9 L^2 continuity

In this section we prove that pseudodifferential operators are bounded in Sobolev spaces. We will first prove the theorem assuming that $\mathrm{Op}\,(a)$ has a symbol in the class $\mathcal{S}^0$; subsequently we will state an improved version of the the theorem which requires the symbol a only to have a a finite numbers of derivatives bounded.

In the course of the proof we will use some general results which are interesting by themselves, and we prove them here.

9.1 Shur test

Let $K(x,y) \in L^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$. Define the linear operator with kernel K as

$$[A_K u](x) := \int_{\mathbb{R}^d} K(x,y) u(y)\,\mathrm{d}y.$$

We introduce

$$\|A\|_{L^\infty_x L^1_y} := \sup_x \int |K(x,y)|\,\mathrm{d}y,$$

$$\|A\|_{L^\infty_y L^1_x} := \sup_y \int |K(x,y)|\,\mathrm{d}x.$$

Proposition 9.1 (Schur test)**.** *$\forall p \in [1,+\infty]$, $\forall u \in L^p(\mathbb{R}^d)$ we have*

$$\|Au\|_{L^p} \le \|A\|^{1-\frac{1}{p}}_{L^\infty_x L^1_y} \|A\|^{\frac{1}{p}}_{L^\infty_y L^1_x} \|u\|_{L^p}.$$

Proof. If $p = \infty$ the estimate is obvious. If $p < \infty$, from Hölder inequality we get

$$\int |K(x,y)||u(y)|dy = \int \underbrace{|K(x,y)|^{1-\frac{1}{p}}}_{L^{p^*}} \underbrace{|K(x,y)|^{\frac{1}{p}}|u(y)|}_{L^p}\,\mathrm{d}y$$
$$\le \left(\int |K(x,y)|dy\right)^{1-\frac{1}{p}} \left(\int |K(x,y)||u(y)|^p dy\right)^{\frac{1}{p}},$$

hence

$$\left(\int |K(x,y)||u(y)|dy\right)^p \le \|A\|^{p-1}_{L^\infty_x L^1_y} \int |K(x,y)||u(y)|^p dy.$$

It follows that

$$\|Au\|^p_{L^p} \le \int \left(\int |K(x,y)||u(y)|dy\right)^p dx \le \|A\|^{p-1}_{L^\infty_x L^1_y} \int\int |K(x,y)||u(y)|^p dydx.$$

Changing the order of integration we obtain the result. □

9.2 Cotlar-Stein theorem

We start with some motivation. We have a linear operator $T\colon X \to X$ and we want to compute $\|T\|$. In many cases it is possibile to decompose the operator in pieces $T = \sum_i T_i$ in such a way that it is easier to compute the norm of the single pieces T_i. However the gain in the decomposition is lost if one estimates brutally with triangular inequality $\|T\| \le \sum_i \|T_i\|$, except in the case the norms $\|T_i\|$ enjoy decay properties.

There are however better cases: to show them consider for the moment the case $T\colon \mathbb{R}^n \to \mathbb{R}^n$ has a block diagonal structure

$$T = \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_m \end{pmatrix}$$

with Λ_i a $m_i \times m_i$ matrix. Then we have

$$T = \sum_i T_i, \quad T_i := \Pi_i \, \Lambda_i \Pi_i$$

where Π_i is the orthogonal projector on the i-th block; clearly

$$\|T\| = \max_i \|\Lambda_i\|.$$

The improvement is due to the fact that the decomposition is orthogonal:

$$T_j^* T_i = (\Pi_j \Lambda_j \Pi_j)^* \, \Pi_i \Lambda_i \Pi_i = \Pi_j^* \Lambda_j \Pi_j^* \Pi_i \Lambda_i \Pi_i = 0 \ , \qquad \forall i \neq j$$

thus

$$\begin{aligned} \|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^* Tx, x \rangle &= \sum_{i,j} \left\langle T_j^* T_i x, x \right\rangle \\ = \sum_i \langle T_i^* T_i x, x \rangle &= \sum_i \|T_i \Pi_i x\|^2 \leq \sup_i \|T_i\| \|x\| \end{aligned}$$

The crucial point is that $T_j^* T_i = T_j T_i^* = \delta_{ji}$. The idea is simply to replace this strong vanishing requirement by a condition that ensures sufficient decay in $|j - k|$.

Theorem 9.2 (Cotlar-Stein)**.** *Consider $(A_j)_{j \in \mathbb{N}}$ a family of bounded operators on a Hilbert space $\mathcal{H}$. Assume that there exists $M > 0$ such that*

$$\sup_j \sum_k \|A_j^* A_k\|^{1/2} \leq M, \qquad \sup_k \sum_j \|A_k A_j^*\|^{1/2} \leq M.$$

Then defining

$$S_N = \sum_{j \leq N} A_j$$

we have

$$\|S_N\| \leq M, \quad \forall N \in \mathbb{N}. \tag{9.1}$$

Before proving the theorem, we recall a preliminary result:

Lemma 9.3 (TT^* lemma)**.** *Let A be a linear bounded operator on an Hilbert space $\mathcal{H}$. Then*

$$\|A\| = \|A^*\| = \|A^* A\|^{1/2} = \|AA^*\|^{1/2} \tag{9.2}$$

Proof. Take $u \in \mathcal{H}$ with $\|u\| \leq 1$. We have that

$$\begin{aligned} \|Au\|^2 &= \langle Au, Au \rangle \\ &= \langle A^* Au, a \rangle \\ &\leq \|A^* A\| \|u\|^2 \\ &\leq \|A^* A\| \\ &\leq \|A^*\| \|A\| \end{aligned}$$

We obtain that

$$\|A\|^2 \leq \|A^*A\| \leq \|A^*\|\|A\| \tag{9.3}$$

from which we also deduce

$$\|A\| \leq \|A^*\|.$$

By substituting A with A^* and using that $(A^*)^* = A$, we obtain the reverse inequality

$$\|A^*\|^2 \leq \|AA^*\| \leq \|A\|\|A^*\| \tag{9.4}$$

which implies also $\|A^*\| \leq \|A\|$. Thus all the inequalities are actually equalities. □

Proof of Cotlar-Stein theorem. By the previous lemma applied to A_i we get $\|A_i\| = \|A_i^*A_i\|^{1/2} \leq M$ which gives

$$\|\sum_{j\leq N} A_j\| \leq NM;$$

clearly this bound is rough since it depends on N.

To eliminate this dependence we use the "power trick", which consists in writing the norm of S_N as the norm of a power of $S_N^*S_N$ or $S_NS_N^*$. In particular

$$\begin{aligned}
&\|S_N^*S_N\| = \|S_N\|^2 \\
&\|(S_N^*S_N)^2\| = \|(S_N^*S_N)^*\,(S_N^*S_N)\| = \|S_N^*S_N\|^2 = \|S_N\|^4
\end{aligned}$$

and iterating we have that

$$\|(S_N^*S_N)^m\| = \|S_N\|^{2m}, \qquad \forall m \in \mathbb{N}$$

which we write as

$$\|S_N\| = \|(S_N^*S_N)^m\|^{1/2m}, \qquad \forall m \in \mathbb{N}.$$

But now we can exploit almost orthogonality:

$$(S_N^*S_N)^m = \sum_{j_1,k_1,\ldots,j_m,k_m} A_{j_1}^*A_{k_1}\cdots A_{j_m}^*A_{k_m}$$

Now we estimate each term in the sum in two different ways: on one hand we have

$$\|A_{j_1}^*A_{k_1}\cdots A_{j_m}^*A_{k_m}\| \leq \|A_{j_1}^*A_{k_1}\|\cdots\|A_{j_m}^*A_{k_m}\|. \tag{9.5}$$

On the other hand we have

$$\|A_{j_1}^*A_{k_1}\cdots A_{j_m}^*A_{k_m}\| \leq \|A_{j_1}^*\|\|A_{k_1}A_{j_2}^*\|\cdots\|A_{k_{m-1}}A_{j_m}^*\|\,\|A_{k_m}\| \tag{9.6}$$

Using $\min(a,b) \leq (ab)^{1/2}$ for any $a,b \geq 0$, we get

$$\begin{aligned}
\|(S_N^*S_N)^m\| &\leq \sum_{\substack{j_1,\ldots,j_m\\k_1,\ldots,k_m}} \|A_{j_1}\|^{1/2}\,\|A_{j_1}^*A_{k_1}\|^{1/2}\,\|A_{k_1}A_{j_2}^*\|^{1/2}\ldots\|A_{k_{m-1}}A_{j_m}^*\|^{1/2}\|A_{j_m}^*A_{k_m}\|^{1/2}\|A_{k_m}\|^{1/2} \\
&\leq M\sum_{j_1,k_1}\|A_{j_1}^*A_{k_1}\|^{1/2}\sum_{j_2}\|A_{k_1}A_{j_2}^*\|^{1/2}\cdots\sum_{j_m}\|A_{k_{m-1}}A_{j_m}^*\|^{1/2}\sum_{k_m}\|A_{j_m}^*A_{k_m}\|^{1/2} \\
&\leq M^{2m}\sum_{j_1}1 \leq M^{2m}N
\end{aligned}$$

hence we get that

$$\|S_N\| = \|(S_N^*S_N)^m\|^{1/2m} \leq MN^{1/2m}, \qquad \forall m \in \mathbb{N}.$$

Taking $m \gg 1$ gives the thesis. □

We state now a useful corollary of Cotlar-Stein theorem

Corollary 9.4. *With the same assumptions of Cotlar-Stein theorem, the series* $S := \sum_j A_j$ *converges in the strong operator topology, namely*

$$\forall u \in \mathcal{H}, \qquad \exists \lim_{N\to\infty} S_N u =: Su. \tag{9.7}$$

Proof. Take first $u \in \operatorname{Ran} A_k^*$, namely $\exists y \in \mathcal{H}$ such that $u = A_k^* y$. We show that $\{S_N u\}_N$ is a Cauchy sequence. Indeed

$$\|(S_N - S_{N'})u\| = \| \sum_{j=N'}^{N} A_j A_k^* y\| \le \sum_j \|A_j A_k^*\| \|y\| \le \sup_k \sum_{j=N'}^{N} \|A_k^* A_j\| \|y\| \to 0$$

as $N, N' \to \infty$ by the assumptions of Cotlar-Stein. Therefore $\sum_j A_j u$ converges to an element of $\mathcal{H}$.

Denote by $U = \cup_k \operatorname{Ran} A_k^*$. The previous argument shows that $\sum_j A_j u$ converges for any $u \in U$. We show now that it converges for any element $u \in \overline{U}$. Again we show that $\{S_N u\}_N$ is a Cauchy sequence. Take $\epsilon > 0$ arbitrary and $y \in U$ so that $\|x - y\| \le \epsilon$. Then using the previous step and the conclusion of Cotlar-Stein (9.1) we get

$$\begin{aligned}\|(S_N - S_{N'})u\| &\le \|(S_N - S_{N'})y\| + \|S_N(x-y)\| + \|S_{N'}(x-y)\| \\ &\le C\epsilon + 2M\epsilon\end{aligned}$$

Finally take $x \in \overline{U}^{\perp}$. Indeed take $x \in \overline{U}^{\perp}$. In particular $\langle x, A_k^* y\rangle = 0$ for any $k \in \mathbb{N}$ and $y \in \mathcal{H}$. It follows that $x \in \ker A_k$ for any k. Thus $\sum_j A_j x = 0$.

□

Remark 9.5. *Remark that it is not true that* S_N *converges to* S *in the operator norm: as an example take* $\mathcal{H} = \ell^2(\mathbb{N})$, $A_j := \Pi_j = \langle \cdot, \mathbf{e}_j\rangle \mathbf{e}_j$ *the projection on the* j*-th element of the basis. Then* $\Pi_j^* \Pi_k = \delta_{j,k}$*, so the assumptions of Cotlar-Stein are fulfilled. Indeed one has that* $S_N x = \sum_{j\le N} \Pi_j x$ *converges to* x $\forall x$*, but* S_N *does not converges to the identity in the operator topology, as* $\|\mathbb{1} - S_N\| = 1$ *for any* N.

9.3 A first result on boundedness on L^2

We shall prove the following result:

Theorem 9.6. *Let* $a \in \mathcal{S}^0$*. Then* $\operatorname{Op}(a)$ *extends to a bounded operator from* $L^2 \to L^2$ *with the following estimate: there exist* $K \in \mathbb{N}$ *and* $C > 0$ *(independent of* a*) such that*

$$\|\operatorname{Op}(a)\,\psi\|_{L^2} \le C \wp_K^0(a) \|\psi\|_{L^2}, \qquad \forall \psi \in L^2.$$

We assume that Theorem 9.6 holds true and announce immediately the continuity in Sobolev spaces, which is an easy corollary.

Corollary 9.7. *Let* $a \in \mathcal{S}^m$, $m \in \mathbb{R}$*. Then* $\operatorname{Op}(a)$ *extends to a bounded operator* $H^s \to H^{s-m}$ *with the estimate:* $\forall s$ *there exist* $K_s \in \mathbb{N}$ *and* $C_s > 0$ *that*

$$\|\operatorname{Op}(a)\,\psi\|_{H^{s-m}} \le C_s\, \wp_{K_s}^m(a) \|\psi\|_{H^s}, \quad \forall \psi \in H^s.$$

Remark 9.8. *In particular if the symbol has a positive order then* $\operatorname{Op}(a)$ *loses regularity, and if the order is negative then* $\operatorname{Op}(a)$ *gains regularity.*

Proof. We know that $\langle\xi\rangle^s \in \mathcal{S}^s\ \forall s \in \mathbb{R}$. Let $\langle D\rangle^s = \mathrm{Op}\,(\langle\xi\rangle^s)$. We know that

$$\|\psi\|_{H^s} = \|\,\langle\xi\rangle^s\,\widehat{\psi}\|_{L^2} = \|\,\langle D\rangle^s\,\psi\|_{L^2}.$$

Moreover $\langle D\rangle^s$ is invertible with inverse $\langle D\rangle^{-s}$ (it is a Fourier multiplier). Thus

$$\|\mathrm{Op}\,(a)\,\psi\|_{H^{s-m}} = \|\,\langle D\rangle^{s-m}\,\mathrm{Op}\,(a)\,\psi\|_{L^2} = \|\,\langle D\rangle^{s-m}\,\mathrm{Op}\,(a)\,\langle D\rangle^{-s}\ \langle D\rangle^s\,\psi\|_{L^2}.$$

Now $\langle D\rangle^{s-m}\,\mathrm{Op}\,(a)\,\langle D\rangle^{-s}$ has symbol $\langle\xi\rangle^{s-m}\,\#a\#\,\langle\xi\rangle^{-s} \in \mathcal{S}^0$, therefore by L^2 continuity theorem

$$\|\mathrm{Op}\,(a)\,\psi\|_{H^{s-m}} \leq C\,\wp^0_{K_0}(\langle\xi\rangle^{s-m}\,\#a\#\,\langle\xi\rangle^{-s})\,\|\,\langle D\rangle^s\,\psi\|_{L^2} \leq C_s\wp^m_{K_s}(a)\|\,\langle D\rangle^s\,\psi\|_{L^2}.$$

□

Proof of Theorem 9.6 We split the proof into different cases.

(i) Case $a \in \mathcal{S}^{-n-1}$. Then $\mathrm{Op}\,(a)$ has integral kernel

$$K(x,y) = \frac{1}{(2\pi)^n}\int e^{\mathrm{i}(x-y)\xi}a(x,\xi)\,\mathrm{d}\xi,$$

and one has

$$|K(x,y)| \leq (2\pi)^{-n}\int |a(x,\xi)|\,\mathrm{d}\xi \leq (2\pi)^{-n}C\int \langle\xi\rangle^{-n-1}\,\mathrm{d}\xi < +\infty.$$

Therefore, by the dominated convergence theorem, $K \in C^0(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R})$ is a bounded and continuous function.

It can be easily verified that $(x-y)^\alpha K(x,y)$ is the integral kernel of $\mathrm{Op}\left(\mathrm{i}^{|\alpha|}\partial^\alpha_\xi a(x,\xi)\right)$, hence

$$|(x-y)^\alpha K(x,y)| \leq \left|(2\pi)^{-n}\int e^{\mathrm{i}(x-y)\xi}\mathrm{i}^{|\alpha|}(\partial^\alpha_\xi a)(x,\xi)\mathrm{d}\xi\right|$$

$$\leq \int |\partial^\alpha_\xi a(x,\xi)|\mathrm{d}\xi \leq \wp^{-n-1}_{|\alpha|}(a)\int \frac{d\xi}{\langle\xi\rangle^{n+1+|\alpha|}} \leq C\,\wp^{-n-1}_K(a) < \infty.$$

Therefor we get off - diagonal decay :

$$|K(x,y)| \leq \frac{C\,\wp^{-n-1}_K(a)}{1+|x-y|^{n+1}}, \qquad \forall x,y \in \mathbb{R}^n \tag{9.8}$$

One concludes using Schur test, as $\sup_x \int |K(x,y)|\mathrm{d}y = \sup_y \int |K(x,y)|\mathrm{d}x < \infty$.

(ii) Case $a \in \mathcal{S}^m$, $m < 0$. We observe that

$$\|\mathrm{Op}\,(a)\,u\|^2_{L^2} = (\mathrm{Op}\,(a)\,u, \mathrm{Op}\,(a)\,u)_{L^2} = (\mathrm{Op}\,(a)^*\,\mathrm{Op}\,(a)\,u,u)_{L^2} = (\mathrm{Op}\,(a^*\#a)\,u,u)_{L^2}.$$

If $m \leq -\frac{n+1}{2}$ then $a^*\#a \in \mathcal{S}^{2m} \subset \mathcal{S}^{-n-1}$. Hence from point (i) we have that $\mathrm{Op}\,(a^*\#a)$ is bounded $L^2 \to L^2$ and

$$\|a(x,D)u\|^2_{L^2} \leq \|u\|_{L^2}\|\mathrm{Op}\,(a^*\#a)\,u\|_{L^2} \leq C\,\wp^{2m}_{K'}(a^*\#a)\|u\|^2_{L^2} \leq C\,\wp^m_K(a)^2\|u\|^2_{L^2}$$

by symbolic calculus.

Now we iterate. If $m \le -\frac{n+1}{4}$, we obtain

$$Opau\|_{L^2}^2 \le \|u\|_{L^2}\| \underbrace{\mathrm{Op}\,(a^*\#a)}_{\in \mathcal{S}^{2m} \subset \mathcal{S}^{-\frac{n+1}{2}}} u\|_{L^2}^2 \le C\wp_K^m(a)\|u\|_{L^2}.$$

We continue iterating in this way with $-\frac{n+1}{8}, -\frac{n+1}{16}, \cdots$ and find the estimate $\forall m < 0$.

(iii) Case $a \in \mathcal{S}^0$. Let $M > 2\sup_{\mathbb{R}^{2n}} |a(x,\xi)|^2$. Then $M - |a(x,\xi)|^2 \ge M/2 > 0$ and hence the function

$$c(x,\xi) := \sqrt{M - |a(x,\xi)|^2} \in \mathcal{S}^0,$$

since $f(t) = \sqrt{t}$ is C^∞ for $t \ge M/2 > 0$. Now we have that

$$\mathrm{Op}\,(c)^*\,\mathrm{Op}\,(c) = \mathrm{Op}\,(c^*\#c)$$

and

$$\begin{aligned} c^*\#c &= c^*c + \mathcal{S}^{-1} \\ &= \bar{c}c + \mathcal{S}^{-1} \\ &= |c|^2 + \mathcal{S}^{-1} \\ &= M - |a|^2 + \mathcal{S}^{-1} \\ &= M - a^*a + \mathcal{S}^{-1}, \end{aligned}$$

So we obtain

$$0 \le \|\mathrm{Op}\,(c)\,u\|_{L^2}^2 = (\mathrm{Op}\,(c)^*\,\mathrm{Op}\,(c)\,u,u) = (\mathrm{Op}\,(c^*\#c)\,u,u) = ((\mathrm{Op}\,(M) - \mathrm{Op}\,(a^*\#a) + r_{-1}(x,D))u,u),$$

and using item (ii) we finally get

$$\|\mathrm{Op}\,(a)\,u\|_{L^2}^2 \le M\|u\|_{L^2} + \|r_{-1}(x,D)u\|_{L^2}\|u\|_{L^2} \overset{\text{step }(ii)}{\le} (M+C)\|u\|_{L^2}^2.$$

□

9.4 Calderón - Vaillancourt theorem

In this section we improve Theorem 9.6 by showing that it is enough to require boundedness on a finite number of derivatives of the symbol. In particular we will prove the following result:

Theorem 9.9 (Calderón - Vaillancourt)**.** *Assume that* $a \in C^{2d+1}(\mathbb{R}^d \times \mathbb{R}^d)$ *fulfills*

$$|a|_{2d+1} := \sum_{|\alpha+\beta| \le 2d+1} \sup_{x,\xi \in \mathbb{R}^d} \left(|\partial_x^\alpha a(x,\xi)| + \left|\partial_\xi^\beta a(x,\xi)\right| \right) < \infty. \tag{9.9}$$

Then there exists a constant $C_d > 0$ *such that*

$$\|\mathrm{Op}\,(a)\,\|_{\mathcal{L}(L^2)} \le C_d|a|_{2d+1}. \tag{9.10}$$

Before proving Theorem 9.9 we need the following lemma:

Lemma 9.10. *There exists* $\chi \in C_0^\infty(\mathbb{R}^d)$ *with* $\mathrm{supp}\,\chi \subset [-\frac{2}{3}, \frac{2}{3}]^d$ *such that*

$$\sum_{j \in \mathbb{Z}^d} \chi(x-j) = 1.$$

Proof. Take $\theta_0 \in C_0^\infty(\mathbb{R})$, $\theta_0 \geq 0$, $\theta_0 \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$, $\operatorname{supp} \theta_0 \subset [-\frac{2}{3}, \frac{2}{3}]$. Put

$$\theta(x) = \sum_{j \in \mathbb{Z}} \theta_0(x - j), \qquad x \in \mathbb{R}.$$

Then θ is periodic on $\mathbb{Z}$, i.e. $\theta(x + k) = \theta(x)$ $\forall k \in \mathbb{Z}$, smooth, $\theta(x) \geq 1$ $\forall x$ (indeed, for any x, there exists $j \in \mathbb{N}$ s.t. $x - j \in [-\frac{1}{2}, \frac{1}{2}]$). Define

$$\chi(x_1, \ldots, x_d) = \prod_{n=1}^{d} \frac{\theta_0(x_n)}{\theta(x_n)}.$$

One checks that χ fulfills the wanted properties. □

We prove now Theorem 9.9.

Proof of Theorem 9.9. It is sufficient to show that the operator

$$Au(x) = \int e^{\mathrm{i}x\xi} \, a(x, \xi) u(\xi) \mathrm{d}\xi$$

maps continuously $L^2 \to L^2$. Indeed $\mathrm{Op}\,(a) = A \circ \mathcal{F}$ and the result follows by the continuity of $\mathcal{F}$ on L^2.

The idea is to decompose the operator A in almost orthogonal packets, and then use Cotlar-Stein to bound the norm.

We make a partition of the phase space $\mathbb{R}^{2d}$ as following:

$$1 = \sum_{\ell, k \in \mathbb{Z}^d} \chi_{k,\ell}(x, \xi), \qquad \chi_{k,\ell}(x, \xi) := \chi(x - k) \, \chi(\xi - \ell)$$

where χ is the function of Lemma 9.10. Define

$$[A_{k,\ell} u](x) := \int_{\mathbb{R}^d} e^{\mathrm{i}x\xi} \, a(x, \xi) \chi_{k,\ell}(x, \xi) u(\xi) \mathrm{d}\xi, \qquad \forall u \in \mathcal{S}$$

First by Schur criterium we have

$$\sup_{k,\ell} \|A_{k,\ell}\| \leq C \sup |a(x, \xi)|, \tag{9.11}$$

thanks to the fact that each $\chi_{k,\ell}$ has compact support.

Now we claim that

$$\begin{aligned} \|A_{k,\ell}^* \, A_{k',\ell'}\| &\leq \frac{C_d \, |a|_{2d+1}^2}{\langle k - k' \rangle^{2d+1} \, \langle \ell - \ell' \rangle^{2d+1}} \\ \|A_{k,\ell} \, A_{k',\ell'}^*\| &\leq \frac{C_d \, |a|_{2d+1}^2}{\langle k - k' \rangle^{2d+1} \, \langle \ell - \ell' \rangle^{2d+1}} \end{aligned} \tag{9.12}$$

for all $k, k', \ell, \ell' \in \mathbb{Z}$.

Since the square roots of (9.12) are summable, we apply Cotlar-Stein theorem 9.2 and get that for any integer $N \in \mathbb{N}$

$$\| \sum_{|k| \leq N} \sum_{|\ell| \leq N} A_{k,\ell} u \| \leq C_d \, |a|_{2d+1} \|u\|_{L^2}, \qquad \forall u \in \mathcal{S}(\mathbb{R}^d) \tag{9.13}$$

Then, denoting

$$A_N u := \sum_{|k|,|\ell|\le N} A_{k,\ell} u, \qquad u \in \mathcal{S}(\mathbb{R}^d),$$

we have that

$$(A_N u)(x) \to (Au)(x)$$

by dominated convergence theorem, since $1 = \sum_{\ell,k} \chi_{k,\ell}(x,\xi)$. Hence by Fatou lemma

$$\|Au\| = \| \lim_{N\to\infty} Au\| \le \liminf_{N\to\infty} \|A_N u\| \le C_d\, |a|_{2d+1} \|u\|_{L^2}, \qquad \forall u \in \mathcal{S}(\mathbb{R}^d)$$

which by density of $\mathcal{S}(\mathbb{R}^d)$ into L^2 concludes the proof.

To prove (9.12) we note that

$$(A^*_{k,\ell} g)(\xi) = \int e^{-\mathrm{i}\xi x}\, \overline{a_{k,\ell}(x,\xi)}\, g(x)\, \mathrm{d}x.$$

It follows that

$$(A^*_{k,\ell}\, A_{k',\ell'} g)(\xi) = \int K^{k,\ell}_{k',\ell'}(\xi,\eta)\, g(\eta) \mathrm{d}\eta$$

with integral kernel

$$\begin{aligned} K^{k,\ell}_{k',\ell'}(\xi,\eta) &= \int e^{-\mathrm{i}x(\xi-\eta)} \overline{a(x,\xi)} \chi_{k,\ell}(x,\xi)\, a(x,\eta) \chi_{k',\ell'}(x,\eta) \mathrm{d}x \\ &= \int e^{-\mathrm{i}x(\xi-\eta)} \overline{a(x,\xi)}\, a(x,\eta)\, \chi(x-k)\, \chi(\xi-\ell)\, \chi(x-k')\, \chi(\eta-\ell') \mathrm{d}x \end{aligned}$$

We notice immediately that the kernel is zero if

$$|k-k'| \ge 4 \tag{9.14}$$

because the supports of $\chi(x-k)$ and $\chi(x-k')$ are disjoints. So we restrict to $|k-k'| \le 4$.

So we need to prove that the kernel has decay in $|\ell-\ell'|$. Remark immediately that the kernel is zero also for

$$|\xi-\ell| \ge 1, \quad |\eta-\ell'| \ge 1.$$

It follows that we can restrict to $|\xi-\ell| \le 1$, $|\eta-\ell'| \le 1$. Now given ℓ, ℓ' with $|\ell-\ell'| \ge 4$, we have the bound

$$|\xi-\eta| \ge |\ell-\ell'| - |\xi-\ell| - |\eta-\ell'| \ge \frac{|\ell-\ell'|}{2} > 0.$$

In particular we can exploit the usual trick: on the support of the kernel, the operator

$$L = \mathrm{i} \frac{\xi-\eta}{|\xi-\eta|^2} \cdot \nabla_x$$

is well defined and $L\left(e^{-\mathrm{i}x(\xi-\eta)}\right) = e^{-\mathrm{i}x(\xi-\eta)}$. It follows that

$$K^{k,\ell}_{k',\ell'}(\xi,\eta) = \int e^{-\mathrm{i}x(\xi-\eta)}\, (L^*)^{2d+1} \left[\overline{a(x,\xi)}\, a(x,\eta)\, \chi(x-k)\ \chi(x-k')\right]\, \chi(\eta-\ell')\, \chi(\xi-\ell) \mathrm{d}x$$

from which we deduce (using also condition (9.14))

$$\begin{aligned}\left|K^{k,\ell}_{k',\ell'}(\xi,\eta)\right| &\le C_d \frac{1}{|\xi-\eta|^{2d+1}}\left(\sum_{|\alpha|\le 2d+1}\sup_{x,\xi}|\partial_x^\alpha a(x,\xi)|\right)^2 \chi(\eta-\ell')\,\chi(\xi-\ell)\\ &\le C_d\frac{|a|^2_{2d+1}}{\langle k-k'\rangle^{2d+1}\,\langle \ell-\ell'\rangle^{2d+1}}\,\chi(\eta-\ell')\,\chi(\xi-\ell)\end{aligned}$$

for any $|\ell-\ell'|\ge 4$. But this estimate is clearly satisfied also if $|\ell-\ell'|\le 4$, as one sees directly from the expression of $K^{k,\ell}_{k',\ell'}(\xi,\eta)$ (without integrating by parts).

As the support of $K^{k,\ell}_{k',\ell'}(\xi,\eta)$ in the variables η,ξ is bounded uniformly in ℓ,ℓ', we get

$$\sup_\xi\int\left|K^{k,\ell}_{k',\ell'}(\xi,\eta)\right|\mathrm{d}\eta=\sup_\eta\int\left|K^{k,\ell}_{k',\ell'}(\xi,\eta)\right|\mathrm{d}\xi\le\le C_d\frac{|a|^2_{2d+1}}{\langle k-k'\rangle^{2d+1}\,\langle\ell-\ell'\rangle^{2d+1}},$$

one concludes by Shur test that the first of (9.12) holds true. The second one is proved similarly, and we skip the details. ☐

9.5 An application

An immediate application of L^2 continuity is the following: if $a\in\mathcal{S}^m$, $m>0$ is an elliptic symbol, then we can construct a parametrix:

$$\mathrm{Op}\,(b)\,\mathrm{Op}\,(a)=\mathbb{1}+R,\qquad b\in\mathcal{S}^{-m},\quad R\in\mathcal{S}^{-\infty}.$$

This implies in particular that, for any $s\in\mathbb{R}$,

$$\begin{aligned}\|u\|_{s+m}&=\|\mathrm{Op}\,(b)\,\mathrm{Op}\,(a)\,u-Ru\|_{s+m}\le C_s\|\mathrm{Op}\,(a)\,u\|_s+\|Ru\|_{s+m}\\&\le C_s\|\mathrm{Op}\,(a)\,u\|_s+C_N\|u\|_{-N}\end{aligned}$$

for any arbitrary $N\in\mathbb{N}$.

The inequality means that any elliptic operator controls a number of derivatives equal to its order.

Moreover, if u solves $\mathrm{Op}\,(a)\,u=f$ and $f\in H^{s_0}$, then we have

$$\|u\|_{s_0+m}\le C\|f\|_{s_0}$$

namely elliptic estimates.