Lemma 1. *Let K be a subset of $\mathbb{R}^d$, and consider a relative open subset V of K. There exists a sequence $\{f_n\}_{n\in\mathbb{R}^d} \subseteq C(K)$ such that:*

- $\lim_n f_n(x) = \mathbb{1}_V(x)$ *pointwise;*
- $\|f_n\|_\infty \leq 1$.

Proof. By definition of subspace topology, V is open in K if and only if there exists U open in $\mathbb{R}^d$ such that $V = U \cap K$, so it is enough to assume $K = \mathbb{R}^d$ (if f is continuos on $\mathbb{R}^d$ then it will be continous on K by defintion of relative topology).

Consider now the set:

$$U^{\frac{1}{n}} = \left\{ x \in U : d(x, \partial U) \geq \frac{1}{n} \right\} \subseteq U,$$

and the functions:

$$\varphi_n(x) = \begin{cases} 1, \quad \text{if } x \leq 0; \\ -nx+1, \quad \text{if } x \in [0, \frac{1}{n}]; \\ 0, \quad x \geq \frac{1}{n}. \end{cases}$$

$$f_n(x) = \varphi_n(d(x, U^{\frac{1}{n}})).$$

- If $x \in U$ then there exists $n \in \mathbb{N}$ such that $d(x, \partial U) \geq \frac{1}{n}$ and so $x \in U^{\frac{1}{m}}$ and $f_m(x) = 1$ for all $m \geq n$;
- If $x \notin U$, then for any $n \in \mathbb{N}$: $d(x, U^{\frac{1}{n}}) \geq \frac{1}{n}$ and so $f_n(x) = 0$.

$\{f_n\}_{n\in\mathbb{N}}$ is the desidered sequence. □

Theorem 1. *Let K be a compact subset of $\mathbb{R}^d$ (in particolar K can be a compact subset of $\mathbb{C}$), and let $B(K)$ be the set of borel bounded functions on K. Consider a subset $U \subseteq B(K)$ such that:*

1. $C(K) \subseteq U$;
2. *U is closed under* bounded pointwise convergence *(BPC) i.e: given a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq U$ such that:*
   (a) $\sup_n \|f_n\|_\infty = M < \infty$;
   (b) $\lim_n f_n(t) = f(t) \quad \forall t \in K$;

   the pointwise limit function f belongs to U.

Then $U = B(K)$.

Proof. Consider the set Ω defined by:

$$\Omega = \{C(K) \subseteq S \subseteq B(K) : S \text{ is closed under BPC}\},$$

this set is nonempty ($U \in \Omega$). Consider:

$$V = \bigcap_{S \in \Omega} S.$$

Of course we have $V \supseteq C(K)$.
CLAIM: V is a vector space.
Let $f_0 \in V$, define the set:

$$V_{f_0} = \{g \in B(K); f_0 + g \in V\}.$$

If $f_0 \in C(K)$ we have that:

- $C(K) \subseteq V_{f_0}$: infact given $g \in C(K)$ then: $g + f_0 \in C(K) \subseteq V$ and so $g \in V_{f_0}$;
- V_{f_0} is closed under BPC: infact given a sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq V_{f_0}$ such that $\sup_n \|g_n\|_\infty = M < \infty$ and $g_n \to g$ pointwise, we know that V is closed under BPC and that $g_n + f \in V$ so that:

$$g_n + f_0 \to g + f_0 \quad \text{pointwise};$$
$$\|g_n + f_0\|_\infty \leq M + \|f_0\|_\infty, \quad \forall n \in \mathbb{N};$$

  and so $g + f_0 \in V$ and $g \in V_{f_0}$.

So $V_{f_0} \in \Omega$, and $V \subseteq V_{f_0}$ for any $f_0 \in C(K)$, hence if $f_0 \in C(K)$ and $g \in V$ we have $f_0 + g \in V$.

Consider now $g_0 \in V$, from what we just showed we have $C(K) \subseteq V_{g_0}$. It can be proved (as we did before) that V_{g_0} is closed under BPC. So we have $V_{g_0} \supseteq V$, and so for any $f \in V$ we have $f + g_0 \in V$. Given the arbitrariness of g_0 we have that V is closed under addition.

Similarly we can show that given $\alpha \in \mathbb{C}$ we have $f \in V$ implies $\alpha f \in V$, so that V is a complex vector space.

Consider now the borel σ-algebra Σ.

Claim: For any $F \in \Sigma$ we have $\mathbb{1}_F \in V$.

Assuming the claim true, we have that all simple[1] function on K are in V, because V is a vector space. But the set of simple functions is dense in $B(K)$

[1] A function from $s : K \to \mathbb{C}$ is said to be a *simple function* if there exist scalars $a_1, \ldots, a_n \in \mathbb{C}$ and measurable sets $F_1, \ldots, F_n$ such that:

$$s = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}.$$

with respect to $\|\cdot\|_\infty$, by a standard resault in Measure Theory, and so we can conclude that $V = B(K)$ by BPC closeness, and since $V \subseteq U$ we are done.

To prove the claim define the set:

$$\Delta = \{E \in \Sigma; \mathbb{1}_E \in V\}$$

and observe that given the charectestic function of an open set, we can approximate it (pointwisely) with continous functions with infinity norm less then one (by Lemma 1). So we have that $A \in \Delta$ for any open set A.

If we show that Δ is a σ-algebra, we are done. By a standard[2] result in Measure Theory it is enough to show that:

1. If $E, F \in \Delta$ and $E \subseteq F$ then $F \setminus E \in \Sigma$;
2. Se$E_1, \ldots, E_n, \dot{\in} \Delta$ is a family of pairwise disjoint sets then $E = \bigcup_n E_n \in \Delta$.

In our case we have that:

1. $\mathbb{1}_{F \setminus E} = \mathbb{1}_F - \mathbb{1}_E$ and so, given the fact that V is a vector space, we have $F \setminus E \in \Delta$ if $E, F \in \Delta$;
2. Let $E_1, \ldots, E_n, \cdots \in \Delta$ pairwise disjoint then we have:

$$\mathbb{1}_{\bigcup_n E_n} = \sum_{n=1}^{\infty} \mathbb{1}_{E_n},$$

and so by BPC closeness we have $\bigcup_n E_n \in \Delta$.

$\square$

[2]A collection of sets satisfying the conditions 1) and 2) is called a *Dynkin system*.