}$. Prove that this implies that $\mathcal{L}$ is also invariant under $R(z, A)$ for all z in the unbounded component of $\mathbf{C} \backslash \mathrm{Spec}(A)$. Give an example in which $\mathcal{L}$ is not invariant under $R(z, A)$ for some other $z \notin \mathrm{Spec}(A)$. $\square$

1.3 Topologies on vector spaces

We define a topological vector space (TVS) to be a complex vector space $\mathcal{V}$ provided with a topology $\mathcal{T}$ such that the map $\{\alpha, \beta, u, v\} \to \alpha u + \beta v$ is jointly continuous from $\mathbf{C} \times \mathbf{C} \times \mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$. All of the TVSs in this book are generated by a family of seminorms $\{p_a\}_{a\in A}$ in the sense that every open set $U \in \mathcal{T}$ is the union of basic open neighbourhoods

$$\bigcap_{r=1}^{n} \{v : p_{a(r)}(v-u) < \varepsilon_r\}$$

of some central point $u \in \mathcal{V}$. In addition we will assume that if $p_a(u) = 0$ for all $a \in A$ then $u = 0$.[9]

Problem 1.3.1 Prove that the topology generated by a family of seminorms turns $\mathcal{V}$ into a TVS as defined above. □

Problem 1.3.2 Prove that the topology on $\mathcal{V}$ generated by a countable family of seminorms $\{p_n\}_{n=1}^{\infty}$ coincides with the topology for the metric

$$d(u, v) := \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(u-v)}{1+p_n(u-v)}. \qquad \square$$

One says that a TVS $\mathcal{V}$ is a Fréchet space if $\mathcal{T}$ is generated by a countable family of seminorms and the metric d above is complete.

Every Banach space $\mathcal{B}$ has a weak topology in addition to its norm topology. This is defined as the smallest topology on $\mathcal{B}$ for which the bounded linear functionals $\phi \in \mathcal{B}^*$ are all continuous. It is generated by the family of seminorms $p_\phi(f) := |\phi(f)|$. We will write

$$\operatorname*{w-lim}_{n\to\infty} f_n = f \qquad \text{or} \qquad f_n \xrightarrow{\text{w}} f$$

to indicate that the sequence $f_n \in \mathcal{B}$ converges weakly to $f \in \mathcal{B}$, that is

$$\lim_{n\to\infty} \langle f_n, \phi\rangle = \langle f, \phi\rangle$$

for all $\phi \in \mathcal{B}^*$.

Problem 1.3.3 Use the Hahn-Banach theorem to prove that a linear subspace L of a Banach space $\mathcal{B}$ is norm closed if and only if it is weakly closed. □

[9] Systematic accounts of the theory of TVSs are given in [Narici and Beckenstein 1985, Treves 1967, Wilansky 1978].

Our next result is called the uniform boundedness theorem and also the Banach-Steinhaus theorem.[10]

Theorem 1.3.4 *Let $\mathcal{B}$, $\mathcal{C}$ be two Banach spaces and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of bounded linear operators from $\mathcal{B}$ to $\mathcal{C}$. Then the following are equivalent.*

(i) $\sup_{\lambda\in\Lambda} \|A_\lambda\| < \infty$;
(ii) $\sup_{\lambda\in\Lambda} \|A_\lambda x\| < \infty$ *for every* $x \in \mathcal{B}$;
(iii) $\sup_{\lambda\in\Lambda} |\phi(A_\lambda x)| < \infty$ *for every* $x \in \mathcal{B}$ *and* $\phi \in \mathcal{C}^*$.

Proof. Clearly (i)⇒(ii)⇒(iii). Suppose that (ii) holds but (i) does not. We construct sequences $x_n \in \mathcal{B}$ and $\lambda(n) \in \Lambda$ as follows. Let x_1 be any vector satisfying $\|x_1\| = 1/4$. Given $x_1, \ldots, x_{n-1} \in \mathcal{B}$ satisfying $\|x_r\| = 4^{-r}$ for all $r \in \{1, \ldots, n-1\}$, let

$$c_{n-1} := \sup_{\lambda\in\Lambda} \|A_\lambda(x_1 + \cdots + x_{n-1})\|.$$

Since (i) is false there exists $\lambda(n)$ such that

$$\|A_{\lambda(n)}\| \geq 4^{n+1}(n + c_{n-1}).$$

There also exists $x_n \in \mathcal{B}$ such that $\|x_n\| = 4^{-n}$ and

$$\|A_{\lambda(n)} x_n\| \geq \frac{2}{3}\|A_{\lambda(n)}\|\,\|x_n\|.$$

The series $x := \sum_{n=1}^\infty x_n$ is norm convergent and

$$\begin{aligned}
\|A_{\lambda(n)} x\| &\geq \|A_{\lambda(n)} x_n\| - \|A_{\lambda(n)}(x_1 + \cdots + x_{n-1})\| - \|A_{\lambda(n)}\| \sum_{r=n+1}^\infty \|x_r\| \\
&\geq \frac{2}{3}\|A_{\lambda(n)}\| 4^{-n} - c_{n-1} - \frac{1}{3}\|A_{\lambda(n)}\| 4^{-n} \\
&\geq \|A_{\lambda(n)}\| 4^{-n-1} - c_{n-1} \\
&\geq n.
\end{aligned}$$

The contradiction implies (i).

The proof of (iii)⇒(ii) uses (ii)⇒(i) twice, with appropriate choices of $\mathcal{B}$ and $\mathcal{C}$. □

[10] According to [Carothers 2005, p. 53], whom we follow, the proof below was first published by Hausdorff in 1932, but the ‘sliding hump’ idea was already well-known. Most texts give a longer proof based on the Baire category theorem. The sliding hump argument is also used in Theorem 3.3.11.

Corollary 1.3.5 *If the sequence $f_n \in \mathcal{B}$ converges weakly to $f \in \mathcal{B}$ as $n \to \infty$, then there exists a constant c such that $\|f_n\| \le c$ for all n.*

In applications, the hypothesis of the corollary is usually harder to prove than the conclusion. Indeed the boundedness of a sequence of vectors or operators is often one of the ingredients used when proving its convergence, as in the following problem.

Problem 1.3.6 Let A_t be a bounded operator on the Banach space $\mathcal{B}$ for every $t \in [a, b]$, and let $\mathcal{D}$ be a dense linear subspace of $\mathcal{B}$. If $\|A_t\| \le c < \infty$ for all $t \in [a, b]$ and $t \to A_t f$ is norm continuous for all $f \in \mathcal{D}$, prove that $(t, f) \to A_t f$ is a jointly continuous function from $[a, b] \times \mathcal{B}$ to $\mathcal{B}$. □

We define the weak* topology of $\mathcal{B}^*$ to be the smallest topology for which all of the functionals $\phi \to \langle f, \phi \rangle$ are continuous, where $f \in \mathcal{B}$. It is generated by the family of seminorms $p_f(\phi) := |\phi(f)|$ where $f \in \mathcal{B}$. If $\mathcal{B}$ is reflexive the weak and weak* topologies on $\mathcal{B}^*$ coincide, but generally they do not.

Theorem 1.3.7 *(Banach-Alaoglu) Every norm bounded set in $\mathcal{B}^*$ is relatively compact for the weak* topology, in the sense that its weak* closure is weak* compact.*

Proof. It is sufficient to prove that the ball

$$B_1^* := \{\phi \in \mathcal{B}^* : \|\phi\| \le 1\}$$

is compact. We first note that the topological product

$$S := \prod_{f \in \mathcal{B}} \{z \in \mathbf{C} : |z| \le \|f\|\}$$

is a compact Hausdorff space. It is routine to prove that the map $\tau : B_1^* \to S$ defined by

$$\{\tau(\phi)\}(f) := \langle f, \phi \rangle$$

is a homeomorphism of B_1^* onto a closed subset of S. □

Problem 1.3.8 Prove that the unit ball B_1^* of B^* provided with the weak* topology is metrizable if and only if $\mathcal{B}$ is separable. □

Problem 1.3.9 Suppose that $f_n \in l^p(\mathbf{Z})$ and that $\|f_n\|_p \le 1$ for all $n = 1, 2, \ldots$. Prove that if $1 < p < \infty$ then the sequence f_n converges weakly to 0 if and only if the functions converge pointwise to 0, but that if $p = 1$ this is not always the case. Deduce that the unit ball in $l^1(\mathbf{Z})$ is not weakly compact, so $l^1(\mathbf{Z})$ cannot be reflexive. □

Bounded operators between two Banach spaces $\mathcal{B}$ and $\mathcal{C}$ can converge in three different senses. Given a sequence of operators $A_n : \mathcal{B} \to \mathcal{C}$, we will write $A_n \xrightarrow{\text{n}} A$, $A_n \xrightarrow{\text{s}} A$ and $A_n \xrightarrow{\text{w}} A$ respectively in place of

$$\lim_{n\to\infty} \|A_n - A\| = 0,$$

$$\lim_{n\to\infty} \|A_n f - Af\| = 0 \text{ for all } f \in \mathcal{B},$$

$$\lim_{n\to\infty} \langle A_n f, \phi\rangle = \langle Af, \phi\rangle \text{ for all } f \in \mathcal{B},\ \phi \in \mathcal{C}^*.$$

Another notation is $\lim_{n\to\infty} A_n = A$, $\text{s-lim}_{n\to\infty} A_n = A$, $\text{w-lim}_{n\to\infty} A_n = A$.

Problem 1.3.10 Let A, A_n be bounded operators on the Banach space $\mathcal{B}$ and let $\mathcal{D}$ be a dense linear subspace of $\mathcal{B}$. Use the uniform boundedness theorem to prove that $A_n \xrightarrow{\text{s}} A$ if and only if there exists a constant c such that $\|A_n\| \le c$ for all n and $\lim_{n\to\infty} A_n f = Af$ for all $f \in \mathcal{D}$. □

Problem 1.3.11 Given two sequences of operators $A_n : \mathcal{B} \to \mathcal{C}$ and $B_n : \mathcal{C} \to \mathcal{D}$, prove the following results:
(a) If $A_n \xrightarrow{\text{s}} A$ and $B_n \xrightarrow{\text{s}} B$ then $B_n A_n \xrightarrow{\text{s}} BA$.
(b) If $A_n \xrightarrow{\text{s}} A$ and $B_n \xrightarrow{\text{w}} B$ then $B_n A_n \xrightarrow{\text{w}} BA$.
(c) If $A_n \xrightarrow{\text{w}} A$ and $B_n \xrightarrow{\text{w}} B$ then $B_n A_n \xrightarrow{\text{w}} BA$ may be false.
Prove or give counterexamples to all other combinations of these types of convergence. □

From the point of view of applications, norm convergence is the best, but it is too strong to be true in many situations; weak convergence is the easiest to prove, but it does not have good enough properties to prove many theorems. One is left with strong convergence as the most useful concept.

Problem 1.3.12 Let P_n be a sequence of projections on $\mathcal{B}$, i.e. operators such that $P_n^2 = P_n$ for all n. Prove that if $P_n \xrightarrow{\text{s}} P$ then P is a projection, and give a counterexample to this statement if one replaces strong convergence by weak convergence. □

Problem 1.3.13 Let A, A_n be operators on the Hilbert space $\mathcal{H}$. Prove that if $A_n \xrightarrow{\text{s}} A$ then $A_n^* \xrightarrow{\text{w}} A^*$, and give an example in which A_n^* does not converge strongly to A^*. □

One sometimes says that A_n converges in the strong* sense to A if $A_n \xrightarrow{\text{s}} A$ *and* $A_n^* \xrightarrow{\text{s}} A^*$.

1.4 Differentiation of vector-valued functions

We discuss various notions of differentiability for two functions $f : [a, b] \to \mathcal{B}$ and $\phi : [a, b] \to \mathcal{B}^*$. We write C^n to denote the space of n times continuously differentiable functions if $n \geq 1$, and the space of continuous functions if $n = 0$.

Lemma 1.4.1 *If $\langle f(t), \psi \rangle$ is C^1 for all $\psi \in \mathcal{B}^*$ then $f(t)$ is C^0. Similarly, if $\langle g, \phi(t) \rangle$ is C^1 for all $g \in \mathcal{B}$ then $\phi(t)$ is C^0.*

Proof. By the uniform boundedness theorem there is a constant N such that $\|f(t)\| \leq N$ for all $t \in [a, b]$. If $a \leq c \leq b$ then

$$\lim_{\delta \to 0} \langle \delta^{-1}\{f(c+\delta) - f(c)\}, \psi \rangle = \frac{\mathrm{d}}{\mathrm{d}c} \langle f(c), \psi \rangle,$$

so using the uniform boundedness theorem again there exists a constant M such that

$$\|\delta^{-1}\{f(c+\delta) - f(c)\}\| \leq M$$

for all small enough $\delta \neq 0$. This implies that

$$\lim_{\delta \to 0} \|f(c+\delta) - f(c)\| = 0.$$

The other part of the lemma has a similar proof. $\square$

Lemma 1.4.2 *If $\langle f(t), \psi \rangle$ is C^2 for all $\psi \in \mathcal{B}^*$ then $f(t)$ is C^1. Similarly, if $\langle g, \phi(t) \rangle$ is C^2 for all $g \in \mathcal{B}$ then $\phi(t)$ is C^1.*

Proof. By the uniform boundedness theorem there exist $g(t) \in \mathcal{B}^{**}$ for each $t \in [a, b]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f(t), \psi \rangle = \langle g(t), \psi \rangle.$$

Moreover $\langle g(t), \psi \rangle$ is C^1 for all $\psi \in \mathcal{B}^*$, so by Lemma 1.4.1 $g(t)$ depends norm continuously on t. Therefore

$$\int_a^t g(s)\,\mathrm{d}s$$

is defined as an element of $\mathcal{B}^{**}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle f(t) - f(a) - \int_a^t g(s)\,\mathrm{d}s, \psi \right\rangle = 0$$

for all $t \in [a, b]$ and $\psi \in \mathcal{B}^*$. It follows that

$$f(t) - f(a) = \int_a^t g(s)\,\mathrm{d}s$$

for all $t \in [a, b]$. We deduce that

$$g(t) = \lim_{h \to 0} h^{-1}\{f(t+h) - f(t)\}$$

the limit being taken in the norm sense. Therefore $g(t) \in \mathcal{B}$, and $f(t)$ is C^1. The proof for $\phi(t)$ is similar. $\square$

Corollary 1.4.3 *If $\langle f(t), \psi\rangle$ is C^∞ for all $\psi \in \mathcal{B}^*$ then $f(t)$ is C^∞. Similarly, if $\langle g, \phi(t)\rangle$ is C^∞ for all $g \in \mathcal{B}$ then $\phi(t)$ is C^∞.*

Proof. One shows inductively that if $\langle f(t), \psi\rangle$ is C^{n+1} for all $\psi \in \mathcal{B}^*$ then $f(t)$ is C^n. $\square$

We will need the following technical lemma later in the book.

Lemma 1.4.4 *(i) If $f : [0, \infty) \to \mathbf{R}$ is continuous and for all $x \geq 0$ there exists a strictly monotonic decreasing sequence x_n such that*

$$\lim_{n\to\infty} x_n = x, \qquad \limsup_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} \leq 0$$

then f is non-increasing on $[0, \infty)$.

(ii) If $f : [0, \infty) \to \mathcal{B}$ is norm continuous and for all $x \geq 0$ there exists a strictly monotonic decreasing sequence x_n such that

$$\lim_{n\to\infty} x_n = x, \qquad \lim_{n\to\infty} \left\langle \frac{f(x_n) - f(x)}{x_n - x}, \phi \right\rangle = 0$$

for all $\phi \in \mathcal{B}^*$ then f is constant on $[0, \infty)$.

Proof. (i) If $\alpha > 0$, $a \geq 0$ and

$$S_{\alpha,a} := \{x \geq a : f(x) \leq f(a) + \alpha(x - a)\}$$

then $S_{\alpha,a}$ is closed, contains a, and for all $x \in S_{\alpha,a}$ and $\varepsilon > 0$ there exists $t \in S_{\alpha,a}$ such that $x < t < x + \varepsilon$. If $u > a$ then there exists a largest number $s \in S_{\alpha,a}$ satisfying $s \leq u$. The above property of $S_{\alpha,a}$ implies that $s = u$. We deduce that $S_{\alpha,a} = [a, \infty)$ for every $\alpha > 0$, and then that $f(x) \leq f(a)$ for all $x \geq a$.

(ii) We apply part (i) to $\mathrm{Re}\left\{e^{i\theta}\langle f(x), \phi\rangle\right\}$ for every $\phi \in \mathcal{B}^*$ and every $\theta \in \mathbf{R}$ to deduce that $\langle f(x), \phi\rangle = 0$ for all $x \geq 0$. Since $\phi \in \mathcal{B}^*$ is arbitrary we deduce that $f(x)$ is constant. $\square$

All of the above ideas can be extended to operator-valued functions. We omit a systematic treatment of the various topologies for which one can define differentiability, but mention three results.

Problem 1.4.5 Prove that if $A,\ B : [a, b] \to \mathcal{L}(\mathcal{B})$ are continuously differentiable in the strong operator topology then they are norm continuous. Moreover $A(t)B(t)$ is continuously differentiable in the same sense and

$$\frac{\mathrm{d}}{\mathrm{d}t}\{A(t)B(t)\} = A(t)'B(t) + A(t)B(t)'$$

for all $t \in [a, b]$. □

Problem 1.4.6 Prove that if $A : [a, b] \to \mathcal{L}(\mathcal{B})$ is differentiable in the strong operator topology then $A(t)^{-1}$ is strongly differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t)^{-1} = -A(t)^{-1}A(t)'A(t)^{-1}$$

for all $t \in [a, b]$. □

Problem 1.4.7 Prove that if $A(t)$ is a differentiable family of $m \times m$ matrices for $t \in [a, b]$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t)^n \neq nA(t)'A(t)^{n-1}$$

in general, but nevertheless

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}[A(t)^n] = n\,\mathrm{tr}[A(t)'A(t)^{n-1}]. \qquad \square$$

We now turn to the study of analytic functions. Let $f(z)$ be a function from the region (connected open subset) U of the complex plane $\mathbf{C}$ taking values in the complex Banach space $\mathcal{B}$. We say that f is analytic on U if it is infinitely differentiable in the norm topology at every point of U.

Lemma 1.4.8 *If $\langle f(z), \phi\rangle$ is analytic on U for all $\phi \in \mathcal{B}^*$ then $f(z)$ is analytic on U.*

Proof. We first note that by a complex variables version of Lemma 1.4.1, $z \to f(z)$ is norm continuous. If γ is the boundary of a disc inside U then

$$\begin{aligned}\langle f(z), \phi\rangle &= \frac{1}{2\pi i}\int_\gamma \frac{\langle f(w), \phi\rangle}{w - z}\,\mathrm{d}w \\ &= \left\langle \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}\,\mathrm{d}w, \phi\right\rangle\end{aligned}$$

for all $\phi \in \mathcal{B}^*$. This implies the vector-valued Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w, \tag{1.9}$$

the right-hand side of which is clearly an analytic function of z. $\square$

Problem 1.4.9 Prove a vector-valued Liouville's theorem: namely if $f : \mathbf{C} \to \mathcal{B}$ is uniformly bounded in norm and analytic then it is constant. $\square$

Lemma 1.4.10 *Let $f_n \in \mathcal{B}$ and suppose that*

$$\sum_{n=0}^{\infty} \langle f_n, \phi \rangle z^n$$

converges for all $\phi \in \mathcal{B}^*$ and all $|z| < R$. Then the power series

$$\sum_{n=0}^{\infty} f_n z^n \tag{1.10}$$

is norm convergent for all $|z| < R$, and the limit is a $\mathcal{B}$-valued analytic function.

Proof. We define the linear functional $f(z)$ on $\mathcal{B}^*$ by

$$\langle f(z), \phi \rangle := \sum_{n=0}^{\infty} \langle f_n, \phi \rangle z^n.$$

The uniform boundedness theorem implies that $f(z) \in \mathcal{B}^{**}$ for all $|z| < R$. An argument similar to that of Lemma 1.4.1 establishes that $z \to f(z)$ is norm continuous, and an application of the Cauchy integral formula as in Lemma 1.4.8 shows that $f(z)$ is norm analytic. A routine modification of the usual proof for the case $\mathcal{B} = \mathbf{C}$ now establishes that the series (1.10) is norm convergent, so we finally see that $f(z) \in \mathcal{B}$ for all $|z| < R$. $\square$

If $a_n \in \mathcal{B}$ for $n = 0, 1, 2, \ldots$ then the power series $\sum_{n=0}^{\infty} a_n z^n$ defines a $\mathcal{B}$-valued analytic function for all z for which the series converges. The radius of convergence R is defined as the radius of the largest circle with centre at 0 within which the series converges. As in the scalar case $R = 0$ and $R = +\infty$ are allowed.

Problem 1.4.11 Prove that

$$R = \sup\{\rho : \{\|a_n\| \rho^n\}_n \text{ is a bounded sequence}\}.$$

Alternatively

$$R^{-1} = \limsup_{n\to\infty} \|a_n\|^{1/n}. \qquad \square$$

The following theorem establishes that the powers series of an analytic function converges on the maximal possible ball $B(0, r) := \{z : |z| < r\}$.

Theorem 1.4.12 *Let $f : B(0, r) \to \mathcal{B}$ be an analytic function which cannot be analytically continued to a larger ball. Then the power series of f has radius of convergence r.*

Proof. If we denote the radius of convergence by R, then it follows immediately from Problem 1.4.11 that $R \le r$. If $|z| < r$ and $t = (r + |z|)/2$ then by adapting the classical proof (which depends on using (1.9)) we obtain

$$f(z) = f(0) + f'(0)\frac{z}{1!} + \cdots + f^{(n)}(0)\frac{z^n}{n!} + \mathrm{Rem}(n)$$

where

$$\mathrm{Rem}(n) := \frac{1}{2\pi i}\int_{|w|=t} \frac{f(w)}{w-z}\left(\frac{z}{w}\right)^{n+1} \mathrm{d}w.$$

This implies that

$$\|\mathrm{Rem}(n)\| \le c_{z,t}(z/t)^{n+1}$$

which converges to 0 as $n \to \infty$. Therefore the power series converges for every z such that $|z| < r$. This implies that $R \ge r$. $\square$

All of the results above can be extended to operator-valued analytic functions. Since the space $\mathcal{L}(\mathcal{B})$ is itself a Banach space with respect to the operator norm, the only new issue is dealing with weaker topologies.

Problem 1.4.13 Prove that if $A(z)$ is an operator-valued function on $U \subseteq \mathbf{C}$, and $z \to \langle A(z)f, \phi\rangle$ is analytic for all $f \in \mathcal{B}$ and $\phi \in \mathcal{B}^*$, then $A(z)$ is an analytic function of z. $\square$

1.5 The holomorphic functional calculus

The material in this section was developed by Hilbert, E. H. Moore, F. Riesz and others early in the twentieth century. A functional calculus is a procedure for defining an operator $f(A)$, given an operator A and some class of complex-valued functions f defined on the spectrum of A. One requires $f(A)$ to satisfy certain properties, including (1.11) below. The following theorem defines a holomorphic functional calculus for bounded linear operators. Several of the proofs in this section apply with minimal changes to unbounded operators, and we will take advantage of that fact later in the book.

Theorem 1.5.1 *Let S be a compact component of the spectrum* $\mathrm{Spec}(A)$ *of the operator A acting on $\mathcal{B}$, and let $f(\cdot)$ be a function which is analytic on a neighbourhood U of S. Let γ be a closed curve in U such that S is inside γ and* $\mathrm{Spec}(A)\backslash S$ *is outside γ. Then*

$$B := \frac{1}{2\pi i}\int_\gamma f(z)R(z,A)\mathrm{d}z$$

is a bounded operator commuting with A. It is independent of the choice of γ, subject to the above conditions. Writing B in the form $f(A)$ we have

$$f(A)g(A) = (fg)(A) \tag{1.11}$$

for any two functions f, g of the stated type. The map $f \to f(A)$ is norm continuous from the stated class of functions with $\|f\| := \max\{|f(z)| : z \in \gamma\}$ to $\mathcal{L}(\mathcal{B})$.

Proof. It is immediate from its definition that

$$BR(w,A) = R(w,A)B$$

for all $w \notin \mathrm{Spec}(A)$. This implies that B commutes with A. In the following argument we label B according to the contour used to define it. If σ is a second contour with the same properties as γ, and we put $\delta := \gamma - \sigma$, then

$$B_\gamma - B_\sigma = B_\delta = 0$$

by the operator version of Cauchy's Theorem.

To prove (1.11), let γ, σ be two curves satisfying the stated conditions, with σ inside γ. Then

$$\begin{aligned}
f(A)g(A) &= -\frac{1}{4\pi^2}\int_\sigma\int_\gamma f(z)g(w)R(z,A)R(w,A)\,\mathrm{d}z\,\mathrm{d}w\\
&= -\frac{1}{4\pi^2}\int_\sigma\int_\gamma \frac{f(z)g(w)}{z-w}(R(w,A)-R(z,A))\,\mathrm{d}z\,\mathrm{d}w\\
&= \frac{1}{2\pi i}\int_\sigma f(w)g(w)R(w,A)\,\mathrm{d}w\\
&= (fg)(A).
\end{aligned}$$

□

Problem 1.5.2 Let A be a bounded operator on $\mathcal{B}$ and let γ be the closed curve $\theta \to re^{i\theta}$ where $r > \|A\|$. Prove that if $p(z) := \sum_{m=0}^{n} a_m z^m$ then

$$p(A) = \sum_{m=0}^{n} a_m A^m. \qquad \square$$

Example 1.5.3 Let A be a bounded operator on $\mathcal{B}$ and suppose that $\mathrm{Spec}(A)$ does not intersect $(-\infty, 0]$. Then there exists a closed contour γ that winds around $\mathrm{Spec}(A)$ and which does not intersect $(-\infty, 0]$. If $t > 0$ then the function z^t is holomorphic on and inside γ, so one may use the holomorphic functional calculus to define A^t. However, one should not suppose that $\|A^t\|$ must be of the same order of magnitude as $\|A\|$ for $0 < t < 1$. Figure 1.1 displays the norms of A^t for $n := 100$, $c := 0.6$ and $0 < t < 2$, where A is the $n \times n$ matrix

$$A_{r,s} := \begin{cases} r/n & \text{if } s = r+1, \\ c & \text{if } r = s, \\ 0 & \text{otherwise.} \end{cases}$$

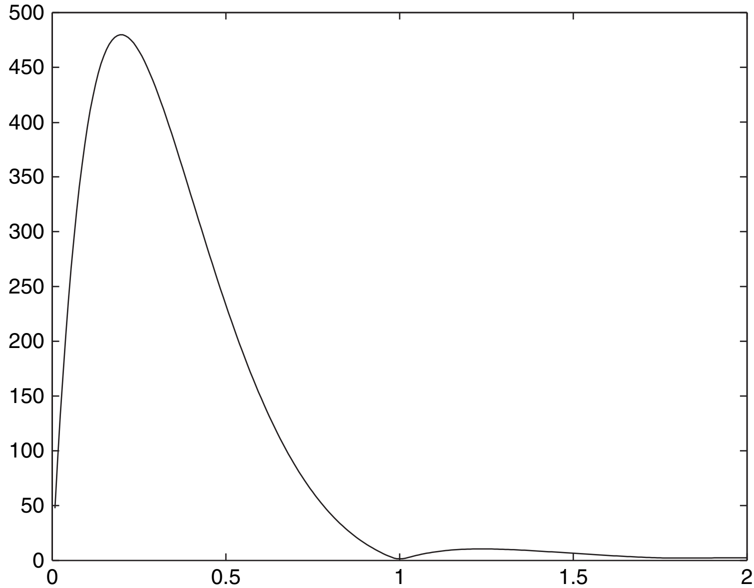

Figure 1.1: Norms of fractional powers in Example 1.5.3

Note that $\|A^t\|$ is of order 1 for $t = 0, 1, 2$. It can be much larger for other t because the resolvent norm must be extremely large on a portion of the contour γ, for any contour satisfying the stated conditions. See also Example 10.2.1. $\square$

Theorem 1.5.4 *(Riesz) Let γ be a closed contour enclosing the compact component S of the spectrum of the bounded operator A acting in $\mathcal{B}$, and suppose that $T = \mathrm{Spec}(A)\backslash S$ is outside γ. Then*

$$P := \frac{1}{2\pi i}\int_\gamma R(z, A)\,\mathrm{d}z$$

is a bounded projection commuting with A. The restriction of A to $P\mathcal{B}$ has spectrum S and the restriction of A to $(I-P)\mathcal{B}$ has spectrum T. P is said to be the spectral projection of A associated with S.

Proof. It follows from Theorem 1.5.1 with $f = g = 1$ that $P^2 = P$. If we put $\mathcal{B}_0 = \mathrm{Ran}(P)$ and $\mathcal{B}_1 = \mathrm{Ker}(P)$ then $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ and $A(\mathcal{B}_i) \subseteq \mathcal{B}_i$ for $i = 0, 1$. If A_i denotes the restriction of A to $\mathcal{B}_i$ then

$$\mathrm{Spec}(A) = \mathrm{Spec}(A_0) \cup \mathrm{Spec}(A_1).$$

The proof is completed by showing that

$$\mathrm{Spec}(A_0) \cap T = \emptyset, \qquad \mathrm{Spec}(A_1) \cap S = \emptyset.$$

If w is in T, then it is outside γ, and we put

$$C_w := \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z} R(z, A)\,\mathrm{d}z.$$

Theorem 1.5.1 implies that

$$\begin{aligned} C_w P = P C_w &= C_w, \\ (wI - A)C_w = C_w(wI - A) &= P. \end{aligned}$$

Therefore $w \notin \mathrm{Spec}(A_0)$. Hence $\mathrm{Spec}(A_0) \cap T = \emptyset$.

Now let τ be the circle with centre 0 and radius $(\|A\| + 1)$. By expanding the resolvent on powers of $1/z$ we see that

$$I = \frac{1}{2\pi i}\int_\tau R(z, A)\,\mathrm{d}z.$$

If σ denotes the curve $(\tau - \gamma)$ then we deduce that

$$I - P = \frac{1}{2\pi i} \int_\sigma R(z, A)\, \mathrm{d}z.$$

By following the same argument as in the first paragraph we see that if w is in S, then it is inside γ and outside σ, so $w \notin \mathrm{Spec}(A_1)$. Hence $\mathrm{Spec}(A_1) \cap S = \emptyset$. $\square$

If S consists of a single point z then the restriction of A to $\mathcal{B}_0 = \mathrm{Ran}(P)$ has spectrum equal to $\{z\}$, but this does not imply that $\mathcal{B}_0$ consists entirely of eigenvectors of A. Even if $\mathcal{B}_0$ is finite-dimensional, the restriction of A to $\mathcal{B}_0$ may have a non-trivial Jordan form. The full theory of what happens under small perturbations of A is beyond the scope of this book, but the next theorem is often useful. Its proof depends upon the following lemma. The properties of orthogonal projections on a Hilbert space are studied more thoroughly in Section 5.3. We define the rank of an operator to be the possibly infinite dimension of its range.

Lemma 1.5.5 *If P and Q are two bounded projections and $\|P - Q\| < 1$ then*

$$\mathrm{rank}(P) = \mathrm{rank}(Q).$$

Proof. If $0 \neq x \in \mathrm{Ran}(P)$ then $\|Qx - x\| = \|(Q - P)x\| < \|x\|$, so $Qx \neq 0$. Therefore Q maps $\mathrm{Ran}(P)$ one-one into $\mathrm{Ran}(Q)$ and $\mathrm{rank}(P) \leq \mathrm{rank}(Q)$. The converse has a similar proof. $\square$

A more general version of the following theorem is given in Theorem 11.1.6, but even that is less general than the case treated by Rellich, in which one simply assumes that the operator depends analytically on a complex parameter z.[11]

Theorem 1.5.6 *(Rellich) Suppose that λ is an isolated eigenvalue of A and that the associated spectral projection P has rank 1. Then for any operator B and all small enough $w \in \mathbf{C}$, $(A + wB)$ has a single eigenvalue $\lambda(w)$ near to λ, and this eigenvalue depends analytically upon w.*

Proof. Let γ be a circle enclosing λ and no other point of $\mathrm{Spec}(A)$, and let P be defined as in Theorem 1.5.4. If

$$|w| < \|B\|^{-1} \min\{\|R(z, A)\|^{-1} : z \in \gamma\}$$

[11] A systematic treatment of the perturbation of eigenvalues of higher multiplicity is given in [Kato 1966A].

then $(zI-(A+wB))$ is invertible for all $z\in\gamma$ by Theorem 1.2.9. By examining the expansion (1.5) one sees that $(zI-(A+wB))^{-1}$ depends analytically upon w for every $z\in\gamma$. It follows that the projections

$$P_w := \frac{1}{2\pi i}\int_\gamma (zI-(A+wB))^{-1}\mathrm{d}z$$

depend analytically upon w. By Lemma 1.5.5 P_w has rank 1 for all such w.

If $f\in \mathrm{Ran}(P)$ then $f_w := P_w f$ depends analytically upon w and lies in the range of P_w for all w. Assuming $f\neq 0$ it follows that $f_w\neq 0$ for all small enough w. Therefore f_w is the eigenvector of $(A+wB)$ associated with the eigenvalue lying within γ for all small enough w. The corresponding eigenvalue satisfies

$$\langle (A+wB)f_w,\phi\rangle = \lambda_w\langle f_w,\phi\rangle$$

where ϕ is any vector in $\mathcal{B}^*$ which satisfies $\langle f,\phi\rangle = 1$. The analytic dependence of λ_w on w for all small enough w follows from this equation. $\square$

Example 1.5.7 The following example shows that the eigenvalues of non-self-adjoint operators may behave in counter-intuitive ways (for those brought up in self-adjoint environments). Let H be a self-adjoint $n\times n$ matrix and let $Bf := \langle f,\phi\rangle\phi$, where ϕ is a fixed vector of norm 1 in $\mathbf{C}^n$. If $A_s := H+isB$ then $\mathrm{Im}\langle A_s f, f\rangle$ is a monotone increasing function of $s\in\mathbf{R}$ for all $f\in\mathbf{C}^n$, and this implies that every eigenvalue of A_s has a positive imaginary part for all $s>0$. If $\{\lambda_{r,s}\}_{r=1}^n$ are the eigenvalues of A_s then

$$\sum_{r=1}^n \lambda_{r,s} = \mathrm{tr}(A_s) = \mathrm{tr}(H)+is$$

for all s. All these facts (wrongly) suggest that the imaginary part of each individual eigenvalue is a positive, monotonically increasing function of s for $s\geq 0$.

More careful theoretical arguments show that the eigenvalues of such an operator move from the real axis into the upper half plane as s increases from 0. All except one then turn around and converge back to the real axis as $s\to+\infty$. For $n=2$ the calculations are elementary, but the case

$$A_s := \begin{pmatrix} -1+is & is & is \\ is & is & is \\ is & is & 1+is \end{pmatrix} \tag{1.12}$$

is more typical.[12] $\square$

[12] See Lemma 11.2.9 for further examples of a similar type.

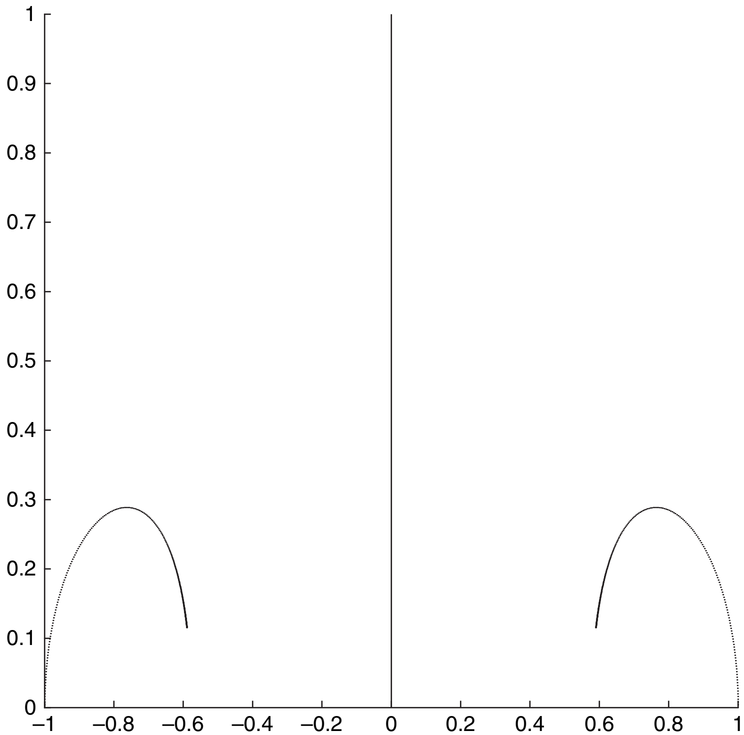

Figure 1.2: Eigenvalues of (1.12) for $0 \le s \le 1$

If an operator $A(z)$ has several eigenvalues $\lambda_r(z)$, all of which depend analytically on z, then generically they will only coincide in pairs, and this will happen for certain discrete values of z. One can analyze the z-dependence of two such eigenvalues by restricting attention to the two-dimensional linear span of the corresponding eigenvectors. The following example illustrates what can happen.

Example 1.5.8 If

$$A(z) := \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

where a, b, c, d are all analytic functions, then the eigenvalues of $A(z)$ are given by

$$\lambda_\pm(z) := (a(z)+d(z))/2 \pm \left\{(a(z)-d(z))^2/4 + b(z)c(z)\right\}^{1/2}.$$

For most values of z the two branches are analytic functions of z, but for certain special z they coincide and one has a square root singularity. In the typical case

$$A(z) := \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$$

one has $\lambda(z) = \pm\sqrt{z}$. The two eigenvalues coincide for $z = 0$, but when this happens the matrix has a non-trivial Jordan form and the eigenvalue 0 has multiplicity 1. $\square$