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On time dependent Schrödinger equations: Global well-posedness and growth of Sobolev norms

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ABSTRACT

In this paper we consider time dependent Schrödinger linear PDEs of the form $i\partial_t \psi = L(t)\psi$, where $L(t)$ is a continuous family of self-adjoint operators. We give conditions for well-posedness and polynomial growth for the evolution in abstract Sobolev spaces.

If $L(t) = H + V(t)$ where $V(t)$ is a perturbation smooth in time and H is a self-adjoint positive operator whose spectrum can be enclosed in spectral clusters whose distance is increasing, we prove that the Sobolev norms of the solution grow at most as t^ϵ when $t \mapsto \infty$, for any $\epsilon > 0$. If $V(t)$ is analytic in time we improve the bound to $(\log t)^\gamma$, for some $\gamma > 0$. The proof follows the strategy, due to Howland, Joye and Nenciu, of the adiabatic approximation of the flow. We recover most of known results and obtain new estimates for several models including 1-degree of freedom Schrödinger operators on $\mathbb{R}$ and Schrödinger operators on Zoll manifolds.

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1. Introduction and statement of the main results

1.1. Introduction

In this paper we study properties of time dependent Schrödinger-type linear partial differential equations defined on scales of Hilbert spaces. Our aim is twofold: (i) to put in a unified setting several results only known in particular cases concerning well-posedness and growth of norms for large time and (ii) to generalize and extend such results to new models.

More precisely, given a scale of Hilbert spaces $\{\mathcal{H}^k\}_{k\in\mathbb{R}}$, we denote by $\langle\cdot,\cdot\rangle_0$ the scalar product of $\mathcal{H}^0$, and we consider Cauchy problems of the form

$$\begin{cases} \mathrm{i}\partial_t\psi(t) = L(t)\psi(t) \\ \psi|_{t=s} = \psi_s \in \mathcal{H}^k\ , \quad s\in\mathbb{R} \end{cases} \tag{1.1}$$

where $L(t)$ is a time-dependent, linear, symmetric (w.r.t. $\langle\cdot,\cdot\rangle_0$) and unbounded operator in $\mathcal{H}^0$. We want here to establish a list of simple criteria which ensure the global in time well-posedness, the unitarity in the base space $\mathcal{H}^0$, as well as giving bounds on the growth of the $\mathcal{H}^k$-norms for the solution of (1.1). In all the paper we assume that the spaces $\mathcal{H}^k$ are defined as the domains of the powers of a positive self-adjoint operator H, i.e. $\mathcal{H}^k \equiv D(H^{k/2})$.

Our first result concerns a very general class of operators $L(t)$. Roughly speaking, under the condition that the commutator $[L(t), H]$ is H^τ-bounded for some $\tau < 1$, we will prove that the flow $\mathcal{U}(t,s)$ of (1.1) exists globally in time in $\mathcal{H}^k$ and its norm grows at most polynomially in time as $t\to\infty$, and more precisely we prove the upper bound

$$\|\mathcal{U}(t,s)\psi\|_{\mathcal{L}(\mathcal{H}^k)} \le C\ \langle t-s\rangle^{\frac{k}{2(1-\tau)}} \tag{1.2}$$

for some constant C independent of t. Here $\langle x\rangle = (1+x^2)^{1/2}$.

It is remarkable that such a bound, in the case $\tau = 0$, is optimal, since there exist operators $L(t)$, H with $[L(t), H]$ bounded s.t. the solution of (1.1) fulfills $\|\mathcal{U}(t,0)\psi\|_{\mathcal{H}^k} \ge C\ \langle t\rangle^{\frac{k}{2}}$. Such an example was constructed by Delort in [10], choosing $L(t) = H + V(t)$ where $H = -\Delta + |x|^2$ on $\mathbb{R}$ is the harmonic oscillator and $V(t)$ is an ad-hoc pseudodifferential operator of order 0 (see Remark 1.6 for more details).

However, with stronger assumptions on $L(t)$ and H, one might hope to improve the bound (1.2). Indeed it is well known that in many interesting situations the norm of flow of (1.1) grows much more slowly, in particular at most as t^ϵ when $t\to\infty$, for any $\epsilon > 0$. This is the case for example for equation (1.1) on $\mathbb{T}$ with $L(t) = -\Delta + V(t,x)$, as proved by Bourgain in [5]. Here Δ is the Laplacian and $V(t,x)$ is a smooth potential. The same bound holds also when $L(t) = -\Delta + V(t,x)$ is defined on Zoll manifolds, as proved by Delort [9].

The crucial feature of these examples is a spectral property of the principal operator $-\Delta$ on Zoll manifolds. Indeed its spectrum can be enclosed in clusters whose distance is increasing (we will refer to such property as *increasing spectral gap condition*). Note that, in the example of Remark 1.6, the harmonic oscillator $-\Delta + |x|^2$ on $\mathbb{R}$ does not fulfill the increasing spectral gap condition.

Such property motivates our second result. In order to improve the upper bound (1.2), we put ourselves in the situation where $L(t)$ is of the form $L(t) = H + V(t)$ and we assume that H has increasing spectral gaps. Then provided that $V(t)$ is smooth in time, we prove that for every $\varepsilon > 0$ the bound

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \leq C_{k,\varepsilon} \langle t-s\rangle^{\varepsilon}, \quad \forall t,s \in \mathbb{R} \tag{1.3}$$

holds. This is essentially the content of Theorem 1.9 below. It is important to note that we allow $V(t)$ to be an *unbounded* perturbation. More precisely we can take $V(t)$ to be H^ν-bounded, where $\nu < 1$ depends only on the spectral properties of H.

In the case where $t \mapsto V(t)$ is analytic, we are able to further improve the bound (1.3), obtaining the logarithmic estimate

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \leq \gamma(\log \langle t-s\rangle)^{\sigma k}, \quad \forall t,s \in \mathbb{R} \tag{1.4}$$

where the constant $\sigma > 0$ can be explicitly calculated. This is the content of Theorem 1.10 below. Once again when $V(t)$ is a bounded perturbation the exponent σ that we find is optimal (see Remark 1.11 below).

Finally we apply our abstract theorems to several different models, including one degree of freedom Schrödinger operators, perturbations of the Laplacian on compact manifolds, Dirac equations, a discrete NLS model and some classes of pseudodifferential operators. We recover many known results proven with different techniques, often improving such results (allowing e.g. unbounded perturbations) but also obtaining new results. More details and references will be given in Section 5.

The problem of estimating the growth of higher norms for equation (1.1) is very old, and goes back to the pioneering works initiated by Howland [20] and developed by Joye [21,22], Nenciu [29] and Barbaroux–Joye [3].

Such authors, roughly speaking, under the increasing spectral gaps condition on H and the assumption that the perturbation $V(t)$ is smooth in time and *bounded*, use the method of adiabatic approximation to prove that for every $\varepsilon > 0$ we have

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^1)} \leq C_{1,\varepsilon} \langle t-s\rangle^{\varepsilon}, \quad \forall t,s \in \mathbb{R}.$$

Our aim here is to extend the adiabatic approximation schema of Joye and Nenciu to a class of *unbounded* perturbations $V(t)$ and to control the growth of the $\mathcal{H}^k$-norm $\forall k > 0$.

As a final remark, we would like to mention some situations in which it is possible to prove better bounds, and in particular to prove that $\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^k)}$ is uniformly bounded in time $\forall k \geq 0$. Such results can be obtained for instance provided that the perturbation V fulfills some stronger assumptions, for example being quasi-periodic in time and small in size. Indeed in these cases one might try to apply KAM methods to conjugate $L(t)$ to a diagonal operator with constant coefficients, which in turn implies that the $\mathcal{H}^k$-norms are uniformly bounded in time $\forall k \geq 0$. The problem of the existence of such a conjugation goes in the literature under the name of *reducibility* and had a tremendous development in the last 20 years. To list the achievements of such theory is out of the scope of this manuscript: we limit ourselves to state the latest results in the various models considered in Section 5.

1.2. Main result

We start to make more precise assumptions. We ask that the scale of Hilbert spaces is generated by a positive self-adjoint operator H in $\mathcal{H}$, in the following sense: first H has a dense domain $D(H) \equiv \mathcal{H}^2$. Then, defining for every $k \geq 0$ the operator H^k by functional calculus (spectral decomposition), we demand that $\mathcal{H}^k \equiv D(H^{k/2})$. For $k < 0$, $\mathcal{H}^k$ is defined by duality as the completion of $\mathcal{H}$ with respect to the norm $\|u\|_k = \sup\{|\langle v, u\rangle|,\ \|v\|_{-k} \leq 1\}$. Notice that for every $m \in \mathbb{R}$ and $k \in \mathbb{R}$, H^m is an isometry from $\mathcal{H}^{k+2m}$ onto $\mathcal{H}^k$. Denote by $\mathcal{H}^\infty := \cap_{k\in\mathbb{R}}\mathcal{H}^k$.

Let us denote by $\|\cdot\|_k$ the natural norm on $\mathcal{H}^k$, which in turns is equivalent to $\|H^{k/2}\cdot\|_0$. Finally given a Banach space $\mathcal{B}$, we denote by $C_b(\mathbb{R}, \mathcal{B})$ the Banach space of continuous and bounded maps $f : \mathbb{R} \mapsto \mathcal{B}$ with the usual sup norm $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|_{\mathcal{B}}$. We denote by $C_b^\infty(\mathbb{R}, \mathcal{B})$ the space of maps $f : \mathbb{R} \to \mathcal{B}$ smooth.

Given $\mathcal{A}, \mathcal{B}$, Banach spaces, we will denote by $\mathcal{L}(\mathcal{A}, \mathcal{B})$ the set of linear bounded maps from $\mathcal{A}$ to $\mathcal{B}$. In case $\mathcal{A} \equiv \mathcal{B}$ we will simply write $\mathcal{L}(\mathcal{A})$.

Given an operator A, we say that A is H^ν-bounded if $AH^{-\nu}$ is a bounded operator on $\mathcal{H}^0$.

Remark 1.1. Recall that $\mathcal{H}^\infty$ is dense in $\mathcal{H}^k$ $\forall k \in [0, \infty[$. This follows from the spectral decomposition of H: $H = \int_0^\infty \lambda dE_H(\lambda)$ (see [30]). Let $E_H[a, b] = \int_a^b dE_H(\lambda)$ be the spectral projector on $[a, b]$. If $\psi \in D(H^{k/2})$ then $E_H[0, N] \in \mathcal{H}^\infty$ for all $N > 0$ and $\lim\limits_{N\to\infty} \|H^{k/2}(\psi - \psi_N)\|_0 = 0$.

Let us introduce now a time dependent family of operators $L(t)$ and the following conditions:

(H0) There exist integers $\mathtt{m} \geq 0$ and $k_0 > 2\mathtt{m}$ such that $t \mapsto L(t) \in C_b(\mathbb{R}, \mathcal{L}(\mathcal{H}^{k+2\mathtt{m}}, \mathcal{H}^k))$ for $0 \leq k \leq k_0$.

(H1) For every $t \in \mathbb{R}$, $L(t)$ is symmetric on $\mathcal{H}^{k_0+2\mathtt{m}}$ w.r.t. the scalar product of $\mathcal{H}^0$ i.e.

$$\langle L(t)\psi, \phi\rangle_0 = \langle \psi, L(t)\phi\rangle_0\,, \qquad \forall \psi, \phi \in \mathcal{H}^{k_0+2\mathtt{m}}\ .$$

(H2) There exists $k_1 > 2\mathtt{m}$ such that $[L(\cdot), H]H^{-1} \in C_b(\mathbb{R}, \mathcal{L}(\mathcal{H}^k))$ for $0 \leq k \leq 2k_1$.

The first theorem concerns existence of a global in time flow of equation (1.1):

Theorem 1.2. *Assume that $L(t)$ fulfills the assumptions* (H0), (H1), (H2)*. Then for all k with $0 \leq k \leq \min(k_0, k_1) - 4\mathtt{m}$, equation* (1.1) *admits a unique propagator $\mathcal{U}(t,s) \in C^0\left(\mathbb{R}\times\mathbb{R},\ \mathcal{L}(\mathcal{H}^k)\right)$ fulfilling*

- *(i)* Well-posedness: *for every initial datum $\psi_s \in \mathcal{H}^{k+2\mathtt{m}}$, there exists a unique global solution $\psi(t) := \mathcal{U}(t,s)\psi_s \in \mathcal{H}^k$ of* (1.1) *such that $\psi(\cdot) \in C^0(\mathbb{R}, \mathcal{H}^{k+2\mathtt{m}}) \cap C^1(\mathbb{R}, \mathcal{H}^k)$.*
- *(ii)* Unitarity: *for every initial datum $\psi_s \in \mathcal{H}^k$, the $\mathcal{H}^0$ norm is preserved by the flow, $\|\psi(t)\|_{\mathcal{H}^0} = \|\psi_s\|_{\mathcal{H}^0}$, $\forall t \in \mathbb{R}$.*
- *(iii)* Group property: $\forall t, r, s \in \mathbb{R}$

$$\mathcal{U}(t,s) = \mathcal{U}(t,r)\mathcal{U}(r,s), \quad \mathcal{U}(s,s) = \mathbb{I}\ . \tag{1.5}$$

- *(iv)* Upper bound on growth: *for every $k \geq 0$, there exists $C_k > 0$ such that*

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \leq C_k \mathrm{e}^{C_k|t-s|}, \quad \forall t, s \in \mathbb{R}. \tag{1.6}$$

In particular $\mathcal{U}(t,s)$ extends to a unitary operator in $\mathcal{H}^0$ fulfilling the group property (*iii*)*.*

Furthermore for every $t \in \mathbb{R}$ and every $k \geq 2\mathtt{m}$, $(L(t), \mathcal{H}^k, \mathcal{H}^0)$ is essentially self-adjoint.

It is remarkable that the assumptions of Theorem 1.2 are the time-dependent assumptions of Nelson commutator theorem to prove essentially self-adjointness, see Proposition A.2. We shall see later that Theorem 1.2 has many applications for proving existence and uniqueness for time dependent Schrödinger equations with time dependent Hamiltonians.

Remark 1.3. Some variants or special cases of Theorem 1.2 are more or less known in the literature. For example the results, at least in the special case $m = 1$ and $k_0 = 4 = k_1$, follow from a classical theorem by Kato [23, Theorem 6.1], as was pointed out to us by J. Schmid [33].

Furthermore for $k = 1$ similar results are proved also in [25, Appendix A], [34, Theorem II.27] and [24].

Remark 1.4. At this level of generality, the estimate on the growth of Sobolev norms of Theorem 1.2 (iv) is optimal. Indeed one example is the following. Let $H = -\frac{d^2}{dx^2} + x^2$ be the harmonic oscillator and $L = x\frac{d}{\mathrm{i}dx} + \frac{d}{\mathrm{i}dx}x$ on $L^2(\mathbb{R})$. We have $[H, L] = -2\left(\frac{d^2}{dx^2} + x^2\right)$ and the assumptions (H0)–(H2) are satisfied. But we have

$$\mathcal{U}(t,0)u(x) = \mathrm{e}^{t/2}u(\mathrm{e}^t x).$$

So we get

$$\int_{\mathbb{R}} |\frac{d}{dx} e^{itL} u(x)|^2 dx = e^t \| \frac{d}{dx} f(x) \|_{L^2(\mathbb{R})}.$$

A first improvement on the growth (1.6) can be obtained by asking that the commutator $[L(t), H]$ is more regular than what is assumed in (H2). More precisely we introduce the following assumption:

(H3) There exist $k_1 > 2\mathtt{m}$ and $\tau < 1$ real such that $[L(\cdot), H]H^{-\tau} \in C_b\left(\mathbb{R}, \mathcal{L}(\mathcal{H}^k)\right)$ for every $0 \le k \le 2k_1$.

Theorem 1.5. *(i) Assume that $L(t)$ satisfies the properties* (H0), (H1) *and* (H3). *Let $0 \le k \le \min\{k_0, 2k_1\} - 4\mathtt{m}$, and $p \in \mathbb{N}$ such that*

$$\frac{k}{1-\tau} \le p \ .$$

Then there exists a positive constant $C_{k,\nu,p}$, independent of t, such that

$$\|\mathcal{U}(t,s)\psi_s\|_{2k} \le C_{k,\tau,p} \ \langle t-s \rangle^p \ \|\psi_s\|_{2k}, \ \forall \psi_s \in \mathcal{H}^{2k} \ . \tag{1.7}$$

(ii) Assume that (H3) *is satisfied for every $k \in \mathbb{N}$ and that $\tau < 1$ is rational. Then for every real $r > 0$ we have*

$$\|\mathcal{U}(t,s)\psi_s\|_r \le C_r(\|\psi_s\|_r + \langle t-s \rangle^{\frac{r}{2(1-\tau)}} \|\psi_s\|_0) \le C'_r \ \langle t-s \rangle^{\frac{r}{2(1-\tau)}} \|\psi_s\|_r \ . \tag{1.8}$$

This result shows that if $[L, H]$ is H^τ-bounded with $\tau < 1$, then the growth of the Sobolev norm is at most polynomial in time. Furthermore, if $1-\tau$ can be chosen arbitrary large then for every $\epsilon > 0$ we have

$$\|\mathcal{U}(t,s)\psi_s\|_r \le C_{r,\epsilon} \ \langle t-s \rangle^{\epsilon} \ \|\psi_s\|_r \ .$$

Remark 1.6. At this level of generality, the bound obtained in (1.7) is optimal, at least for $\tau = 0$. Indeed Delort [10] proved that there exists a time-dependent pseudodifferential operator $V(t)$ of order 0 and $\forall r > 0$ an initial datum $\psi_r \in \mathcal{H}^r$ s.t. the solution of $i\dot{\psi} = (-\Delta + |x|^2)\psi + V(t)\psi$, $x \in \mathbb{R}$, with $\psi(0) = \psi_r$ fulfills $\|\mathcal{U}(t,0)\psi_r\|_{\mathcal{H}^r} \sim \langle t \rangle^{r/2}$ (where $\mathcal{H}^r := D((-\Delta + |x|^2)^{r/2})$). In such example, $H = -\Delta + |x|^2$, and condition (H3) is fulfilled with $\tau = 0$. Then one sees that (1.7) is optimal.

In order to improve further the polynomial growth in (1.7), we make more restrictive assumptions on the structure of $L(t)$. First we ask that $L(t)$ is a perturbation of H, i.e. $L(t) = H + V(t)$, where $V(t)$ is a time-dependent self-adjoint operator. Clearly we assume that $L(t)$ satisfies (H0), (H1) and (H2) (in particular we can take $\mathtt{m} = 1$). Then we know from Theorem 1.2 that the Hamiltonian $L(t) := H + V(t)$ generates a propagator in each space $\mathcal{H}^k$, $k \in \mathbb{N}$, $k \le \min\{k_0, k_1\} - 2$, which is unitary in $\mathcal{H}^0$.

We make two further assumptions. The first one concerns the structure of the spectrum of H, which is asked to fulfill the following condition on increasing spectral gaps:

(Hgap) The spectrum $\sigma(H)$ of H can be enclosed in clusters $\{\sigma_j\}_{j\ \geq 1}$,

$$\sigma(H) \subseteq \bigcup_{1\leq j<\infty} \sigma_j \ , \tag{1.9}$$

where each σ_j is a bounded interval of $\mathbb{R}$ (we assume that they are listed in increasing order). Define[1]

$$\Delta_j := \text{dist}(\sigma_{j+1}, \sigma_j) \ , \quad \delta_j := \sup_{\lambda_1,\lambda_2\in\sigma_j} |\lambda_1 - \lambda_2| \ .$$

Then there exist $\mu > 0$ and positive constants α, β (independent of j) such that

$$\frac{1}{\alpha}\, j^{\mu} \leq \Delta_j \leq \alpha\, j^{\mu}, \qquad \delta_j \leq \beta\, j^{\mu} \ , \quad \forall j \geq 1 \ . \tag{1.10}$$

Remark 1.7. If H fulfills *(Hgap)*, then its spectrum is localized in the following sense: there exist positive constants C_1, C_2 (independent of j) such that

$$C_1\, j^{\mu+1} \leq \min \sigma_j \leq \max \sigma_j \leq C_2\, j^{\mu+1} \ , \quad \forall j \geq 1 \ . \tag{1.11}$$

In particular

$$\max \sigma_j \leq \min \sigma_{j+1}, \qquad \max \sigma_j \leq \frac{C_2}{C_1} \min \sigma_j \ , \quad \forall j\ \geq 1 \ .$$

The second assumption concerns the regularity in time of the perturbation $V(t)$:

$(\mathrm{Vc})_n$ Let $n \geq 1$. There exists ν with[2]

$$0 \leq \nu < \frac{\mu}{\mu+1}$$

such that $V(\cdot)H^{-\nu}$ belongs to $C_b^r(\mathbb{R}, \mathcal{L}(\mathcal{H}^k))$, $r \geq 2$, for all $0 \leq k \leq 2n$. In particular $\forall 0 \leq \ell \leq r$, there exists a positive $R_{n,\ell}$ s.t.

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell V(t)\ H^{-p-\nu}\|_{\mathcal{L}(\mathcal{H}^0)} \leq R_{n,\ell} \ , \quad \forall 0 \leq p \leq n \ , \quad \forall\, 0 \leq \ell \leq r \ . \tag{1.12}$$

The following result is an extension of Theorem 2 of [29].

[1] Clearly Δ_j are the distances between of the spectral clusters, while δ_j are their diameters.
[2] Here μ is the rate of growth of the spectral gap as defined in (1.10).

Theorem 1.8. *Fix an arbitrary* $n \geq 0$. *Assume that* $H + V(t)$ *fulfills* (H0), (H1), (H2), (Hgap) *and* (Vc)$_n$. *Then for any real* $0 < k \leq 2n$ *there exists* $C_{k,r}$, *independent of* t, *such that*

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \leq C_{k,r} \langle t-s\rangle^{\frac{k}{2r}\left(\frac{\mu}{\mu+1}-\nu\right)^{-1}}, \quad \forall t,s \in \mathbb{R}. \tag{1.13}$$

A natural question is if increasing the regularity in time of $V(t)$ leads to better estimates. The answer is positive: assume

(Vs)$_n$ Let $n \geq 1$. There exists ν with[3] $0 \leq \nu < \frac{\mu}{\mu+1}$ such that $V(\cdot)H^{-\nu}$ belongs to $C_b^\infty(\mathbb{R}, \mathcal{L}(\mathcal{H}^k))$ for all $0 \leq k \leq 2n$. In particular $\forall \ell \geq 0$, there exists a positive $R_{n,\ell}$ s.t.

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^\ell V(t)\, H^{-p-\nu}\|_{\mathcal{L}(\mathcal{H}^0)} \leq R_{n,\ell}\,, \quad \forall 0 \leq p \leq n\,, \qquad \forall \ell \geq 0\,. \tag{1.14}$$

Theorem 1.9. *Fix an arbitrary* $n \geq 0$. *Assume that* $H + V(t)$ *fulfills* (H0), (H1), (H2), (Hgap) *and* (Vs)$_n$. *Then for any real* $0 < k \leq 2n$ *and every* $\varepsilon > 0$ *there exists* $C_{k,\varepsilon}$, *independent of* t, *such that*

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \leq C_{k,\varepsilon} \langle t-s\rangle^{\varepsilon}, \quad \forall t,s \in \mathbb{R}. \tag{1.15}$$

If one assumes that $V(t)$ is analytic in time, better estimates were proved for 1-D Hamiltonians [36] or for perturbations of the Laplace operator on the torus [6,12]. We are able to extend such results to our more general situation, provided V fulfills the following analytic estimates:

(Va)$_n$ Let $n \geq 0$. There exists ν with $0 \leq \nu < \frac{\mu}{\mu+1}$ such that $V(\cdot)H^{-\nu}$ is an operator in $\mathcal{L}(\mathcal{H}^k)$, $\forall 0 \leq k \leq 2n$, analytic in time. In particular there exist $c_{0,n}, c_{1,n} > 0$ such that $\forall \ell \geq 0$

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^\ell V(t)\, H^{-p-\nu}\|_{\mathcal{L}(\mathcal{H}^0)} \leq c_{0,n}\, c_{1,n}^\ell\, \ell!\,, \quad \forall 0 \leq p \leq n\,. \tag{1.16}$$

Then we have

Theorem 1.10. *Fix an arbitrary* $n \geq 0$. *Assume that* $H + V(t)$ *fulfills* (H0), (H1), (H2), (Hgap) *and* (Va)$_n$. *Then for any real* $0 < k \leq 2n$ *there exists a positive* γ, *independent of* t, *s.t.*

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \leq \gamma\,(\log\langle t-s\rangle)^{\frac{k}{2}\left(\frac{\mu}{\mu+1}-\nu\right)^{-1}}, \qquad \forall t,s \in \mathbb{R}. \tag{1.17}$$

[3] Here μ is the rate of growth of the spectral gap as defined in (1.10).

Notice that Theorem 1.9 and Theorem 1.10 hold true with only time regularity on $V(t)$ and a limited amount of regularity in the scale spaces $\mathcal{H}^k$. On the contrary, all the previous results deal with potentials which are smooth or analytic in the scale of spaces $\mathcal{H}^k$. In particular in [36,12] the authors assume analyticity in t and x. Here we only need analyticity in t and some finite amount of regularity in x.

Remark 1.11. In the case of the Schrödinger equation on the circle, Bourgain [5] showed that the logarithmic growth factor for $t \to \infty$ can not be avoided. More precisely he proved the following. Fix $s > 0$; there exist a sequence of real potentials $V_j(x,t)$, a sequence of initial conditions ψ_0^j and a sequence of times $t_j \to \infty$ as $j \to \infty$, s.t. the solutions $\psi_j(t)$ of $\mathrm{i}\dot{\psi} = -\partial_{xx}\psi + V_j(x,t)\psi$, $x \in \mathbb{T}$, fulfill $\|\psi_j(t_j)\|_{H^s} \geq C_s(\log\langle t_j\rangle)^s$. Since $H \equiv -\partial_{xx}$ fulfills (Hgap) with $\mu = 1$ and V fulfills $(\mathrm{Va})_n$ with $\nu = 0$, we see that the bound in (1.17) is optimal.

Remark 1.12. Theorem 1.10 could be extended, with a different exponent, replacing analytic estimates $(\mathrm{Va})_n$ by Gevrey estimates:

$(\mathrm{Vg})_n$ Fix $n \geq 0$. There exist $0 \leq \nu < \frac{\mu}{\mu+1}$ and $s > 1$ s.t.

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^\ell V(t)\, H^{-p-\nu}\|_{\mathcal{L}(\mathcal{H}^0)} \leq c_{0,m}\, C_{1,m}^\ell\, (\ell!)^s, \ \forall \ell \geq 0,\ 0 \leq p \leq n\,. \tag{1.18}$$

1.3. Scheme of the proof

The proof proceeds essentially in three steps. First we prove Theorem 1.2. The strategy is to regularize the operator $L(t)$ obtaining a sequence of bounded operators $L_N(t)$ for which we are able to prove uniform estimates on the flow they generate, and then to pass to the limit. This in turn is possible thanks to the boundedness of $[L(t), H]H^{-1}$. Theorem 1.5 then follows easily by a recursive argument.

The strategy to prove Theorem 1.9 and Theorem 1.10 is to extend the scheme of Nenciu [29] to deal with unbounded perturbations. The idea is to construct an adiabatic approximation $\mathcal{U}_{ad}(t,s)$ of the flow $\mathcal{U}(t,s)$, for which the norms $\mathcal{H}^k$ are bounded uniformly in time. In case of time-analytic perturbations, special care is needed in order to perform estimates.

Organization of the paper: In Section 2 we prove Theorem 1.2 and Theorem 1.5. In Section 3 we prove the control of the growth of the Sobolev norms in case of perturbations depending smoothly in time, namely we prove Theorem 1.9. Theorem 1.8 will be deduced during the proof of Theorem 1.9. In Section 4 we consider perturbations depending analytically in time and we prove Theorem 1.10. In Section 5 we apply the abstract theorems to different kind of Schrödinger equations.

2. Existence of the propagator

The aim of this section is to prove Theorem 1.2 and Theorem 1.5. It is technically more convenient to consider the integral form of equation (1.1)

$$\psi(t) = \psi_s + \mathrm{i}^{-1} \int_s^t L(r)\psi(r)dr \tag{2.1}$$

We begin with an easy lemma:

Lemma 2.1. *Assume that the condition* (H3) *is satisfied. Let* $\theta := 1 - \tau$. *Then*

(i) For $k \in \mathbb{N}$, $1 \le k \le k_1$, *we have* $[L, H^k]H^{-k+\theta} \in C_b\left(\mathbb{R}, \mathcal{L}(\mathcal{H}^0)\right)$.
(ii) For any $\theta' < \theta$ *and any real* p *such that* $0 < p < k_1$ *we have* $[L, H^p]H^{-p+\theta'} \in C_b\left(\mathbb{R}, \mathcal{L}(\mathcal{H}^0)\right)$.

Proof. (i) The proof is by induction on k. First write $[L, H^{k+1}] = [L, H^k]\, H - H^k\,[H, L]$, which shows that

$$[L, H^{k+1}]H^{-k-1+\theta} = [L, H^k]\, H^{-k+\theta} - H^k\,[H, L]H^{-1+\theta}H^{-k}\ .$$

The inductive assumption and the hypothesis $[H, L]H^{-1+\theta}$ bounded as an operator from $\mathcal{H}^\ell \to \mathcal{H}^\ell$, $\forall 0 \le \ell \le 2k_1$, imply the inductive assumption.

(ii) For simplicity let us give the proof for $0 < p < 1$. We use the following Cauchy formula

$$H^p\psi = \frac{1}{2\mathrm{i}\pi} \oint_\Gamma z^{p-1}(H - z)^{-1}H\psi dz$$

for a suitable complex contour Γ. Using that $[L, (H - z)^{-1}] = (H - z)^{-1}[L, H](H - z)^{-1}$ we get

$$\begin{aligned}[L, H^p] &= \frac{1}{2\mathrm{i}\pi} \oint_\Gamma z^{p-1}(H - z)^{-1}[L, H]H(H - z)^{-1}Hdz + \frac{1}{2\mathrm{i}\pi} \oint_\Gamma z^{p-1}(H - z)^{-1}[L, H]dz \\ &= I + II \end{aligned} \tag{2.2}$$

We have

$$IIH^{-p+\theta'} = \frac{1}{2\mathrm{i}\pi} \oint_\Gamma z^{p-1}(H - z)^{-1}H^{1-p-\theta+\theta'}H^{-s}[L, H]H^{-1+\theta}H^s dz,$$

where $s = \theta' - \theta + 1 - p$. It results that $IIH^{-p+\theta'}$ is bounded on $\mathcal{H}$ if $\theta' < \theta$.

Using the same trick we get that $IH^{-p+\theta'}$ is bounded on $\mathcal{H}$ if $\theta' < \theta$.

The same proof can be done for $k < p < k+1$. □

Remark 2.2. It is not clear that the above estimate can be proved under assumption (H3) with $\theta' = \theta$ if p is not an integer.

Let $\mathtt{m}$ as in Theorem 1.2 and suppose further that $\mathtt{m} > 0$ (the case $\mathtt{m} = 0$ corresponds to bounded $L(t)$). The main idea of the proof is to regularize $L(t)$ is such a way that it becomes a bounded operator, for which it is possible to construct a unitary flow. To do so, for any $N \geq 1$ introduce the smoothing operator

$$R_N := \left(1 + \frac{H^{\mathtt{m}}}{N}\right)^{-1} .$$

The following lemma describes the main properties of the smoothing operator R_N.

Lemma 2.3. *There exists a positive $C_{\mathtt{m}}$ such that $\forall k, N > 0$ one has:*

(i) $R_N : \mathcal{H}^k \to \mathcal{H}^{k+2\mathtt{m}}$ *and* $\|R_N\|_{\mathcal{L}(\mathcal{H}^k, \mathcal{H}^{k+2\mathtt{m}})} \leq N$.
(ii) $\|R_N\|_{\mathcal{L}(\mathcal{H}^k, \mathcal{H}^k)} \leq C_{\mathtt{m}}$.
(iii) $\|R_N - \mathbb{I}\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}}, \mathcal{H}^k)} \leq \frac{C_{\mathtt{m}}}{N}$.
(iv) $\|R_N - \mathbb{I}\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}\eta}, \mathcal{H}^k)} \leq \frac{C_{\mathtt{m},\eta}}{N^\eta}$, $\forall \eta \in]0,1]$.

Proof. The proof is an easy computation, and it is skipped. Notice that (iv) follows from (ii) and (iii) using interpolation. □

Now we regularize the operator $L(t)$ by defining

$$L_N(t) := R_N \, L(t) \, R_N \ .$$

Lemma 2.4. *For every $N \geq 1$, $L_N(t)$ is symmetric on $\mathcal{H}^0$ and bounded on $\mathcal{H}^k$ for $0 \leq k \leq k_0 - 2\mathtt{m}$. Furthermore for every $\eta \in]0,1]$ there exists $C_\eta > 0$ such that for $0 \leq k \leq k_0 - 2\mathtt{m}$ we have:*

$$\|L_N(t) - L(t)\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}(1+\eta)}, \mathcal{H}^k)} \leq \frac{C_\eta}{N^\eta}, \quad N \geq 1, \quad t \in \mathbb{R} \ . \tag{2.3}$$

Proof. We prove only the estimate. By Lemma 2.3 one has

$$\begin{aligned}
\|L_N(t) - L(t)\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}(1+\eta)}, \mathcal{H}^k)} &\leq \|R_N L(t)(R_N - \mathbb{I})\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}(1+\eta)}, \mathcal{H}^k)} \\
&\quad + \|(R_N - \mathbb{I})L(t)\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}(1+\eta)}, \mathcal{H}^k)} \\
&\leq \left(\|L(t)\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}}, \mathcal{H}^k)} + \|L(t)\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}(1+\eta)}, \mathcal{H}^{k+2\mathtt{m}\eta})}\right) \frac{C_{\mathtt{m},\eta}}{N^\eta} \\
&\leq \frac{C_\eta}{N^\eta} \ ,
\end{aligned}$$

where the last inequality follows from (H0) using that $k + 2\mathtt{m} \leq k_0$. □

We use $L_N(t)$ as a propagator for a regularized differential equation. More precisely consider the regularized Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t \psi = L_N(t)\psi \\ \psi|_{t=s} = \psi_s \ , \quad s \in \mathbb{R} \end{cases} \tag{2.4}$$

Since the operator $L_N(t)$ is bounded on $\mathcal{H}^k$ to itself for every k, $0 \le k \le k_1$, it generates a flow $\mathcal{U}_N(t,s) \in C(\mathbb{R}\times\mathbb{R}, \mathcal{L}(\mathcal{H}^k))$ for $0 \le k \le k_1$, which is unitary in $\mathcal{H}^0$.

Lemma 2.5. *For any* $0 < k \le 2k_1$, *there exists a positive constant* C_k, *independent of* N, *such that*

$$\|\mathcal{U}_N(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \le \ e^{C_k|t-s|} \ , \quad \forall N > 0 \ .$$

Proof. First we control $\|\mathcal{U}_N(t,s)\|_{\mathcal{L}(\mathcal{H}^{2k})}$ for $0 \le k \le k_1$. We must show that $H^k\,\mathcal{U}_N(t,s)\,H^{-k}$ is bounded uniformly in N as an operator from $\mathcal{H}^0$ to itself. Remark that, due to the unitarity of $\mathcal{U}_N(t,s)$ in $\mathcal{H}^0$, one has

$$\|H^k\,\mathcal{U}_N(t,s)\,H^{-k}\|_{\mathcal{L}(\mathcal{H}^0)} = \|\mathcal{U}_N(t,s)^*\,H^k\,\mathcal{U}_N(t,s)\,H^{-k}\|_{\mathcal{L}(\mathcal{H}^0)} \ .$$

Now one has

$$\begin{aligned} \mathcal{U}_N(t,s)^*\,H^k\,\mathcal{U}_N(t,s)\,H^{-k} &= \mathbb{I} + \int_s^t \mathcal{U}_N(r,s)^*\,[L_N(r), H^k]\,\mathcal{U}_N(r,s)\,H^{-k}dr \\ &= \mathbb{I} + \int_s^t \mathcal{U}_N(r,s)^*\,R_N\,[L(r), H^k]\,H^{-k}\,R_N\,H^k\,\mathcal{U}_N(r,s)\,H^{-k}dr \end{aligned}$$

where we used that $[R_N, H^k] = 0$ and

$$[L_N(t), H^k] = R_N\,[L(t), H^k]\,R_N \ .$$

By Lemma 2.1, for $0 \le k \le k_1$, one has the bound $\|[L(t), H^k]\,H^{-k}\|_{\mathcal{L}(\mathcal{H}^0)} \le C_k$ for some positive constant C_k, thus it follows (using also Lemma 2.3 (*ii*)) that uniformly in N

$$\|R_N\,[L(t), H^k]\,H^{-k}\,R_N\|_{\mathcal{L}(\mathcal{H}^0)} \le C_k \ , \quad \forall N > 0 \ , \qquad 0 \le k \le k_1 \ .$$

Such estimate combined with the unitarity of $\mathcal{U}_N(t,s)$ in $\mathcal{L}(\mathcal{H}^0)$ gives

$$\begin{aligned} &\|H^k\,\mathcal{U}_N(t,s)\,H^{-k}\|_{\mathcal{L}(\mathcal{H}^0)} \\ &\quad \le 1 + \int_s^t \|\mathcal{U}_N(r,s)^*\,R_N\,[L(r), H^k]\,H^{-k}\,R_N\,H^k\,\mathcal{U}_N(r,s)\,H^{-k}\|_{\mathcal{L}(\mathcal{H}^0)}dr \\ &\quad \le 1 + C_k \int_s^t \|H^k\,\mathcal{U}_N(r,s)\,H^{-k}\|_{\mathcal{L}(\mathcal{H}^0)}dr \end{aligned}$$

which by Gronwall inequality allows us to conclude that

$$\|\mathcal{U}_N(t,s)\|_{\mathcal{L}(\mathcal{H}^{2k})} \leq e^{C_k|t-s|} , \qquad \forall 0 \leq k \leq k_1 .$$

Interpolating with the trivial bound $\|\mathcal{U}_N(t,s)\|_{\mathcal{L}(\mathcal{H}^0)} = 1$ gives the result for general k. □

Proof of Theorem 1.2. Fix arbitrary $t, s \in \mathbb{R}$. Choose $\eta > 0$ small enough. The first step is to show that for every $\psi \in \mathcal{H}^{k+2\mathtt{m}(1+\eta)}$, the sequence $\{\mathcal{U}_N(t,s)\psi\}_N$ is a Cauchy sequence in the space $\mathcal{H}^k$. For $k \leq k_2 \equiv \min\{k_0, 2k_1\} - 4\mathtt{m}$ one has

$$\begin{aligned}
&\|\mathcal{U}_N(t,s)\psi - \mathcal{U}_{N'}(t,s)\psi\|_k \\
&= \| \int_s^t \partial_r(\mathcal{U}_{N'}(t,r)\mathcal{U}_N(r,s)\psi)\, dr\|_k \\
&= \| \int_s^t \mathcal{U}_{N'}(t,r)\ (L_N(r) - L_{N'}(r))\mathcal{U}_N(r,s)\psi\, dr\|_k \\
&\leq |t-s| \sup_{r\in[s,t]} \|\mathcal{U}_{N'}(t,r)\|_{\mathcal{L}(\mathcal{H}^k)} \|L_N(r) - L_{N'}(r)\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}(1+\eta)},\mathcal{H}^k)} \\
&\quad \times \|\mathcal{U}_N(r,s)\|_{\mathcal{L}(\mathcal{H}^{k+2\mathtt{m}(1+\eta)})} \|\psi\|_{k+2\mathtt{m}(1+\eta)} \\
&\leq C\left(\frac{1}{N^\eta} + \frac{1}{(N')^\eta}\right)\ |t-s|\ e^{(C_k + C_{k+2\mathtt{m}(1+\eta)})|t-s|}\ \|\psi\|_{k+2\mathtt{m}(1+\eta)},
\end{aligned}$$

where in the last inequality we used a easy variant of estimate (2.3) in Lemma 2.4. For any t, s in a bounded interval, and $\psi \in \mathcal{H}^{k+2\mathtt{m}(1+\eta)}$, the sequence $\{\mathcal{U}_N(t,s)\psi\}_N \subset \mathcal{H}^k$ is a Cauchy sequence. Since $\mathcal{H}^{k+2\mathtt{m}(1+\eta)}$ is dense in $\mathcal{H}^k$ and $\|\mathcal{U}_N(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \leq e^{C_k|t-s|}$ uniformly in N, by an easy density argument one shows that for any $\psi \in \mathcal{H}^k$ the sequence $\{\mathcal{U}_N(t,s)\psi\}_N$ is also Cauchy in $\mathcal{H}^k$, $k \leq k_2$. Thus for every $\psi \in \mathcal{H}^k$ the limit

$$\mathcal{U}(t,s)\psi := \lim_{N\to\infty} \mathcal{U}_N(t,s)\psi$$

exists in $\mathcal{H}^k$, $k < k_2$. Moreover we have the following error estimate, for $N > 0$ large enough,

$$\|\mathcal{U}(t,s)\psi - \mathcal{U}_N(t,s)\psi\|_k \leq \frac{C}{N^\eta}\,|t-s|\,\mathrm{e}^{C|t-s|}\,\|\psi\|_{k+2\mathtt{m}(1+\eta)} , \quad 0 \leq k \leq k_2 . \tag{2.5}$$

By the principle of uniform boundedness (Banach–Steinhaus Theorem), $\mathcal{U}(t,s) \in \mathcal{L}(\mathcal{H}^k)$. Since $\mathcal{U}_N(t,s)$ is an isometry in $\mathcal{H}^0$,

$$\|\mathcal{U}(t,s)\psi\|_0 = \lim_{N\to\infty} \|\mathcal{U}_N(t,s)\psi\|_0 = \|\psi\|_0$$

which shows that $\mathcal{U}(t,s)$ is an isometry on $\mathcal{H}^0$.

Let us prove now that $\psi(t) = \mathcal{U}(t,s)\psi_s$ satisfies the integral equation (2.1). Denote $\psi^N(t) = \mathcal{U}_N(t,s)\psi_s$. Then we have

$$\psi^N(t) = \psi_s + \mathrm{i}^{-1}\int_s^t L_N(r)\psi^N(r)dr \ . \tag{2.6}$$

Using Lemma 2.4 and estimate (2.5) there exists $C > 0$, depending on a, b, k but not on N, such that for $a \le s \le r \le t \le b$, $k \le k_2$ we have

$$\|L_N(r)\psi^N(r) - L(r)\psi(r)\|_k \le C\left(\|\psi(r) - \psi^N(r)\|_{k+2\mathtt{m}} + \frac{1}{N^\eta}\|\psi\|_{k+2\mathtt{m}(1+\eta)}\right).$$

So we can pass to the limit in (2.6) and we get

$$\psi(t) = \psi_s + \mathrm{i}^{-1}\int_s^t L(r)\psi(r)dr \ . \tag{2.7}$$

In particular if $\psi_s \in \mathcal{H}^{k+2\mathtt{m}(1+\eta)}$ then $t \mapsto \psi(t)$ is strongly derivable from $\mathbb{R}$ into $\mathcal{H}^k$ and satisfies the Schrödinger equation (1.1). Furthermore

$$\mathcal{U}(t,s)\psi = \lim_{N\to\infty} \mathcal{U}_N(t,s)\psi = \lim_{N\to\infty} \mathcal{U}_N(t,r)\mathcal{U}_N(r,s)\psi = \mathcal{U}(t,r)\mathcal{U}(r,s)\psi$$

where the limits are in the $\mathcal{H}^k$ topology. This shows the group property.

Finally we have shown that $(t,s) \mapsto \mathcal{U}(t,s) \in \mathcal{L}(\mathcal{H}^{k+2\mathtt{m}}, \mathcal{H}^k)$ is strongly continuously differentiable with strong derivatives

$$\partial_t \mathcal{U}(t,s) = -\mathrm{i}L(t)\mathcal{U}(t,s) \ .$$

With the same proof we get also

$$\partial_s \mathcal{U}(t,s) = \mathrm{i}\,\mathcal{U}(t,s)L(s) \ . \quad \square$$

We now prove the second theorem, concerning the growth of the norms.

Proof of Theorem 1.5. (i) It is enough to prove (1.7) for $\psi_s \in \mathcal{H}^\infty$. We have proved in Theorem 1.2 that $\mathcal{U}(t,s)$ is an isometry in $\mathcal{H}^0$ so we have

$$\|\mathcal{U}(t,s)\psi_s\|_{2k} = \|\mathcal{U}^*(t,s)\, H^k\, \mathcal{U}(t,s)\psi_s\|_0.$$

But we have

$$\mathcal{U}^*(t,s)\, H^k\, \mathcal{U}(t,s)\psi_s = H^k\psi_s + \mathrm{i}^{-1}\int_s^t \mathcal{U}^*(r,s)\,[L(r), H^k]\,\mathcal{U}(r,s)\psi_s\, dr$$

Hence using assumption (H3) and Lemma 2.1, we get the first estimate

$$\|\mathcal{U}(t,s)\psi_s\|_{2k} \leq \|\psi_s\|_{2k} + C_k \int_s^t \|\mathcal{U}(r,s)\psi_s\|_{2(k-\theta)} dr, \quad \theta = 1-\tau. \tag{2.8}$$

After m iterations of (2.8), with another constant $C_{k,m}$, we get

$$\|\mathcal{U}(t,s)\psi_s\|_{2k} \leq C_{k,m} \left(\|\psi_s\|_{2k} + |t-s|(\|\psi_s\|_{2(k-\theta)} + \cdots + |t-s|^{m-1}\|\psi_s\|_{2(k-(m-1)\theta)}\right)$$
$$+ C_{k,m} \int_s^t \int_s^{t_1} \cdots \int_s^{t_{m-1}} \|\mathcal{U}(t_m,s)\psi_s\|_{2(k-m\theta)} dt_m dt_{m-1} \cdots dt_1. \tag{2.9}$$

Now choose m such that $m\theta \geq k$ in such a way that $\|\mathcal{U}(t_m,s)\psi_s\|_{2(k-m\theta)} \leq \|\mathcal{U}(t_m,s)\psi_s\|_0$. Then use the unitarity of $\mathcal{U}(t,s)$ in $\mathcal{H}^0$ to obtain the bound (1.7).

(ii) If $\theta = \frac{p}{q}$ we get the inequality for $r = 2k$ with $k = p\ell$, $m = \ell q$ from Theorem 1.5. We conclude by an usual interpolation argument. □

With very similar arguments one can prove the following result about convergence of flows.

Theorem 2.6. *Let $L(t)$ be an operator fulfilling (H0)–(H2) with $k_0 = k_1 = \infty$. Let $\{L_n(t)\}_{n\geq 1}$ be a sequence of operators fulfilling (H0), (H1) and (H2) with $k_0 = k_1 = \infty$ uniformly in n, namely $\forall k \geq 0$, there exists $C_k > 0$ s.t.*

$$\sup_{t\in\mathbb{R}} \|[L_n(t), H]H^{-1}\|_{\mathcal{L}(\mathcal{H}^k)} \leq C_k \ , \quad \forall n \ . \tag{2.10}$$

Assume that there exists $m \geq 0$ s.t. $\forall k \geq 0$

$$\sup_{t\in\mathbb{R}} \|L_n(t) - L(t)\|_{\mathcal{L}(\mathcal{H}^{k+m},\mathcal{H}^k)} \to 0, \quad n \to \infty \ . \tag{2.11}$$

Denote by $\mathcal{U}_n(t,s)$ the propagator of $L_n(t)$ and by $\mathcal{U}(t,s)$ the propagator of $L(t)$. Then for every $\psi \in \mathcal{H}^{k+m}$, for every $t, s \in \mathbb{R}$ fixed, one has

$$\|\mathcal{U}_n(t,s)\psi - \mathcal{U}(t,s)\psi\|_k \to 0, \quad n \to \infty \ . \tag{2.12}$$

Proof. By Theorem 1.2 the flows $\mathcal{U}_n(t,s)$ and $\mathcal{U}(t,s)$ are well defined and fulfill (i)–(iv) of Theorem 1.2. We claim that for every $k \geq 0$, $\exists \widetilde{C}_k > 0$ s.t.

$$\|\mathcal{U}_n(t,s)\|_{\mathcal{L}(\mathcal{H}^k)} \leq e^{\widetilde{C}_k|t-s|} \ , \quad \forall n \geq 0 \ . \tag{2.13}$$

Such estimate follows by arguing similarly to the proof of Lemma 2.5 and using estimate (2.10) to estimate $[L_n(t), H^k]H^{-k}$. We skip the details. Now we have

$$\|\mathcal{U}_n(t,s)\psi - \mathcal{U}(t,s)\psi\|_k$$
$$= \|\int_s^t \mathcal{U}(t,r)\ (L_n(r) - L(r))\,\mathcal{U}_N(r,s)\psi\, dr\|_k$$
$$\le |t-s| \sup_{r\in[s,t]} \|\mathcal{U}(t,r)\|_{\mathcal{L}(\mathcal{H}^k)}\, \|L_n(r)-L(r)\|_{\mathcal{L}(\mathcal{H}^{k+m},\mathcal{H}^k)} \|\mathcal{U}_n(r,s)\|_{\mathcal{L}(\mathcal{H}^{k+m})} \|\psi\|_{k+m}$$
$$\le \sup_{r\in[s,t]} \|L_n(r)-L(r)\|_{\mathcal{L}(\mathcal{H}^{k+m},\mathcal{H}^k)}\ |t-s|\ e^{(C_k+\widetilde{C}_{k+m})|t-s|}\ \|\psi\|_{k+m},$$

which converges to 0 by (2.11). □

3. Growth of norms for perturbations smooth in time

In this section we prove Theorem 1.9, and at the end of the Section we show how to prove Theorem 1.8. First we show that under assumptions (Hgap) and $(\mathrm{Vs})_n$, the operator $H + V(t)$ satisfies a spectral gap property. Then we describe the algorithm which will allow us to construct an adiabatic approximation $\mathcal{U}_{ad}(t,s)$ of the flow of the operator $H + V(t)$. Here we follow the strategy of [29], adding analytic estimates to the construction. Finally we show how to use the adiabatic approximation $\mathcal{U}_{ad}(t,s)$ to control the growth of the Sobolev norm.

3.1. Spectral properties of $H + V(t)$

It is more convenient to have dyadic gaps between the clusters, so we define a new sequence of clusters as follows. Fix a large integer $\mathtt{J} \ge 1$ (to be chosen later on). Define the new clusters

$$\widetilde{\sigma}_1 := \bigcup_{1\le l\le 2^{\mathtt{J}}} \sigma_l\ , \qquad \widetilde{\sigma}_j = \bigcup_{2^{\mathtt{J}+j-2}+1\le l\le 2^{\mathtt{J}+j-1}} \sigma_l \qquad \text{for } j\ge 2\ . \tag{3.1}$$

We define as well

$$\widetilde{\Delta}_j := \mathrm{dist}(\widetilde{\sigma}_{j+1},\widetilde{\sigma}_j)\ , \qquad \widetilde{\delta}_j := \sup_{\lambda_1,\lambda_2\in\widetilde{\sigma}_j} |\lambda_1-\lambda_2|\ , \tag{3.2}$$

$$\lambda_j^+ = \max_{\lambda\in\widetilde{\sigma}_j} \lambda\ , \quad \lambda_j^- = \min_{\lambda\in\widetilde{\sigma}_j} \lambda\ . \tag{3.3}$$

So condition (Hgap) is written now as

$(\widetilde{\mathrm{Hgap}})$ The spectrum of H fulfills $\sigma(H) \subseteq \bigcup_{1\le j<+\infty} \widetilde{\sigma}_j$ and there exist positive constants $\widetilde{\alpha}, \widetilde{\beta}$ (independent of $\mathtt{J}$) s.t. $\forall j \ge 1$

$$\widetilde{\alpha}^{-1}\, 2^{(\mathtt{J}+j-1)\mu} \le \widetilde{\Delta}_j \le \widetilde{\alpha}\, 2^{(\mathtt{J}+j-1)\mu}, \qquad \widetilde{\delta}_j \le \widetilde{\beta}\, 2^{(\mathtt{J}+j-1)(\mu+1)}\ . \tag{3.4}$$

Remark 3.1. Let $H(t)$ be an operator fulfilling $\widetilde{\text{(Hgap)}}$ uniformly in time $t \in \mathbb{R}$. Then there exist positive constants $\widetilde{C}_1, \widetilde{C}_2$ (independent of $\mathtt{J}, j$) such that

$$\begin{aligned} &\lambda_1^+ \leq \widetilde{C}_2 \, 2^{\mathtt{J}(\mu+1)} \ , \\ &\widetilde{C}_1 \, 2^{(\mathtt{J}+j-1)(\mu+1)} \leq \lambda_j^- \leq \lambda_j^+ \leq \widetilde{C}_2 \, 2^{(\mathtt{J}+j-1)(\mu+1)} \ , \quad \forall j \geq 2 \ . \end{aligned} \tag{3.5}$$

In particular we have the very useful property

$$\max \widetilde{\sigma}_j \leq \frac{\widetilde{C}_2}{\widetilde{C}_1} \min \widetilde{\sigma}_j \ , \qquad \forall j \in \mathbb{N} \ . \tag{3.6}$$

We will denote by Γ_j, $j \geq 1$, an anti clock-wise oriented rectangle in the complex plane which isolates the cluster $\widetilde{\sigma}_j$, that is Γ_j contains only $\widetilde{\sigma}_j$ at its interior. We fix such contours so that

$$\inf_{\lambda \in \Gamma_1} \operatorname{dist}(\lambda, \sigma(H)) \geq \frac{\widetilde{\Delta}_1}{2} \ , \qquad \inf_{\lambda \in \Gamma_j} \operatorname{dist}(\lambda, \sigma(H)) \geq \frac{\widetilde{\Delta}_{j-1}}{2} \ , \qquad j \geq 2 \ . \tag{3.7}$$

Finally define

$$\delta := 1 - \frac{\mu+1}{\mu} \nu \ . \tag{3.8}$$

It is important to remark that by our assumptions $0 < \delta \leq 1$.

We prove now a perturbative result. It is in this lemma which enters into play the restriction $\nu < \frac{\mu}{\mu+1}$. This is indeed the condition which guarantees that the operator $H + V(t)$ has a spectrum with increasing spectral gaps.

Lemma 3.2. *Let H satisfy* (Hgap) *and $V(t)$ satisfy* $(\text{Vs})_n$ *for some $n \geq 0$. There exists a constant C_H (depending only on H), such that if $\mathtt{J}$ is large enough to fulfill*

$$2^{\mathtt{J}\mu\delta} \geq 2^4 \, C_H \sup_{t \in \mathbb{R}} \| V(t) \, H^{-\nu} \|_{\mathcal{L}(\mathcal{H}^0)} \tag{3.9}$$

then $H + V(t)$ fulfills $\widetilde{\text{(Hgap)}}$ *uniformly in $t \in \mathbb{R}$, with new clusters*

$$\widetilde{\sigma}'_j = [\lambda_j^- - \frac{\widetilde{\Delta}_{j-1}}{4}] \cup \widetilde{\sigma}_j \cup [\lambda_j^+ + \frac{\widetilde{\Delta}_{j-1}}{4}] \ , \quad j \in \mathbb{N} \ . \tag{3.10}$$

Here we defined $\widetilde{\Delta}_0 := \widetilde{\Delta}_1$.

Proof. We show that any $z \in \bigcup_j [\lambda_j^+ + \frac{\widetilde{\Delta}_j}{4}, \lambda_{j+1}^- - \frac{\widetilde{\Delta}_j}{4}]$ belongs to the resolvent set of $H + V(t)$. For $z \in \mathbb{C} \backslash \mathbb{R}$ we have

$$H + V(t) - z = \left(V(t)(H-z)^{-1} + \mathbb{I} \right)(H-z) = \left([V(t) \, H^{-\nu}] \, [H^\nu \, (H-z)^{-1}] + \mathbb{I} \right) \, (H-z) \ .$$

By spectral decomposition

$$H^\nu (H-z)^{-1} = \sum_{j\geq 1} \int_{\Gamma_j} \frac{\zeta^\nu}{\zeta - z}\, dE_H(\zeta) \tag{3.11}$$

where $\{E_H(\zeta)\}_{\zeta\in\mathbb{R}}$ is the spectral decomposition of H. One has

$$\|H^\nu(H-z)^{-1}\|_{\mathcal{L}(\mathcal{H}^0)} = \sup_{\zeta\in\sigma(H)} \left|\frac{\zeta^\nu}{\zeta - z}\right| \leq \left(1 + \frac{z}{\mathrm{dist}(z,\sigma(H))}\right)^\nu \frac{1}{\mathrm{dist}(z,\sigma(H))^{1-\nu}} \ .$$

Fix $z \in \bigcup_j [\lambda_j^+ + \frac{\widetilde{\Delta}_j}{4}, \lambda_{j+1}^- - \frac{\widetilde{\Delta}_j}{4}]$. Then (using also (3.4), (3.5))

$$\|H^\nu(H-z)^{-1}\|_{\mathcal{L}(\mathcal{H}^0)} \leq \frac{4\,(\lambda_{j+1}^-)^\nu}{\widetilde{\Delta}_j} \leq 4\,\widetilde{\alpha}\,\widetilde{C}_2^\nu\, 2^\mu\, 2^{(\mathtt{J}+j)[(\mu+1)\nu-\mu]} \leq C_H\, 2^{-(\mathtt{J}+j)\mu\delta} \ ,$$

where $\delta > 0$ is defined in (3.8). Thus provided (3.9) holds one has

$$\sup_{t\in\mathbb{R}} \|V(t)\, H^{-\nu}\|_{\mathcal{L}(\mathcal{H}^0)}\ \|H^\nu(H-z)^{-1}\|_{\mathcal{L}(\mathcal{H}^0)} \leq 1/2 \tag{3.12}$$

and we can invert $[V(t)\, H^{-\nu}]\,[H^\nu\,(H-z)^{-1}] + \mathbb{I}$ by Neumann series and define the resolvent

$$R_V(t,\lambda) := (H-z)^{-1}\ \left([V(t)\, H^{-\nu}]\,[H^\nu\,(H-z)^{-1}] + \mathbb{I}\right)^{-1} \ .$$

This shows that any $z \in \bigcup_j [\lambda_j^+ + \frac{\widetilde{\Delta}_j}{4}, \lambda_{j+1}^- - \frac{\widetilde{\Delta}_j}{4}]$ belongs to the resolvent set of $H+V(t)$, $\forall t\in\mathbb{R}$. Thus

$$\sigma(H+V(t)) \subset \bigcup_{j\geq 1} \widetilde{\sigma}'_j \ .$$

The lemma follows easily. Notice that we get in particular that for every $t\in\mathbb{R}$, $H+V(t)$ is self-adjoint on the domain $D(H)$ of H. □

Remark 3.3. One has that $\widetilde{\Delta}'_j := \mathrm{dist}(\widetilde{\sigma}'_{j+1}, \widetilde{\sigma}'_j)$, $\widetilde{\delta}'_j := \sup_{\lambda_1,\lambda_2\in\widetilde{\sigma}'_j} |\lambda_1 - \lambda_2|$ fulfill (3.4) with new constants $\widetilde{\alpha}$, $\widetilde{\beta}$.

In the following we will always use the clusters $\widetilde{\sigma}'_j$'s. By abusing the notation we will suppress the up-script $'$ and write only $\widetilde{\sigma}_j \equiv \widetilde{\sigma}'_j$.

Lemma 3.4. *There exists $\widetilde{C}_H > 0$, independent on $j, \mathtt{J}$, such that for all $j \geq 1$*

$$\sup_{z\in\Gamma_j} \|H^\nu(H-z)^{-1}\|_{\mathcal{L}(\mathcal{H}^0)} \leq \frac{\widetilde{C}_H}{\widetilde{\Delta}_{j-1}^\delta} \ , \tag{3.13}$$

where δ is defined in (3.8).

Proof. We show that there exists a constant $\widetilde{C} > 0$, independent on j, $\mathtt{J}$, s.t. for every $z \in \Gamma_j$,

$$\|H^\nu (H - z)^{-1}\|_{\mathcal{L}(\mathcal{H}^0)} \leq \widetilde{C} \frac{2^{(\mathtt{J}+j-1)\nu}}{\mathrm{dist}(z, \sigma(H))^{1-\nu}} \ . \tag{3.14}$$

Then (3.13) follows easily using (3.7) and (3.4).

To prove (3.14), recall that $\|H^\nu (H - z)^{-1}\|_{\mathcal{L}(\mathcal{H}^0)} = \sup_{\zeta \in \sigma(H)} \frac{|\zeta|^\nu}{|\zeta - z|}$ and write

$$\frac{|\zeta|^\nu}{|\zeta - z|} = \left(\frac{|\zeta|}{|\zeta - z|} \right)^\nu \frac{1}{|\zeta - z|^{1-\nu}} \ .$$

Let $z \in \Gamma_j$. If $\zeta \in \widetilde{\sigma}_j$ we have by (3.5) and (3.4)

$$\frac{|\zeta|}{|\zeta - z|} \leq \widetilde{C}_2 \frac{2^{(\mathtt{J}+j-1)(\mu+1)}}{\widetilde{\Delta}_{j-1}} \leq C_3 \, 2^{(\mathtt{J}+j-1)} \ ,$$

where $C_3 > 0$ is independent of $j, \mathtt{J}$.

Now if $\zeta \in \widetilde{\sigma}_{j'}$, $j' \neq j$ then $\zeta \approx 2^{(\mathtt{J}+j'-1)(\mu+1)}$ and there exists $C_4 > 0$ s.t. $|\zeta - z| \geq \widetilde{C}_4 2^{(\mathtt{J}+j'-1)(\mu+1)}$ (notice that $\mathrm{length}(\widetilde{\sigma}_k) \geq c 2^{(\mathtt{J}+k-1)(\mu+1)}$) so

$$\frac{|\zeta|}{|\zeta - z|} \leq \widetilde{C}_4 .$$

Hence (3.14) follows with $\widetilde{C} = \max(C_3, C_4)$. $\square$

3.2. Adiabatic approximation

Let us start now the adiabatic approximation as explained in [27,29,22,21].

We present first the formal construction. In a second step we perform analytic estimates to prove that all the objects are well defined.

The idea is to construct a sequence of operators $B_m(t)$ such that for every $m \geq 0$ the flow $\mathcal{U}_{ad,m}(t, s)$ of $H + V(t) - B_m(t)$ is adiabatic, in particular it preserves the $\mathcal{H}^k$-norm, and $B_m(t)$ is a more and more regularizing operator in a suitable sense. The $B_m(t)$ are constructed step by step such that at each step we have an adiabatic transport for spectral projectors. Let us recall here the adiabatic approximation used at each step following [27,29].

Consider $H_W(t) = L(t) + W(t)$ a perturbation of $L(t) := H + V(t)$ such that $\sigma(H_W) \subseteq \bigcup_{j \geq 1} \sigma_j^W$, a splitting of the spectrum of $H_W(t)$ into clusters σ_j^W, uniform in time $t \in \mathbb{R}$. $\Pi_j^W(t)$ denotes the spectral projector of $H_W(t)$ onto σ_j^W. We are looking for an adiabatic transport for all the $\{\Pi_j^W(t)\}_{j \geq 1}$ which means that we want to find an Hamiltonian $H_{ad}(t) = L(t) - B(t)$ (a "small" perturbation of $L(t)$) such that

$$\Pi_{m,j}(s) = \mathcal{U}^*_{ad,m}(t, s) \, \Pi_{m,j}(t) \, \mathcal{U}_{ad,m}(t, s), \qquad \forall t, s \in \mathbb{R}, j \geq 1. \tag{3.15}$$

Taking the time derivative we see that (3.15) is satisfied if and only if

$$\mathrm{i}[B, \Pi_j^W] = \partial_t \Pi_j^W + \mathrm{i}[L, \Pi_j^W] := F_j. \tag{3.16}$$

It is not difficult to solve the homological equation (3.16) using the decomposition $B = \sum_{k,k' \geq 1} \Pi_k^W B \Pi_{k'}^W$. First note that, by the properties of orthogonal projectors, one has $\Pi_k^W F_j \Pi_{k'}^W = 0 \ \forall k \neq j$, $k' \neq j$ and $\Pi_j^W F_j \Pi_j^W = 0$, hence there are no diagonal terms in the homological equation. We can thus assume that $\Pi_k^W B \Pi_{k'}^W = 0$ if $k, k' \neq j$ and we have $\Pi_j^W B \Pi_{k'}^W = \mathrm{i}\, \Pi_j^W F_j \Pi_{k'}^W$.

So, by a computation using that $\{\Pi_k^W\}_{k \geq 1}$ is a complete family of orthogonal projectors, we get

$$B = \mathrm{i} \left(\sum_{k \geq 1} \Pi_k^W \left(\partial_t \Pi_k^W + \mathrm{i}[L, \Pi_k^W] \right) \right) . \tag{3.17}$$

The Nenciu algorithm [27] is obtained by iterating this formal computation:

$$W \longrightarrow W + B, \quad \Pi_k^W \longrightarrow \Pi_k^{W+B}$$

$$B_{new} = \mathrm{i} \left(\sum_{k \geq 1} \Pi_k^{W+B} \left(\partial_t \Pi_k^{W+B} + \mathrm{i}[L, \Pi_k^{W+B}] \right) \right) . \tag{3.18}$$

We describe now how to construct the $B_m(t)$'s. A sequence $H_m(t)$ of perturbations of $L(t)$ is constructed by induction as follows:

$$H_0(t) := L(t)$$
$$H_{m+1}(t) := H_m(t) + B_m(t) \ , \quad \forall m \geq 0 \ ,$$

where the $B_m(t)$ are obtained from the spectral projectors of $H_m(t)$. More precisely, we will prove that at each step $\sigma(H_m(t)) \subseteq \bigcup_{j \geq 1} \widetilde{\sigma}_j$, where the $\widetilde{\sigma}_j$'s are the ones of (3.10). Denote by $\Pi_{m,j}(t)$ the spectral projector of $H_m(t)$ on the cluster $\widetilde{\sigma}_j$. Then following (3.17) we define

$$B_m(t) := \mathrm{i} \sum_{1 \leq j < +\infty} \Pi_{m,j}(t)\, \partial_{(t,L)} \Pi_{m,j}(t), \tag{3.19}$$

where

$$\partial_{(t,L)} A(t) := \partial_t A(t) + \mathrm{i}[L(t), A(t)]$$

is the Heisenberg derivative of A.

So that according (3.15) and (3.16) the flow $\mathcal{U}_{ad,m}(t,s)$ of $H + V(t) - B_m(t)$ fulfills the adiabatic property

$$\mathcal{U}_{ad,m}(t,s)\Pi_{m,j}(s) = \Pi_{m,j}(t)\mathcal{U}_{ad,m}(t,s) \ , \quad \forall j \geq 1 \ , \ \forall t,s \in \mathbb{R}$$

(see Lemma 3.9 below) and thus it is an adiabatic approximation of the flow $\mathcal{U}(t,s)$ of $H + V(t)$. The reason to iterate the procedure is that at each step the $B_m(t)$'s are more regularizing operators (see Corollary 3.10).

Let us give some technical details to justify this construction under our assumptions. Let us denote

$$B_{m,j}(t) := \Pi_{m,j}(t)\, \partial_{(t,L)}\Pi_{m,j}(t) \ .$$

Notice that $B_{0,j}(t) = \Pi_{0,j}(t)\, \partial_t \Pi_{0,j}(t)$, since $[L(t), \Pi_{0,j}(t)] = 0$.

In the following we shall denote $\| \cdot \| \equiv \| \cdot \|_{\mathcal{L}(\mathcal{H}^0)}$ the operator norm in $\mathcal{L}(\mathcal{H}^0)$.

Lemma 3.5. *Under the same assumptions of Theorem 1.9, fix an arbitrary* $\mathtt{M} \in \mathbb{N}$. *If* $\mathtt{J}$ *is sufficiently large, for every integers* $\ell \geq 0$, $0 \leq m \leq \mathtt{M}$, $0 \leq p \leq n$, *there exists* $C_{m,n,\ell} > 0$, *independent of* $\mathtt{J}$, *such that*

$$\sup_{t\in\mathbb{R}} \|H^p \partial_t^\ell B_{m,j}(t) H^{-p}\| \leq \frac{C_{m,n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta(1+m)}}, \quad \forall j \in \mathbb{N} \ . \tag{3.20}$$

Therefore $B_m(t)$ *in* (3.19) *is well defined,* $B_m(\cdot) \in C_b^\infty(\mathbb{R}, \mathcal{L}(\mathcal{H}^k))$ *for any* $0 \leq k \leq 2n$ *and*

$$\sup_{t\in\mathbb{R}} \|H^p \ \partial_t^\ell B_m(t) \ H^{-p}\| \leq \frac{\widetilde{C}_{m,n,\ell}}{2^{\mathtt{J}\mu\delta(1+m)}} \ , \quad \forall \ell \geq 0, \ 0 \leq p \leq n \ . \tag{3.21}$$

Finally $B_m(t)$ *is a self-adjoint operator in* $\mathcal{H}^0$.

Lemma 3.5 is quite technical and we postpone its proof at the end of the section.

Remark 3.6. In particular $B_m(t)$ satisfies the condition $(\mathrm{Vs})_n$ (with $\nu = 0$).

Define for $m \geq 1$ the operators

$$H_m(t) := H + V(t) + B_0(t) + \cdots + B_{m-1}(t) \equiv L(t) + W_m(t)$$

$$H_{ad,m}(t) := H + V(t) - B_m(t) \equiv L(t) - B_m(t)$$

The following corollary follows immediately from Lemma 3.5:

Corollary 3.7. *Fix* $\mathtt{M} \geq 1$*. Then provided* $\mathtt{J}$ *is sufficiently large, the following holds true:*

(i) For every $1 \leq m \leq \mathtt{M}$ *and* $t \in \mathbb{R}$*, the operators* $H_m(t)$ *and* $H_{ad,m}(t)$ *are self-adjoint operators generating a unitary flow in* $\mathcal{H}^0$*.*

(ii) For every $1 \leq m \leq \mathtt{M}$ *and* $t \in \mathbb{R}$*,* $H_m(t)$ *and* $H_{ad,m}(t)$ *fulfill* $(\widetilde{\text{Hgap}})$ *uniformly in time* $t \in \mathbb{R}$ *with* $\widetilde{\sigma}_j$*'s as in Lemma 3.2.*

Proof. (i) By Lemma 3.5 $\forall 0 \leq m \leq \mathtt{M}$ the operator $B_m(t)$ is a bounded self-adjoint operator. Hence $H_m(t) = L(t) + W_m(t)$ and $H_{ad,m}(t) = L(t) - B_m(t)$ are bounded perturbations of $L(t)$, and thus they are self-adjoint operators generating a unitary flow in $\mathcal{H}^0$.

(ii) Write $H_m(t)$ as $H_m(t) \equiv H + W(t)$ with $W(t) := V(t) + B_0(t) + \cdots + B_m(t)$. By $(\text{Vs})_n$ and (3.21) it fulfills

$$\sup_{t\in\mathbb{R}} \|H^p \, \partial_t^\ell W(t) \, H^{-p-\nu}\| \leq R_{n,\ell} + (m+1)\frac{\widetilde{C}_{m,n,\ell}}{2^{\mathtt{J}\mu\delta}} \leq 2R_{n,\ell} \ ,$$

provided $\mathtt{J}$ is sufficiently large (depending on $\mathtt{M}$). Then Lemma 3.2 gives the claim.

The proof for $H_{ad,m}(t)$ is analogous. □

We will denote by $\mathcal{U}_{ad,m}(t,s)$ the propagator of $H_{ad,m}(t)$. The two key points, proved in Corollary 3.10 below, are the following:

(i) $\mathcal{U}_{ad,m}(t,s)$ is an adiabatic approximation of $\mathcal{U}(t,s)$ which preserves the $\mathcal{H}^k$-norms.
(ii) The operators B_m's are smoothing operators.

In order to prove those two properties it is convenient to measure the $\mathcal{H}^k$-norm with the help of the projectors $\Pi_{m,j}$'s. More precisely perform the construction at order m. Introduce the block diagonal operator

$$\Lambda_m(t) := \sum_{1\leq j<\infty} 2^{(j-1)(\mu+1)} \, \Pi_{m,j}(t) \ .$$

As the $\Pi_{m,j}$'s are orthogonal projectors one has that

$$\|\Lambda_m(t)\psi\|_0^2 = \sum_{j\geq 1} 2^{2(j-1)(\mu+1)} \, \|\Pi_{m,j}(t)\psi\|_0^2 \ , \qquad \forall \psi \in \mathcal{H}^0 \ .$$

The next lemma shows that the norm $\|H^p \cdot\|$ is equivalent to the norm $\|\Lambda_m(t)^p \cdot\|$:

Lemma 3.8. *Fix* $n \in \mathbb{N}$*,* $n \geq 0$*. Assume that* $(\text{Vs})_n$ *is satisfied. Then for any* $1 \leq m \leq \mathtt{M}$*, there exist positive* c_1 *and* c_2*, depending on* $n, V, \sigma(H), \|H^n B_i(t) \ H^{-n}\|$*, such that* $\forall 0 \leq p \leq n$*,* $\forall \psi \in \mathcal{H}^{2p}$*,* $\forall t \in \mathbb{R}$

$$c_1 \|\psi\|_{2p} \leq \|(H_{ad,m}(t) + c_0)^p \psi\|_0 \leq c_2 \|\psi\|_{2p} \tag{3.22}$$

$$c_1\, 2^{\mathtt{J}p(\mu+1)} \, \|\Lambda_m^p(t)\psi\|_0 \leq \|(H_{ad,m}(t) + c_0)^p \psi\|_0 \leq c_2\, 2^{\mathtt{J}p(\mu+1)} \, \|\Lambda_m^p(t)\psi\|_0 \ . \tag{3.23}$$

The proof is postponed in Appendix D.

We prove now some properties of the adiabatic evolution.

Lemma 3.9. *For every integer $0 \leq m \leq \mathtt{M}$ and $j \geq 1$ we have*

$$\Pi_{m,j}(t) = \mathcal{U}_{ad,m}(t,s)\, \Pi_{m,j}(s)\, \mathcal{U}_{ad,m}(t,s)^*, \qquad \forall t, s \in \mathbb{R}. \tag{3.24}$$

Proof. For any propagator $\mathcal{U}(t,s)$ with generator $t \mapsto L(t)$ of class C^1 and any C^1 and bounded operator $A(t)$ we have

$$\partial_t \left(\mathcal{U}(t,s)^*\, A(t)\, \mathcal{U}(t,s)\right) = \mathcal{U}(t,s)^*\, \partial_{(t,L)} A(t)\, \mathcal{U}(t,s).$$

Since the generator of $\mathcal{U}_{ad,m}(t,s)$ is $L(t) - B_m(t)$, it is enough to prove that

$$\partial_{(t,L-B_m)} \Pi_{m,j}(t) = 0, \quad \forall t \in \mathbb{R} \ , \qquad \forall j \geq 1 \ , \ m \geq 0 \ . \tag{3.25}$$

This follows easily using the definition of $B_m(t)$ and properties of orthogonal projectors. □

Corollary 3.10. *(i) For every $0 \leq m \leq \mathtt{M}$, $\mathcal{U}_{ad,m}(t,s)$ preserves the $\mathcal{H}^k$-norms. More precisely for every $0 \leq p \leq n$, there exists $C_p > 0$ s.t.*

$$\|\mathcal{U}_{ad,m}(t,s)\|_{\mathcal{L}(\mathcal{H}^{2p})} \leq C_p \qquad \forall t, s \in \mathbb{R} \ .$$

(ii) For every $0 \leq m \leq \mathtt{M}$, $B_m(t) : \mathcal{H}^0 \mapsto \mathcal{H}^{2p}$ provided $p < \frac{\mu}{\mu+1}\delta(1+m)$.

Proof. (i) First note that by Lemma 3.9 one has that $\Lambda_m(t)\mathcal{U}_{ad,m}(t,s) = \mathcal{U}_{ad,m}(t,s)\Lambda_m(s)$. Then by Lemma 3.8 and the unitarity of $\mathcal{U}_{ad,m}(t,s)$ in $\mathcal{H}^0$ one has

$$\begin{aligned} \|\mathcal{U}_{ad,m}(t,s)\psi_s\|_{2p} &\leq C \|\Lambda_m^p(t)\mathcal{U}_{ad,m}(t,s)\psi_s\|_0 \leq C \|\mathcal{U}_{ad,m}(t,s)\Lambda_m^p(s)\psi_s\|_0 \\ &\leq C \|\Lambda_m^p(s)\psi_s\|_0 \leq C \|\psi_s\|_{2p} \ . \end{aligned}$$

(ii) Recall that $\Pi_{m,j}(t)B_m(t) = B_{m,j}(t)$. Now we have

$$\begin{aligned} \|B_m(t)\psi_s\|_{2p}^2 &\leq C \|\Lambda_m^p(t) B_m(t)\psi_s\|_0^2 \leq C \sum_{j \geq 1} 2^{(j-1)(\mu+1)2p} \|B_{m,j}(t)\psi_s\|_0^2 \\ &\leq C \|\psi_s\|_0 \sum_{j \geq 1} 2^{2(j-1)[(\mu+1)p - \mu\delta(1+m)]} \leq C \|\psi_s\|_0 \end{aligned}$$

provided $p < \frac{\mu}{\mu+1}\delta(1+m)$. □

We are finally ready to prove Theorem 1.9.

Proof of Theorem 1.9. Fix $\epsilon > 0$ and choose $\mathtt{M}$ such that

$$\frac{1}{\epsilon}\frac{(\mu+1)n}{\mu\delta} \leq \mathtt{M}+1 \ . \tag{3.26}$$

Choose $\mathtt{J}$ sufficiently large to perform the construction at step $\mathtt{M}$. As the evolution $\mathcal{U}(t,s)$ is unitary in $\mathcal{H}^0$ and $\Pi_{\mathtt{M},j}(t)$ is a projector we have

$$\|\Pi_{\mathtt{M},j}(t)\,\mathcal{U}(t,s)\psi_s\|_0 \leq \|\psi_s\|_0 \ , \quad \forall j \geq 1, \quad \forall t,s \in \mathbb{R} \ . \tag{3.27}$$

We compare the evolution $\mathcal{U}(t,s)$ with the adiabatic evolution $\mathcal{U}_{ad,\mathtt{M}}(t,s)$ defined above. In order to do this, write

$$\mathrm{i}\dot{\psi} = (H + V(t))\psi = H_{ad,\mathtt{M}}(t)\psi + B_{\mathtt{M}}(t)\psi$$

and use the Duhamel formula

$$\mathcal{U}(t,s) = \mathcal{U}_{ad,\mathtt{M}}(t,s) - i\int_s^t \mathcal{U}_{ad,\mathtt{M}}(t,r)\, B_{\mathtt{M}}(r)\,\mathcal{U}(r,s)dr. \tag{3.28}$$

By equation (3.24), the property $\Pi_{\mathtt{M},j}(t)\, B_{\mathtt{M}}(t) = B_{\mathtt{M},j}(t)$ and Lemma 3.5 one has

$$\begin{aligned}\|\Pi_{\mathtt{M},j}(t)\mathcal{U}(t,s)\psi_s\|_0 &\leq \|\mathcal{U}_{ad,\mathtt{M}}(t,s)\,\Pi_{\mathtt{M},j}(s)\psi_s\|_0 + \|\int_s^t \mathcal{U}_{ad,\mathtt{M}}(t,r)\, B_{\mathtt{M},j}(r)\,\mathcal{U}(r,s)\psi_s dr\|_0 \\ &\leq \|\Pi_{\mathtt{M},j}(s)\psi_s\|_0 + \langle t-s\rangle\, 2^{-(j-1)(\mathtt{M}+1)\mu\delta}\|\psi_s\|_0 \ ,\end{aligned} \tag{3.29}$$

where in the last line we used that, provided $\mathtt{J}$ is sufficiently large,

$$\sup_{t\in\mathbb{R}}\|B_{\mathtt{M},j}(t)\| \leq \frac{C_{\mathtt{M},n,0}}{\widetilde{\Delta}_{j-1}^{(\mathtt{M}+1)\delta}} \leq \frac{1}{2^{(j-1)(\mathtt{M}+1)\mu\delta}} \ , \qquad \forall j \geq 1,\ t \in \mathbb{R}.$$

We compute now the norm of $\mathcal{U}(t,s)\psi_s$ in $\mathcal{H}^{2n}$. Fix $\mathtt{N} \equiv \mathtt{N}(t)$ to be chosen later. By Lemma 3.8

$$\|\mathcal{U}(t,s)\psi_s\|_{2n}^2 \leq \frac{c_2}{c_1}\, 2^{\mathtt{J}(\mu+1)2n}\, \|\Lambda_{\mathtt{M}}^n(t)\,\mathcal{U}(t,s)\psi_s\|_0^2 \leq \frac{c_2}{c_1}\, 2^{\mathtt{J}(\mu+1)2n}(I + II) \ ,$$

where

$$I := \sum_{1\leq j\leq \mathtt{N}} 2^{(j-1)(\mu+1)2n}\|\Pi_{\mathtt{M},j}(t)\mathcal{U}(t,s)\psi_s\|_0^2 \ ,$$

$$II := \sum_{j\geq \mathtt{N}+1} 2^{(j-1)(\mu+1)2n}\|\Pi_{\mathtt{M},j}(t)\mathcal{U}(t,s)\psi_s\|_0^2 \ .$$

To estimate I, use (3.27) to obtain

$$I \leq \|\psi_s\|_0^2 \sum_{0\leq j\leq \mathtt{N}-1} 2^{j(\mu+1)2n} \leq \|\psi_s\|_0^2 \, \frac{2^{\mathtt{N}(\mu+1)2n}-1}{2^{(\mu+1)2n}-1} \leq C\|\psi_s\|_0^2 \, 2^{\mathtt{N}(\mu+1)2n} \ , \tag{3.30}$$

where C depends only on n, μ. To estimate the second summand, we use (3.29) and Lemma 3.8 to obtain

$$\begin{aligned} II &\leq 4\sum_{j\geq \mathtt{N}} 2^{j(\mu+1)2n}\|\Pi_{\mathtt{M},j}(s)\psi_s\|_0^2 + 4\,\langle t-s\rangle^2\,\|\psi_s\|_0^2 \sum_{j\geq\mathtt{N}} 2^{2j[(\mu+1)n-(\mathtt{M}+1)\mu\delta]} \\ &\leq 4\|\psi_s\|_{2n}^2 + 4\,\langle t-s\rangle^2\,\|\psi_s\|_0^2 \, \frac{2^{[(\mu+1)n-(\mathtt{M}+1)\mu\delta]2\mathtt{N}}}{1-2^{2[(\mu+1)n-(\mathtt{M}+1)\mu\delta]}} \end{aligned} \tag{3.31}$$

where we used that $(\mu+1)n/\mu\delta \leq \mathtt{M}+1$. Thus, (3.30) and (3.31) give

$$\|\mathcal{U}(t,s)\psi_s\|_{2n}^2 \leq \widetilde{C}\,2^{\mathtt{J}(\mu+1)2n}\,\|\psi_s\|_{2n}^2\,\left[2^{\mathtt{N}(\mu+1)2n} + \langle t-s\rangle^2\,2^{[(\mu+1)n-(\mathtt{M}+1)\mu\delta]2\mathtt{N}}\right] \ , \tag{3.32}$$

where $\widetilde{C}$ does not depend on $\mathtt{N}$. Now choose $\mathtt{N}(t)$ in such a way to optimize (3.32), i.e. pick

$$\mathtt{N}(t) = \frac{1}{(\mathtt{M}+1)\mu\delta}\log\langle t-s\rangle$$

to obtain

$$\|\mathcal{U}(t,s)\psi_s\|_{2n}^2 \leq C\,2^{\frac{2(\mu+1)n}{(\mathtt{M}+1)\mu\delta}\log\langle t-s\rangle}\,\|\psi_s\|_{2n}^2 \ . \tag{3.33}$$

Using (3.26) one has

$$\|\mathcal{U}(t,s)\psi_s\|_{2n}^2 \leq C\,\langle t-s\rangle^{\frac{2(\mu+1)n}{(\mathtt{M}+1)\mu\delta}}\,\|\psi_s\|_{2n}^2 \leq C\,\langle t-s\rangle^{2\epsilon}\,\|\psi_s\|_{2n}^2 \ ,$$

which is the desired estimate. □

We also get the following application of the adiabatic approximation concerning the spectra of Floquet operators (see [20,29,22]).

Let assume that conditions (Hgap), $(\mathrm{Vs})_n$ are satisfied and suppose that $V(t)$ is periodic with period $T>0$. Denote $\mathcal{F} := \mathcal{U}(T,0)$ the Floquet operator (or monodromy operator). Let us recall that $\mathcal{U}(nT,0) = \mathcal{F}^N$ so the spectrum of $\mathcal{F}$ gives informations on the large time behavior of the propagator.

Theorem 3.11. *Let us assume that conditions* (Hgap), $(\mathrm{Vs})_n$ *are satisfied, V is T-periodic and that $(H+\mathrm{i})^{-N}$ is in the trace class for N large enough. Then the Floquet operator $\mathcal{F}$ has no absolutely continuous spectrum.*

Proof. It results from Lemma 3.5 that $B_m(t)$ is in the trace class for m large enough. So from (3.28) we infer that $\mathcal{U}(T,0) - \mathcal{U}_{ad,m}(T,0)$ is in the trace class. It is easy to see that the Hamiltonians $H_m(t)$ are T-periodic (from the induction construction). So it results from (3.24) that $\mathcal{U}_{ad,m}(T,0)$ commutes with $\Pi_{m,j}(0)$, the spectral projectors of $H_m(T)$. But $(H_m(T)+\mathrm{i})^{-1}$ is a compact operator hence the spectrum of $\mathcal{U}_{ad,m}(T,0)$ is purely discrete. Applying the Birman–Krein–Kato [4] theorem on the stability of the absolutely spectrum under class trace perturbations we get Theorem 3.11. □

3.3. Proof of Lemma 3.5

The proof is by induction. Through all the proof, we will denote by $C_{m,n,\ell}$ some positive constants which depend on m, n, ℓ but not on $j, \mathtt{J}$.

We will prove (3.20) together with the estimate $\forall j \in \mathbb{N}$

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^{\ell+1} \Pi_{m,j}(t)\, H^{-p}\| \ , \ \sup_{t\in\mathbb{R}} \|H^p\, \partial_t^{\ell}\ \partial_{(t,L)} \Pi_{m,j}(t)\, H^{-p}\| \leq \frac{C_{m,n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta}} \ , \quad \forall 0 \leq p \leq n, \ \ell \geq 0 \tag{3.34}$$

Step $m = 0$**.** Recall that $H_0(t) = H + V(t) \equiv L(t)$. Provided $\mathtt{J}$ is sufficiently large, by Lemma B.2 the projectors

$$\Pi_{0,j}(t) := -\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_j} R_0(t,\lambda)\, d\lambda \ , \qquad \forall j \geq 1 \tag{3.35}$$

are well defined and fulfill

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^{\ell+1} \Pi_{0,j}(t)\, H^{-p}\| \leq \frac{C_{0,n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta}} \ , \quad \forall 0 \leq p \leq n, \ \ell \geq 0 \ ,$$

for some constants $C_{0,n,\ell}$ independent of j. This proves (3.34) for $m = 0$. Recall that $B_{0,j}(t) := \Pi_{0,j}(t)\, \partial_t \Pi_{0,j}(t)$. Then by Leibnitz rule and (3.34) it follows immediately (3.20) for $m = 0$.

Step $m \rightsquigarrow m+1$**.** Assume that we performed already m steps, with $m < \mathtt{M}$. Then we constructed the operators $B_i(t) = \sum_j B_{i,j}(t)\ \forall\, 1 \leq i \leq m$. In order to construct $B_{m+1}(t)$, we need the spectral projectors of the operator $H_{m+1}(t) \equiv H + V(t) + B_0(t) + \cdots + B_m(t)$ (see formula (3.19)). By Corollary 3.7 $H_{m+1}(t)$ fulfills $\widetilde{(\mathrm{Hgap})}$ provided $\mathtt{J}$ is sufficiently large. Therefore we can apply Lemma B.2 and obtain that $H_{m+1}(t)$ fulfills $\widetilde{(\mathrm{Hgap})}$ and that the projectors

$$\Pi_{m+1,j}(t) = -\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_j} R_{m+1}(t,\lambda)\, d\lambda \ , \qquad R_{m+1}(t,\lambda) := (H_{m+1}(t) - \lambda)^{-1}$$

are well defined $\forall j \geq 1$ and fulfill

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^{\ell+1}\Pi_{m+1}(t)\ H^{-p}\| \leq \frac{C_{m+1,n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta}}\ , \qquad \forall 0 \leq p \leq n,\ \ell \geq 0\ . \tag{3.36}$$

This proves the first of (3.34) for $m+1$.

We pass to estimate $\partial_t^\ell\, \partial_{(t,L)}\Pi_{m+1}(t)$. Using the definition of H_{m+1} we get

$$\partial_{(t,L)}\Pi_{m+1,j}(t) = \partial_t\Pi_{m+1,j}(t) - \mathrm{i}\sum_{l=0}^{m}[B_l(t),\, \Pi_{m+1,j}(t) - \Pi_{l,j}(t)] - \mathrm{i}\sum_{l=0}^{m}[B_l(t), \Pi_{l,j}(t)]\ . \tag{3.37}$$

Consider the last term in the r.h.s. above. Note that $\partial := \partial_{(t,L)}$ is a derivative in the algebra $\mathcal{L}(\mathcal{H}^0)$. So for any projector Π we have $\Pi\, \partial\Pi = \partial\Pi - \partial\Pi\ \Pi$. Using the definition of B_l and the properties of the projectors, one gets the identity

$$[B_l(t), \Pi_{l,j}(t)] = -\mathrm{i}\partial_{(t,L)}\Pi_{l,j}(t)\ .$$

Therefore using the inductive estimates (3.21) and (3.36) we get

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell[B_l(t), \Pi_{l,j}(t)]H^{-p}\| \leq \frac{C'_{l,n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta}}\ , \quad \forall\ell \geq 0,\ 0 \leq p \leq n\ . \tag{3.38}$$

Consider now the term in the middle of (3.37). To estimate it, remark that

$$\Pi_{m+1,j}(t) - \Pi_{l,j}(t) = -\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_j} R_{m+1}(t,\lambda)\,(H_{m+1}(t) - H_l(t))\,R_l(t,\lambda)\,d\lambda\ . \tag{3.39}$$

As $H_{m+1}(t) - H_l(t) = \sum_{k=l}^{m} B_k(t)$, by (3.21)

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell\,(H_{m+1}(t) - H_l(t))\ \ H^{-p}\| \leq \widetilde{C}_{m,n,\ell}, \quad \forall\ell \geq 0,\ 0 \leq p \leq n\ ,$$

thus we can apply Lemma B.3 and get that

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell\,(\Pi_{m+1,j}(t) - \Pi_{l,j}(t))\ \ H^{-p}\| \leq \frac{\widetilde{C}_{m,n,\ell}}{\widetilde{\Delta}_{j-1}}\ , \quad \forall\ell \geq 0,\ 0 \leq p \leq n\ . \tag{3.40}$$

Therefore by Leibnitz rule and estimates (3.21), (3.40) we find that

$$\sup_{t\in\mathbb{R}} \|H^p \left(\partial_t^\ell \sum_{l=0}^{m}[B_l(t),\, \Pi_{m+1,j}(t) - \Pi_{l,j}(t)]\right)\ H^{-p}\| \leq \frac{C'_{m,n,\ell}}{\widetilde{\Delta}_{j-1}}\ , \quad \forall\ell \geq 0,\ 0 \leq p \leq n\ . \tag{3.41}$$

We come back to the estimate of $\partial_{(t,L)}\Pi_{m+1,j}(t)$. Using (3.36), (3.38) and (3.41) we get

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell\, \partial_{(t,L)}\Pi_{m+1,j}(t)\ H^{-p}\| \leq \frac{C_{m+1,n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta}}\ , \quad \forall 0 \leq p \leq n\ , \ell \geq 0\ ,$$

proving the inductive estimate (3.34).

We define now a series of objects and in the next lemma we give the estimates. Let

$$L_{m,j}(t) := \Pi_{m+1,j}(t) - \Pi_{m,j}(t) \tag{3.42}$$

$$K_{m,j}(t) := \Pi_{m,j}(t)\, L_{m,j}(t) \tag{3.43}$$

$$D_{m,j}(t) := \Pi_{m+1,j}(t)\, L_{m,j}(t) \tag{3.44}$$

Lemma 3.12. *For every integers $\ell \geq 0$, $0 \leq p \leq n$, $0 \leq m < \mathtt{M}$, provided $\mathtt{J}$ is sufficiently large, one has*

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell L_{m,j}(t)\ H^{-p}\| \leq \frac{c_{m,n,\ell}}{\widetilde{\Delta}_{j-1}}\ , \tag{3.45}$$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell K_{m,j}(t)\ H^{-p}\| \leq \frac{\widetilde{c}_{m,n,\ell}}{\widetilde{\Delta}_{j-1}^{(m+1)\delta+1}}\ , \tag{3.46}$$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell D_{m,j}(t)\ H^{-p}\| \leq \frac{\widehat{c}_{m,n,\ell}}{\widetilde{\Delta}_{j-1}^{(m+1)\delta+1}}\ . \tag{3.47}$$

Proof. First we prove (3.45). One has

$$L_{m,j}(t) = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} R_{m+1}(t,\lambda)\ B_m(t)\ R_m(t,\lambda)\ d\lambda\ . \tag{3.48}$$

We apply Lemma B.3 with $P = V + B_0 + \cdots + B_m$, $Q = V + B_0 + \cdots + B_{m-1}$, $B = B_m$, and get

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell L_{m,j}(t)\ H^{-p}\| \leq \frac{c_{m,n,\ell}}{\widetilde{\Delta}_{j-1}}\ , \quad \forall \ell \geq 0,\ 0 \leq p \leq n\ .$$

For later use we study the operator $(1 - L_{m,j}(t))^{-1}$. Provided $m < \mathtt{M}$ and $\mathtt{J}$ is sufficiently large, estimate (3.45) with $\ell = 0$ guarantees that $(1 - L_{m,j}(t))$ is invertible by Neumann series in H^p and

$$\sup_{t\in\mathbb{R}} \|H^p\ (1 - L_{m,j}(t))^{-1}\ H^{-p}\| \leq 2\ , \quad \forall 0 \leq p \leq n\ .$$

To study its derivatives we can proceed as in (B.8), and get

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell (1 - L_{m,j}(t))^{-1}\ H^{-p}\| \leq \frac{\widetilde{c}'_{m,n,\ell}}{\widetilde{\Delta}_{j-1}}\ , \quad \forall 0 \leq p \leq n\ , \ell \geq 1\ , \tag{3.49}$$

provided J is sufficiently large.

Estimate (3.46) follows immediately from the identity

$$\begin{aligned} K_{m,j}(t) &= -\Pi_{m,j}(t) \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} R_m(t,\lambda)\ B_m(t)\ R_{m+1}(t,\lambda)\ d\lambda \\ &= -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} R_m(t,\lambda)\ B_{m,j}(t)\ R_{m+1}(t,\lambda)\ d\lambda \end{aligned} \tag{3.50}$$

and the application of Lemma B.3 with $B = B_{m,j}$, using the bound

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell B_{m,j}(t)\ H^{-p}\| \leq \frac{c_{m,n,\ell}}{\widetilde{\Delta}_{j-1}^{(m+1)\delta}}\ , \quad \forall \ell \geq 0,\ 0 \leq p \leq n\ ,$$

which follows from the inductive assumption.

Finally we prove (3.47). Using $\Pi_{m+1,j}^2 = \Pi_{m+1,j}$ and simple algebraic manipulations one proves that [29, (2.41)]

$$D_{m,j}(t) = \Pi_{m+1,j}(t)\, K_{m,j}(t)\, (1 - L_{m,j}(t))^{-1}\ . \tag{3.51}$$

Then Leibnitz rule, (3.36), (3.46), (3.49) give the claimed estimate. □

We can now conclude the proof by calculating the norm of $B_{m+1,j}$. One has the formula [29, (2.42)]

$$B_{m+1,j}(t) = \mathrm{i} D_{m,j}(t)\, \partial_{(t,L)} \Pi_{m+1,j}(t) + \Pi_{m+1,j}(t)\, \partial_{(t,H-H_{m+1})} K_{m,j}(t)\ . \tag{3.52}$$

Consider first the term $D_{m,j}(t)\partial_{(t,L)}\Pi_{m+1,j}(t)$. Then (3.47) and the inductive assumption give

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell (D_{m,j}(t)\, \partial_{(t,L)} \Pi_{m+1,j}(t))\ H^{-p}\| \leq \frac{c_{m,n,\ell}}{\widetilde{\Delta}_{j-1}^{(m+2)\delta+1}}\ , \qquad \forall 0 \leq p \leq n,\ \ell \geq 0\ . \tag{3.53}$$

To estimate $\partial_{(t,H-H_{m+1})} K_{m,j}$ use that $H - H_{m+1} = -\sum_{i=0}^m B_i$, so that

$$\partial_{(t,H-H_{m+1})} K_{m,j} = \partial_t K_{m,j} - \mathrm{i} \sum_{0\leq i\leq m} [B_i, K_{m,j}]\ .$$

Using once again (3.46) and the inductive assumption we get

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell \partial_{(t,H-H_{m+1})} K_{m,j}(t)\ H^{-p}\| \leq \frac{c_{m,n,\ell}}{\widetilde{\Delta}_{j-1}^{(m+1)\delta+1}}\ , \qquad \forall 0\leq p\leq n,\ \ell\geq 0\ . \tag{3.54}$$

Then (3.53), (3.54) and $0<\delta\leq 1$ give

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell B_{m+1,j}(t)\ H^{-p}\| \leq \frac{\widetilde{c}_{m,n,\ell}}{\widetilde{\Delta}_{j-1}^{(m+1)\delta+1}} \leq \frac{C_{m+1,n,\ell}}{\widetilde{\Delta}_{j-1}^{(m+2)\delta}}\ , \qquad \forall 0\leq p\leq n,\ \ell\geq 0$$

thus proving the inductive step.

The estimate on $B_m(t)$ is trivial. The self-adjointness can be proved using the arguments of [29, Lemma 2].

The proof of Theorem 1.8 follows by exactly the same arguments, but it is easier. In fact it is a simple adaptation of the methods of Nenciu [29], so we just sketch the proof.

Proof of Theorem 1.8. In case $V(t)$ fulfills $(\mathrm{Vc})_n$, it is sufficient to note that in the adiabatic algorithm described above one can perform at most $\mathtt{M}=r-1$ steps. Indeed in this case one verifies, exactly as in [29], that $B_{m,j}\in C_b^{r-m-1}(\mathbb{R},\mathcal{L}(\mathcal{H}^k))$ and estimates (3.20), (3.21) hold for $0\leq\ell\leq r-m-1$. Since at every step of the adiabatic iteration we lose regularity in time, we can perform at most $\mathtt{M}=r-1$ steps. Thus estimate (3.33) with $\mathtt{M}=r-1$ gives

$$\|\mathcal{U}(t,s)\psi_s\|_{2n}^2 \leq C\ \langle t-s\rangle^{\frac{2(\mu+1)n}{r\mu\delta}}\ \|\psi_s\|_{2n}^2 \leq C\ \langle t-s\rangle^{\frac{2n}{r}\left(\frac{\mu}{\mu+1}-\nu\right)^{-1}}\ \|\psi_s\|_{2n}^2\ ,$$

which implies the desidered estimate (1.13). □

4. Growth of norms for perturbations analytic in time

In this section we prove the upper bound on the growth of the norm in case of perturbations which are analytic in time. The proof is essentially the same as in case of perturbations smooth in time, but we need extra attention to compute the dependence of all the constants from the parameters $\mathtt{J}$ and $\mathtt{M}$. Indeed in this case we want to optimize $\mathtt{J}$ and $\mathtt{M}$ by choosing them as a function of $t-s$, so we need to know exactly how all the constants depend on such parameters.

Notice that perturbations analytic in time were considered in [21,28].

First rewrite assumption $(\mathrm{Va})_n$ in the following way: there exist $\mathtt{a}, c, A>0$ such that for any integer $\ell\geq 0$, $0\leq p\leq n$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell V(t)\ H^{-p-\nu}\| \leq \mathtt{a}\ \frac{c^\ell\ \ell!}{A\ (1+\ell)^2}\ . \tag{4.1}$$

Here A is a constant such that

$$(1+\ell)^2 \sum_{n_1+\cdots+n_k=\ell} \frac{1}{(1+n_1)^2}\cdots\frac{1}{(1+n_k)^2} \leq A^{k-1}\ , \tag{4.2}$$

and can be chosen to be $A\geq 2\pi^2/3$.

Note that (4.1) can always be achieved simply by choosing $A\,c_{0,n} \leq \mathtt{a}$, $4\,c_{1,n} \leq c$. The next step is to extend Lemma 3.5 in the analytic setting. Define

$$\mathtt{d} := \frac{2^{\mu\delta}}{2^{\mu\delta}-1}$$

and

$$\mathtt{C}_H := \max(C_H,\ \widetilde{C}_H)\ ,$$

where C_H and $\widetilde{C}_H$ are the constants of Lemma 3.2 respectively Lemma 3.4.

Finally we fix a time $T \gg 1$. We obtain the following

Proposition 4.1. *Fix a positive $\mathtt{M} \in \mathbb{N}$ and choose $\mathtt{J}$ such that*

$$2^{12}\ [\mathtt{C}_H\,(\mathtt{a}+2\mathtt{d})c]\,\mathtt{M} \leq 2^{\mathtt{J}\mu\delta}\ . \tag{4.3}$$

For every integers m, ℓ, p such that

$$0 \leq \ell + m \leq \mathtt{M}\ , \quad 0 \leq p \leq n$$

the following holds true:

(i) The operators $\Pi_{m,j}$ fulfill for every $j \geq 1$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell \Pi_{m,j}(t)\ H^{-p}\| \leq \frac{\ell!\ c^\ell}{(1+\ell)^2}\left(\frac{1}{A\,2^{(j-1)\mu\delta}}\right)^{\min(\ell,1)}\ , \tag{4.4}$$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell \sum_{k=0}^{m} \partial_{(t,L)}\Pi_{k,j}(t)\ H^{-p}\| \leq \frac{\ell!\ c^\ell}{A(1+\ell)^2}\ \frac{1}{2^{(j-1)\mu\delta}}\ \frac{1}{4}\ , \tag{4.5}$$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell \partial_{(t,L)}\Pi_{m,j}(t)\ H^{-p}\| \leq \frac{\ell!\ c^\ell}{A(1+\ell)^2}\,\frac{1}{2^{(j-1)\mu\delta}}\ . \tag{4.6}$$

(ii) The operators $B_{m,j}$ fulfill for every $j \geq 1$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell B_{m,j}(t)\ H^{-p}\| \leq \frac{\ell!\,c^\ell}{A(1+\ell)^2}\,\frac{1}{2^{(j-1)(m+1)\mu\delta}}\,\frac{1}{(1+m)^2}\ . \tag{4.7}$$

(iii) The operators $B_m(t)$ fulfill

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell B_m(t)\ H^{-p}\| \leq \frac{\ell!\,c^\ell}{A(1+\ell)^2}\,\frac{\mathtt{d}}{(1+m)^2}\ . \tag{4.8}$$

(iv) The Hamiltonians $H_m(t)$, $H_{ad,m}(t)$ fulfill $\widetilde{(\mathrm{Hgap})}$ with $\widetilde{\sigma}_j$'s as in (3.10).

Before proving Proposition 4.1, we show how Theorem 1.10 follows.

Proof of Theorem 1.10. Having fixed $T \gg 1$, we consider the evolution $\mathcal{U}(t,0)$ on a time interval $0 \le t \le T$. Choose $\mathtt{J}$ in such a way that (4.3) is fulfilled, namely

$$\mathtt{J} := \frac{1}{\mu\delta}\log(\mathtt{M}+1) + \frac{1}{\mu\delta}\log\left(2^{12}\ [\mathtt{C}_H\,(\mathtt{a}+2\mathtt{d})\,c]\right)\ , \tag{4.9}$$

and choose $\mathtt{M}$ as a function of T:

$$\mathtt{M}+1 = \lfloor \frac{1}{4}\log\langle T\rangle \rfloor\ , \tag{4.10}$$

where $\lfloor\cdot\rfloor$ denotes the integer part.

Now remark that the constants c_1, c_2 of Lemma 3.8 *do not* depend on $\mathtt{M}$ and $\mathtt{J}$. Indeed they depend only on $\sigma(H)$, n, and the norm of $H^{-n}(V(t)+\sum_{i=0}^{\mathtt{M}} B_i(t))H^n$. But by (4.8) it follows easily that such norm depends only on $\mathtt{a}$, $\mathtt{d}$ (see (4.16) below for the precise computation). Hence we can repeat the arguments of the proof of Theorem 1.9 and using estimate (4.7) to estimate $B_{m,j}$, one gets

$$\sup_{t\in[0,T]} \|\mathcal{U}(t,0)\psi_0\|_{2n}^2 \le C\, 2^{\mathtt{J}(\mu+1)2n} 2^{\frac{2(\mu+1)n}{(\mathtt{M}+1)\mu\delta}\log\langle T\rangle}\ \|\psi_0\|_{2n}^2\ , \tag{4.11}$$

where the constant C *does not* depend on $\mathtt{J}$ and $\mathtt{M}$. Now substitute $\mathtt{J}$ as in (4.9) and $\mathtt{M}$ as (4.10) to get

$$\sup_{t\in[0,T]} \|\ \mathcal{U}(t,0)\psi_0\|_{2n}^2 \le \gamma\,(\log\langle T\rangle)^{2n(\mu+1)/\mu\delta}\ \|\psi_0\|_{2n}^2\ , \tag{4.12}$$

for some $\gamma > 0$ which does not depend on T. Since T was arbitrary, the estimate above holds $\forall t$. It is easy to adapt the proof to consider also the case $\mathcal{U}(t,s)$.

Finally interpolating with $k = 0$ gives the general case. The exponent in (1.17) is obtained by simply replacing δ with its definition (3.8). □

4.1. Proof of Proposition 4.1

Step $m = 0$. Define $\Pi_{0,j}(t)$ as in (3.35). We apply Lemma C.2 with $P = V$, $a = \mathtt{a}$, $b = 1$ (it is easy to see that (C.4) is fulfilled) and get that for every $\ell \ge 0$, $0 \le p \le n$

$$\begin{aligned}\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell \Pi_{0,j}(t)\ H^{-p}\| &\le \frac{\ell!\ c^\ell}{(1+\ell)^2}\left(\frac{2^4\,\mathtt{C}_H\,\mathtt{a}}{A\,\widetilde{\Delta}_{j-1}^{\delta}}\right)^{\min(\ell,1)} \\ &\le \frac{\ell!\ c^\ell}{(1+\ell)^2}\left(\frac{1}{A\,2^{(j-1)\mu\delta}}\right)^{\min(\ell,1)}.\end{aligned} \tag{4.13}$$

Consider now $\partial_{(t,L)}\Pi_{0,j}(t) \equiv \partial_t \Pi_{0,j}(t)$. For $\ell+1 \le \mathtt{M}$ one has by (4.3), (4.13)

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell \partial_{(t,L)} \Pi_{0,j}(t)\ H^{-p}\| \le \frac{(\ell+1)!\ c^{\ell+1}}{(1+\ell)^2}\ \frac{2^4\,\mathtt{C}_H\,\mathtt{a}}{A\,\widetilde{\Delta}_{j-1}^\delta} \le \frac{\ell!\ c^\ell}{A(1+\ell)^2}\,\frac{1}{2^{(j-1)\mu\delta}}\,\frac{1}{4}\ . \tag{4.14}$$

Thus we proved (4.4), (4.5) and (4.6) for $m=0$. Consider now $B_{0,j}(t) = \Pi_{0,j}(t)\,\partial_t\Pi_{0,j}(t)$. We apply Lemma C.1 with $P = \Pi_{0,j}$, $a = 1, b = \frac{2^4\,\mathtt{C}_H\,\mathtt{a}}{A\,\widetilde{\Delta}_{j-1}^\delta}$, $k = 0$ and $Q = \partial_t\Pi_{0,j}$, $d = \frac{2^4\,\mathtt{C}_H\,\mathtt{a}}{A\,\widetilde{\Delta}_{j-1}^\delta}$, $f = 1$, $i = 1$ and obtain that for $\ell + 1 \le \mathtt{M}$

$$\begin{aligned}\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell B_{0,j}(t)\ H^{-p}\| &\le \frac{2^4\,\mathtt{C}_H\,\mathtt{a}}{A\,\widetilde{\Delta}_{j-1}^\delta}\left(\frac{2^5\,\mathtt{C}_H\,\mathtt{a}}{A\,\widetilde{\Delta}_{j-1}^\delta}+1\right)^{\min(\ell,1)}\ \frac{(\ell+1)!\,c^{\ell+1}}{A(1+\ell)^2}\\ &\le \frac{2^5\,\mathtt{C}_H\,\mathtt{a}}{\widetilde{\Delta}_{j-1}^\delta}\,\frac{(\ell+1)!\,c^{\ell+1}}{A(1+\ell)^2}\\ &\le \frac{\ell!\,c^\ell}{A(1+\ell)^2}\,\frac{1}{2^{\mu(j-1)\delta}}\end{aligned}$$

provided

$$\max\left(\frac{2^5\,\mathtt{C}_H\,\mathtt{a}}{A},\ 2^5\,\mathtt{C}_H\,\mathtt{a}\,\mathtt{M}\,c\right) \le 2^{\mathtt{J}\mu\delta}\ , \tag{4.15}$$

which is clearly fulfilled using (4.3). This proves (4.7) for $m = 0$.

Step $m \rightsquigarrow m+1$**.** Assume that we performed already $0 < m < \mathtt{M}$ steps. By the inductive assumption $\forall\, 0 \le i \le m$ one has that $B_i(t) = \sum_j B_{i,j}(t)$ fulfills (4.8) $\forall \ell + i + 1 \le \mathtt{M}$.

Thus $H_{m+1}(t) = H + W(t)$, where we defined $W(t) := V(t) + \sum_{i=0}^m B_i(t)$. It fulfills $\forall \ell + m + 1 \le \mathtt{M}$, $\forall 0 \le p \le n$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell W(t)\ H^{-p-\nu}\| \le \frac{\ell!\ c^\ell}{A(1+\ell)^2}\left(\mathtt{a} + \mathtt{d}\sum_{i=0}^m \frac{1}{(1+i)^2}\right) \le \frac{\ell!\ c^\ell}{A(1+\ell)^2}\ (\mathtt{a} + 2\mathtt{d})\ . \tag{4.16}$$

Thus $W(t)$ is a perturbation of H analytic in time which fulfills the conditions of Lemma C.2. Indeed with $a = \mathtt{a} + 2\mathtt{d}$ we have that $2^4\,\mathtt{C}_H\,a \le 2^{\mathtt{J}\mu\delta}$, hence by Lemma C.2 the projectors

$$\Pi_{m+1,j}(t) = -\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_j} R_{m+1}(t,\lambda)\ d\lambda\ , \qquad R_{m+1}(t,\lambda) := (H_{m+1}(t) - \lambda)^{-1}$$

are well defined $\forall j \ge 1$. Furthermore they fulfill $\forall \ell + 1 + m \le \mathtt{M}$, $\forall 0 \le p \le n$

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^\ell \Pi_{m+1,j}(t) H^{-p}\| \leq \frac{\ell!\; c^\ell}{(1+\ell)^2} \left(\frac{2^4\, \mathtt{C}_H}{A \widetilde{\Delta}_{j-1}^{\delta}} \, (\mathtt{a} + 2\mathtt{d}) \right)^{\min(\ell,1)} , \tag{4.17}$$

where we used (4.16) and Lemma C.2.

To estimate $\partial_t^\ell\, \partial_{(t,L)} \Pi_{m+1,j}(t)$ we use again formula (3.37). Consider its last term. Since $[B_l(t), \Pi_{l,j}(t)] = -\mathrm{i}\partial_{(t,L)}\Pi_{l,j}(t)$, one gets the identity

$$\sum_{l=0}^{m+1} \partial_{(t,L)}\Pi_{l,j}(t) = \partial_t \Pi_{m+1,j}(t) - \mathrm{i} \sum_{l=0}^{m} [B_l(t),\, \Pi_{m+1,j}(t) - \Pi_{l,j}(t)] \; . \tag{4.18}$$

This identity allows us to estimate (4.5) at step $m+1$. We estimate the two terms in the r.h.s. above separately. To estimate the second one we use formula (3.39). Since

$$\sup_{t\in[0,T]} \|H^p\, \partial_t^\ell \left(H_{m+1}(t) - H_l(t) \right)\; H^{-p}\| \leq \frac{\ell!\, c^\ell}{A(1+\ell)^2}\, (\mathtt{a} + 2\mathtt{d}) \;,$$

by Lemma C.3 we get that $\forall \ell + m + 1 \leq \mathtt{M}$, $\forall 0 \leq p \leq n$,

$$\sup_{t\in[0,T]} \|H^p\, \partial_t^\ell \left(\Pi_{m+1,j}(t) - \Pi_{l,j}(t) \right)\; H^{-p}\| \leq \frac{\ell!\, c^\ell}{A(1+\ell)^2}\, \frac{(\mathtt{a} + 2\mathtt{d})\; 2^5}{\widetilde{\Delta}_{j-1}} \; . \tag{4.19}$$

Hence by Lemma C.1, (4.8), (4.19) we get

$$\sup_{t\in[0,T]} \|H^p\, \sum_{l=0}^{m} [B_l(t),\, \Pi_{m+1,j}(t) - \Pi_{l,j}(t)]\; H^{-p}\| \leq \frac{\ell!\, c^\ell}{A(1+\ell)^2}\, \frac{(\mathtt{a} + 2\mathtt{d})\; \mathtt{d}\, 2^8}{\widetilde{\Delta}_{j-1}} \; . \tag{4.20}$$

The first term of (4.18) is estimated by (4.17) with $\ell+1$ replacing ℓ. Together with (4.20) we get for $\ell + 1 + m \leq \mathtt{M}$, $\forall 0 \leq p \leq n$

$$\sup_{t\in[0,T]} \|H^p\, \partial_t^\ell \sum_{l=0}^{m+1} \partial_{(t,L)}\Pi_{l,j}(t)\; H^{-p}\| \leq \frac{\ell!\; c^\ell}{A(1+\ell)^2}\, \frac{1}{2^{(j-1)\mu\delta}}\, \frac{1}{4} \tag{4.21}$$

using (4.3). This proves (4.5) at step $m+1$.

Now consider $\partial_{(t,L)}\Pi_{m+1,j}(t)$. Using (4.21) and the inductive assumption (4.5) we get

$$\begin{aligned} \|H^p\, \partial_t^\ell\, \partial_{(t,L)}\Pi_{m+1,j}(t)\; H^{-p}\| &\leq \|H^p\, \partial_t^\ell \sum_{k=0}^{m+1} \partial_{(t,L)}\Pi_{k,j}(t)\; H^{-p}\| \\ &\quad + \|H^p\, \partial_t^\ell \sum_{k=0}^{m} \partial_{(t,L)}\Pi_{k,j}(t)\; H^{-p}\| \\ &\leq \frac{\ell!\; c^\ell}{A(1+\ell)^2}\, \frac{1}{2^{(j-1)\mu\delta}} \end{aligned} \tag{4.22}$$

proving (4.5) at step $m+1$.

Next we estimate $L_{m,j}$, $K_{m,j}$, $D_{m,j}$ defined in (3.42)–(3.44).

Lemma 4.2. *For every* $0 \le p \le n$, $\ell + m + 1 \le \mathtt{M}$ *one has*

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell L_{m,j}(t)\ H^{-p}\| \le \frac{\ell!\,c^\ell}{A(1+\ell)^2}\frac{2^5\,\mathtt{d}}{\widetilde{\Delta}_{j-1}} \tag{4.23}$$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell K_{m,j}(t)\ H^{-p}\| \le \frac{\ell!\,c^\ell}{A(1+\ell)^2}\frac{2^5}{\widetilde{\Delta}_{j-1}}\frac{1}{2^{\mu(j-1)(m+1)\delta}}\frac{1}{(1+m)^2} \tag{4.24}$$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell D_{m,j}(t)\ H^{-p}\| \le \frac{\ell!\,c^\ell}{A(1+\ell)^2}\frac{2^8}{2^{\mu(j-1)(m+1)\delta}\,\widetilde{\Delta}_{j-1}}\frac{1}{(1+m)^2} \tag{4.25}$$

Proof. First we prove (4.23). Using the definition of $L_{m,j}$ given by (3.48), we apply Lemma C.3 with $P = V + B_0 + \cdots + B_m$, $Q = V + B_0 + \cdots + B_{m-1}$, $B = B_m$, $h = \mathtt{d}/(1+m)^2$ and get that for $\ell + 1 + m \le \mathtt{M}$, $\forall 0 \le p \le n$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell L_{m,j}(t)\ H^{-p}\| \le \frac{\ell!\,c^\ell}{A(1+\ell)^2}\frac{2^5\,\mathtt{d}}{\widetilde{\Delta}_{j-1}}\frac{1}{(1+m)^2},$$

provided $2^4\mathtt{C}_H(\mathtt{a}+2\mathtt{d}) \le 2^{\mathtt{J}\mu\delta}$. For later use consider the operator $(1 - L_{m,j}(t))^{-1}$. Provided $\frac{2^5}{A\,\widetilde{\Delta}_{j-1}}\mathtt{d} \le \frac{1}{2}$, the operator $(1 - L_{m,j}(t))$ is invertible by Neumann series and

$$\sup_{t\in[0,T]} \|(1 - L_{m,j}(t))^{-1}\| \le 2\ .$$

To study its derivatives we can proceed as in (B.8), (C.7) to get for $\ell + m + 1 \le \mathtt{M}$, $\ell \ge 1$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell (1 - L_{m,j}(t))^{-1}\ H^{-p}\| \le \frac{\ell!\,c^\ell}{A(1+\ell)^2} 2\sum_{k=1}^{\ell}\left(\frac{2^6\,\mathtt{d}}{\widetilde{\Delta}_{j-1}}\right)^k \le \frac{\ell!\,c^\ell}{A(1+\ell)^2}\frac{\mathtt{d}\,2^8}{\widetilde{\Delta}_{j-1}} \tag{4.26}$$

provided $2^7\,\mathtt{d} \le \widetilde{\Delta}_{j-1}$. Thus for $\ell + m + 1 \le \mathtt{M}$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell (1 - L_{m,j}(t))^{-1}\ H^{-p}\| \le \frac{\ell!\,c^\ell}{(1+\ell)^2}\, 2\left(\frac{2^7\,\mathtt{d}}{A\,\widetilde{\Delta}_{j-1}}\right)^{\min(\ell,1)} . \tag{4.27}$$

Estimate (4.24) follows immediately from the identity (3.50) and Lemma C.3 with $B = B_{m,j}$, $h = 2^{-\mu(j-1)(m+1)\delta}\,(1+m)^{-2}$.

Finally we prove (4.25). Consider formula (3.51). Then Lemma C.1 applied twice and (4.17), (4.24), (4.27) give $\forall \ell + 1 + m \le \mathtt{M}$

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell D_{m,j}(t)\ H^{-p}\| \le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{2^8}{2^{\mu(j-1)(m+1)\delta}\,\widetilde{\Delta}_{j-1}} \frac{1}{(1+m)^2}\ , \quad \forall 0\le p\le n\ , \tag{4.28}$$

where we used that (4.3) implies $\max\left[\frac{2^8\,\mathtt{d}}{A\,\widetilde{\Delta}_{j-1}}\ , \frac{2^4\,\mathtt{c}_H}{A\widetilde{\Delta}_{j-1}^{\delta}}(\mathtt{a}+\mathtt{d})\right] \le 1.$ □

We can now conclude the proof by calculating the norm of $B_{m+1,j}$. We use again (3.52). Consider first the term $D_{m,j}\partial_{(t,L)}\Pi_{m+1,j}$. We can compute its first ℓ derivatives provided $\ell+1+m \le \mathtt{M}$. We apply once again Lemma C.1 with $P = D_{m,j}$, $k = 0$ and $Q = \partial_{(t,L)}\Pi_{m+1,j}$, $i = 0$, and use estimates (4.22), (4.25) to get

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell(D_{m,j}(t)\,\partial_{(t,L)}\Pi_{m+1,j}(t))\ H^{-p}\| \le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{2^{10}}{2^{(j-1)\mu(m+2)\delta}\,\widetilde{\Delta}_{j-1}\,(1+m)^2}\ . \tag{4.29}$$

The other term to estimate is $\Pi_{m+1,j}\partial_{(t,H-H_{m+1})}K_{m,j}$. To estimate $\partial_{(t,H-H_{m+1})}K_{m,j}$ write $\partial_{(t,H-H_{m+1})}K_{m,j} = \partial_t K_{m,j} - \sum_{0\le i\le m}[B_i, K_{m,j}]$. By (4.24)

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell\partial_t K_{m,j}(t)H^{-p}\| \le \frac{(1+\ell)!\, c^{\ell+1}}{A(1+\ell)^2} \frac{1}{2^{\mu(j-1)(m+1)\delta}} \frac{2^5}{\widetilde{\Delta}_{j-1}} \frac{1}{(1+m)^2}\ . \tag{4.30}$$

Consider now $\sum_{i=0}^m [B_i, K_{m,j}]$. For $\ell+1+m \le \mathtt{M}$ we have that Lemma C.1, estimates (4.8) and (4.24) imply that

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell \sum_{i=0}^m [B_i(t), K_{m,j}(t)]\ H^{-p}\| \le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{1}{2^{\mu(j-1)(m+1)\delta}} \frac{2^7\,\mathtt{d}}{\widetilde{\Delta}_{j-1}} \frac{1}{(1+m)^2}\ . \tag{4.31}$$

Then (4.30) and (4.31) imply that, for $\ell+1+m \le \mathtt{M}$,

$$\sup_{t\in[0,T]} \|H^p\ \partial_t^\ell\Big(\Pi_{m+1,j}(t)\partial_{(t,H-H_{m+1})}K_{m,j}(t)\Big)\ H^{-p}\| \le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{1}{2^{\mu(j-1)(m+1)\delta}} \frac{2^5}{\widetilde{\Delta}_{j-1}} \frac{1}{(1+m)^2} (\mathtt{M}c + 4\mathtt{d})\ . \tag{4.32}$$

Then (4.29) and (4.32) give

$$\begin{aligned}
&\sup_{t\in[0,T]} \|H^p\partial_t^\ell B_{m+1,j}(t)\ H^{-p}\| \\
&\le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{1}{2^{\mu(j-1)(m+1)\delta}} \frac{2^5}{\widetilde{\Delta}_{j-1}} \frac{1}{(1+m)^2} \left[\frac{2^5}{2^{(j-1)\mu\delta}} + \mathtt{M}c + 4\mathtt{d}\right] \\
&\le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{1}{2^{\mu(j-1)(m+2)\delta}} \frac{1}{(2+m)^2}
\end{aligned}$$

where we used that

$$2^5 \frac{(m+2)^2}{(m+1)^2}[2^5 + \mathtt{M}c + 4\mathtt{d}] \leq 2^{12}\, c\, \mathtt{d}\, \mathtt{M} \leq 2^{\mathtt{J}\mu\delta} \ . \tag{4.33}$$

The inductive step is proved.

5. Applications

In this section we apply our abstract theorems to different models. We are able to recover many already known results and to prove new estimates.

5.1. One degree of freedom Schrödinger operators

Let us consider here equation (1.1) where $L(t)$ is a time dependent perturbation of the anharmonic oscillator, namely

$$L(t) = -\frac{d^2}{dx^2} + x^{2k} + p(x) + V(t,x) = H_k + V(t,x), \qquad x \in \mathbb{R} \tag{5.1}$$

where $k \in \mathbb{N}$, $p(x)$ is a polynomial of degree less than $2k-1$, and $V(t,x)$ is a real valued time dependent perturbation with a polynomial growth in x of degree $\leq m$ fulfilling $\forall \ell \geq 0$

$$\sup_{t\in\mathbb{R}} |\partial_t^\ell \partial_x^j V(t,x)| \leq C_\ell \, \langle x\rangle^{(m-j)_+} \ , \qquad \forall x \in \mathbb{R} \ , \tag{5.2}$$

where $r_+ := \max(0,r)$ for any $r \geq 0$. Without restriction we can always assume that H_k is positive and invertible. The following lemma is an easy computation

Lemma 5.1. *For every $\mu > 0$ there exists $C_\mu > 0$ such that for every $(j,k) \in \mathbb{N}\times\mathbb{N}$ such that $\frac{j}{2k} + \frac{\ell}{2} \leq \mu$ we have*

$$\| x^j \frac{d^\ell}{dx^\ell} u \|_{L^2(\mathbb{R}^d)} \leq C_\mu \| H_k^\mu u\|_{L^2(\mathbb{R}^d)} \ . \tag{5.3}$$

Under the condition that $m \leq k+1$ we get that the commutator $[V(t,x), H_k]$ is H_k-bounded. By Theorem 1.2 $L(t)$ generates a propagator $\mathcal{U}(t,s)$ in the Hilbert spaces scale $\mathcal{H}_k^r := D((H_k)^{r/2})$. Furthermore if $m < k+1$ then $[V(t,x), H_k]$ is H_k^τ-bounded with $\tau = \frac{m-1+k}{2k} < 1$. Thus Theorem 1.5 can be applied and we get the following polynomial bound for the growth of the $\mathcal{H}_k^r$-norm $\forall r > 0$,

$$\|\mathcal{U}(t,s)\psi_s\|_r \leq C_r \, \langle t-s\rangle^{\frac{kr}{k-m+1}} \, \|\psi_s\|_r \ . \tag{5.4}$$

For $k=1$ and $m=0$ we recover a known bound for time dependent perturbations of the harmonic oscillator:

$$\|\mathcal{U}(t,s)\psi_s\|_r \le C_r \langle t-s\rangle^{\frac{r}{2}} \|\psi_s\|_r \ . \tag{5.5}$$

As mentioned in Remark 1.6, Delort [10] suggests that estimate (5.5) may be sharp for $V(t,x)$ satisfying (5.2) with $m = 0$. Actually the example constructed by him is a zero order pseudo-differential operator. Construct a local potential to saturate the estimate (5.5) is still an open problem.

When $k > 1$ (namely the anharmonic case) we can improve the bound (5.4) by applying Theorem 1.9 and Theorem 1.10. Indeed it is well known (see e.g. [17]) that in this case H_k satisfies (Hgap). Indeed the resolvent of H_k is a compact operator in $L^2(\mathbb{R})$, hence its spectrum is discrete, $\sigma(H_k) = \{\lambda_j\}_{j\ge 1}$ and furthermore it is known to be simple. To verify the gap condition we use the following lemma:

Lemma 5.2. *There exists $c_k > 0$ such that*

$$\lambda_{j+1} - \lambda_j \ge c_k\, j^{\mu_k} \ , \quad \forall j \ge 1 \ ,$$

where $\mu_k = \frac{k-1}{k+1}$.

Proof. It is known that the eigenvalues $\{\lambda_j\}_{j\ge1}$ of H_k are given at all order in j by a Bohr–Sommerfeld rule [17]: one has that

$$\lambda_j^{\frac{k+1}{2k}} = b_k\left(j + \frac{1}{2}\right) + O(1)$$

where b_k is a smooth function such that $b_k(x) = c_0 x + o(x)$. Lemma 5.2 follows easily. □

Lemma 5.2 shows that H_k satisfies (Hgap) defining $\forall j \ge 1$ the clusters $\sigma_j := \{\lambda_j\}$ and $\mu_k = \frac{k-1}{k+1}$.

Consider now the perturbation $V(t,x)$. The critical index to apply Theorem 1.9 is here $\frac{\mu_k}{\mu_k+1} = \frac{k-1}{2k}$. One verifies easily that $V(t,x)$ is $H_k^{\frac{m}{2k}}$-bounded. Hence provided $m < k-1$, we have that $\nu := \frac{m}{2k}$ fulfills $\nu < \frac{\mu_k}{\mu_k+1}$ (such condition appears already in a work by Howland [20] in order to study the Floquet spectrum when $V(t,x)$ is a periodic in time perturbation).

Theorem 5.3 *(Smooth case). Fix an integer $k > 1$ and let $m < k-1$. Assume that V satisfies the estimate (5.2). Then for every $r > 0$, for every $\varepsilon > 0$, there exists a positive $C_{r,\epsilon}$ s.t.*

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \le C_{r,\varepsilon} \langle t-s\rangle^{\varepsilon} \ .$$

Proof. Having fixed $r > 0$, choose an integer n s.t. $r \le 2n$. To apply Theorem 1.9 we have to check that V fulfills assumption $(\mathrm{Vs})_n$. Remark that H_k is a pseudodifferential operator whose symbol is in the class $S^{2k}_{1,k}$ of Definition 5.13, while $V(t)$ belongs to $\widetilde{S}^m_{1,k}$

of Definition 5.12. But under assumption (5.2), $H_k^p \partial_t^\ell V(t) H_k^{-p-\nu}$ is a pseudo-differential operator of order 0 (see the symbolic calculus of Theorem E.1 and Theorem E.2). So applying the Calderon–Vaillancourt theorem (Theorem E.4) we get that $(\mathrm{Vs})_n$ is satisfied. □

Remark 5.4. Under the weaker assumption that V has just $\mathtt{a}$ derivatives uniformly bounded in time,

$$\sup_{t\in\mathbb{R}} |\partial_t^\ell \partial_x^j V(t,x)| \leq C_\ell \langle x \rangle^{(m-j)_+} \ , \qquad \forall x \in \mathbb{R} \ , \quad \forall 0 \leq \ell \leq \mathtt{a} \ , \tag{5.6}$$

we can apply Theorem 1.8 and obtain the estimate

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_{r,\mathtt{a}} \langle t-s \rangle^{\frac{rk}{\mathtt{a}(k-1-m)}} \ .$$

In case $V(t,x)$ is analytic in time, we obtain better estimates:

Theorem 5.5 *(Analytic case). Fix an integer $k > 1$ and let $m < k-1$. Assume that there exist $C_0, C_1 > 1$ such that $\forall \ell, j \geq 0$ we have*

$$\sup_{t\in\mathbb{R}} \| \langle x \rangle^{-(m-j)_+} \partial_t^\ell \partial_x^j V(t,x) \|_{L_x^\infty(\mathbb{R})} \leq C_1 \, C_0^\ell \, \ell! \ . \tag{5.7}$$

Then we have that $\forall r > 0$

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_r \left(\log \langle t-s \rangle\right)^{\frac{rk}{k-1-m}} \ .$$

Proof. We apply Theorem 1.10. Having fixed $r > 0$, we choose an arbitrary integer n with $r \leq 2n$. We check assumption $(\mathrm{Va})_n$ using again the Calderon–Vaillancourt Theorem. □

Comparison with previous results: To the best of our knowledge Theorem 5.3 and Theorem 5.5 are new.

In same cases better estimates on the $\mathcal{H}_k^r$-norm of the flow are known. For example if $V(t,x)$ is a quasi-periodic function of time and small in size, one might try to prove reducibility, which in turn implies that the Sobolev norms are uniformly bounded in time. We mention just the latest results: Bambusi [2,1] proved reducibility for $L(t)$ on $\mathbb{R}$ in several cases, including $k > 1$ and $V(t,x)$ fulfilling (5.2) with $m < k+1$ (in some cases even for $m \leq 2k$). Grébert and Paturel [16] proved reducibility for $L(t)$ on $\mathbb{R}^d$, $d \geq 1$, with $k = 1$ and $V(t,x)$ a small bounded quasi-periodic perturbation.

5.2. Operators on compact manifolds

Let (M,g) be a Riemaniann compact manifold with metric g and let $\triangle_g$ be the Laplace–Beltrami operator. Denote by $S_{cl}^m(M)$ the space of classical symbols of order $m \in \mathbb{R}$ on the cotangent $T^*(M)$ of M (see Hörmander [19] for more details).

Let $H = 1 - \triangle_g$ and $V(t) \equiv V(t, x, D_x)$ be an Hermitian classical pseudodifferential operator of order $m \leq 1$. We want to consider the Schrödinger equation (1.1) with $L(t)$ defined by

$$L(t) = -\triangle_g + 1 + V(t) = H + V(t) \ ,$$

and study its flow in the usual scale of Sobolev spaces $H^k(M) \equiv D(H^{k/2})$.

By semiclassical calculus one verifies that $[L(t), H]H^{-1}$ is a pseudodifferential operator of order 0, hence the assumptions of Theorem 1.2 are satisfied and $L(t)$ has a well defined propagator $\mathcal{U}(t, s)$ in $H^k(M)$ and it is unitary in $L^2(M)$.

Moreover one has that $[L(t), H]H^{-\tau}$, $\tau = \frac{m+1}{2}$ is a pseudodifferential operator of order 0. Provided $m < 1$, one has $\tau < 1$, hence by applying Theorem 1.5 we get for the flow $\mathcal{U}(t, s)$ the following uniform estimate in the space $H^k(M)$:

$$\|\mathcal{U}(t, s)\|_{\mathcal{L}(H^k(M))} \leq C_k \langle t - s\rangle^{\frac{k}{1-m}} \ . \tag{5.8}$$

Better estimates can be obtained if the spectrum of $\triangle_g$ satisfies a gap condition. A typical example is the Laplace–Beltrami operator on Zoll manifolds. We recall that Zoll manifolds are manifolds where all geodesics are closed and have the same period, for examples spheres in any dimension. It is a classical result due to Colin de Verdière [7] that the spectrum of $\sqrt{\triangle}_g$ is concentrated in $\bigcup_{j\geq 1}[j + \sigma - \frac{C}{j}, j + \sigma + \frac{C}{j}]$, where $\sigma \in \mathbb{Z}/4$ and $C > 0$. Defining $\forall j \geq 1$ the cluster $\sigma_j := [(j + \sigma - \frac{C}{j})^2, (j + \sigma + \frac{C}{j})^2]$, one sees immediately that the gap condition is satisfied with $\mu = 1$. Hence H fulfills (Hgap). The critical regularity for V is then $\frac{\mu}{\mu+1} = \frac{1}{2}$.

Theorem 5.6. *Assume that $\forall t \in \mathbb{R}$, $V(t)$ is an Hermitian pseudodifferential operator on M of order $m < 1$. Assume that in local charts its symbol $v(t, x, \xi)$ fulfills the following condition: there exists $C_1 > 0$ s.t. $\forall \ell \geq 0$, for every multi-indices α, β there exists $C_{\alpha\beta} > 0$ such that*

$$\| \langle \xi\rangle^{-m+|\beta|} \partial_x^\alpha \partial_\xi^\beta \partial_t^\ell v(t, x, \xi)\|_{L^\infty(\mathbb{R}_t\times M\times\mathbb{R}^d)} \leq C_{\alpha\beta}\, C_1^\ell\, \ell!. \tag{5.9}$$

Then for any $r > 0$ the propagator $\mathcal{U}(t, s)$ for $H + V(t)$ satisfies

$$\|\mathcal{U}(t, s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_r \left(\log \langle t - s\rangle\right)^{\frac{r}{1-m}} \tag{5.10}$$

Proof. Having fixed $r > 0$, choose an integer n with $r \leq 2n$. We verify that $(\mathrm{Va})_n$ holds. By semiclassical calculus, $V(t)H^{-\frac{m}{2}} \in S^0_{cl}(M)$. For $m < 1$, $\nu := \frac{m}{2}$ is strictly smaller than $\frac{1}{2}$, the critical regularity. To verify that $V(t)$ satisfies $(\mathrm{Va})_n$ it suffices to work in local charts (since M is compact one can considered just a finite number of them). Then by Calderon–Vaillancourt theorem, the norm of $\partial_t^\ell V$ as an operator $H^{n+2\nu}(M) \to H^n(M)$ is controlled by

$$C \sum_{|\alpha|+|\beta|\leq N} \| \langle\xi\rangle^{-m+|\beta|} \partial_x^\alpha \partial_\xi^\beta \partial_t^\ell v(t,x,\xi)\|_{L^\infty(\mathbb{R}_t\times M\times\mathbb{R}^d)}$$

for some universal constants C, N sufficiently large and depending only on n and the dimension of M. Then using (5.9) one verifies that $(\mathrm{Va})_n$ is fulfilled. □

Comparison with previous results: Theorem 5.6 for Zoll manifolds and with unbounded perturbations is a new result.

In case $M = \mathbb{T}$, Theorem 5.6 was proved by Bourgain [5] when $V(t,x)$ is an analytic periodic function in both x and t and extended by Wang [36] for $V(t,x)$ real analytic function with arbitrary dependence on t. Such authors obtained the bound $\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_r (\log\langle t-s\rangle))^{\varsigma r}$, for some constant $\varsigma > 3$. Remark that our Theorem 5.6 improves this estimate: indeed for bounded potentials one can take $m = 0$ in (5.10), leading to the better estimate $\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_r (\log\langle t-s\rangle)^r$.

Later Fang and Zhang [12] extended the results of [36] to the d-dimensional torus $\mathbb{T}^d$, $d > 1$ (such result is not covered by Theorem 5.6 since $-\triangle$ on $\mathbb{T}^d$ does not fulfill (Hgap)).

In case $V(t,x)$ is a smooth function of x and t, the estimate $\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_r \langle t-s\rangle^\epsilon$ was proved by Bourgain [6] for $M = \mathbb{T}^d$, $d \geq 1$, and by Delort when M is a Zoll manifold.

If $V(t)$ is quasi-periodic in time and small in size, some results of reducibility are known. We cite here only the latest results (see their bibliography for more references). In case $M = \mathbb{T}$, Feola and Procesi [14] proved reducibility when $V(t,x)$ is quasi-periodic in time, small in size, and in some class of unbounded operators. In case $M = \mathbb{T}^d$, $d > 1$, Eliasson and Kuksin [11] proved reducibility when $V(t,x)$ is a small analytic potential. For $M = \mathbb{S}^2$ (2-dimensional sphere) reducibility was proved by Corsi, Haus and Procesi [8].

5.3. Time dependent electro-magnetic fields

Consider the Schrödinger equation (1.1) with $L(t) = H_{a,V}(t)$ the time dependent electro-magnetic field

$$H_{a,V}(t) := \frac{1}{2}(D + a(t,x))^2 + V(t,x)\ , \qquad x \in \mathbb{R}^d\ , \quad d \geq 2\ ,$$

where we denoted $D := \mathrm{i}^{-1}\nabla$. Here we assume that the electromagnetic potential $(a(t,x), V(t,x))$ is continuous in $t \in \mathbb{R}$ and smooth in $x \in \mathbb{R}^d$. Furthermore we assume that for every multi-index α we have the following uniform estimate in (t,x):

$$\left|\partial_x^\alpha a(t,x)\right| \leq C_\alpha \langle x\rangle^{(1-|\alpha|)_+}\ , \qquad \left|\partial_x^\alpha V(t,x)\right| \leq C_\alpha \langle x\rangle^{(2-|\alpha|)_+}\ , \qquad \forall t \in \mathbb{R}\ . \quad (5.11)$$

We choose $H = H_{osc}$ where $H_{osc} = \frac{1}{2}\left(-\triangle + |x|^2\right)$ is the harmonic oscillator and define $\forall r \geq 0$ the spaces $\mathcal{H}^r = D(H_{osc}^{r/2})$. By direct computations we can prove that the assumptions of Theorem 1.2 are satisfied. Indeed write first

$$H_{a,V} = -\triangle + V(t,x) + 2a(t,x)\cdot D + \mathrm{i}^{-1}\mathrm{div}(a(t,x)) + a^2(t,x).$$

Denote $\partial_j = \frac{\partial}{\partial x_j}$. Then we get that

$$K := [H_{a,V}, H_{osc}] = \sum_{1\le j,k\le d} \gamma_{j,k}\partial^2_{j,k} + \sum_{1\le j\le d}\gamma_j\partial_j + \gamma_0$$

where for any multi-index α, there exists a $C_\alpha > 0$ s.t. for any $1 \le j,k \le d$

$$|D^\alpha\gamma_{j,k}(t,x)| \le C_\alpha, \quad |D^\alpha\gamma_j(t,x)| \le C_\alpha \langle x\rangle^{(1-|\alpha|)_+}, \quad |D^\alpha\gamma_0(t,x)| \le C_\alpha \langle x\rangle^{(2-|\alpha|)_+}.$$

The following Lemma is well known and can be easily proved by induction:

Lemma 5.7. *For every multi-index α, β we have*

$$\|x^\alpha D^\beta u\|_{L^2(\mathbb{R}^d)} \le C_{\alpha,\beta}\|H_{osc}^{\frac{|\alpha+|\beta|}{2}} u\|_{L^2(\mathbb{R}^d)}, \quad \forall u \in L^2(\mathbb{R}^d). \tag{5.12}$$

From this Lemma it results that K is H_{osc}-bounded. Moreover if $a(t,x)$ does not depend on x and $V(t,x)$ grows at most linearly in x, i.e. $\left|\partial_x^\alpha V(t,x)\right| \le C_\alpha \langle x\rangle^{(1-|\alpha|)_+}$, $\forall t\in\mathbb{R}$, then K is $H_{osc}^{1/2}$-bounded. Then we can apply our general results (Theorems 1.2 and 1.5) to get

Theorem 5.8. *Under assumptions (5.11) we have:*

(i) For each t, the Hamiltonian $H_{a,V}(t)$ is essentially self-adjoint in $L^2(\mathbb{R}^d)$ with core $\mathcal{S}(\mathbb{R}^d)$.
(ii) For every $k\in\mathbb{N}$, the Cauchy problem (1.1) with $L(t)\equiv H_{a,V}(t)$ is globally well-posed in the weighted Sobolev space $\mathcal{H}^k(\mathbb{R}^d) = D(H_{osc}^{k/2})$.
(iii) If furthermore $a(t,x) = a(t)$ depends only on time t and $\left|\partial_x^\alpha V(t,x)\right| \le C_\alpha \langle x\rangle^{(1-|\alpha|)_+}$, $\forall t\in\mathbb{R}$, then for any $r\in\mathbb{N}$, we have the bound:

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \le C_r \langle t-s\rangle^r .$$

Comparison with previous results: Theorem 5.8 (i) and (ii) where proved by Yajima in [37,38] by a different method. We recover them as a consequence of our general results. Notice that $V(t,x)$ has no fixed sign.

5.4. Differential systems of first order

Let us denote by $M_N(\mathbb{C})$ the space of Hermitian $N\times N$ matrices. Let $t\in\mathbb{R}$, $x\in\mathbb{R}^d$, $A_j(t)$, $1\le j\le d$, and $B(t,x)$ belong to $M_N(\mathbb{C})$, the A_j's depend only on time, $A_j \in C_b(\mathbb{R}, M_N(\mathbb{C}))$, while $B(t,x)\in C_b(\mathbb{R}, C^\infty(\mathbb{R}^d, M_N(\mathbb{C}))$ satisfies $\forall$ multi-indexes α

$$|\partial_x^\alpha B(t,x)| \le C_\alpha \langle x \rangle^{(m-|\alpha|)_+} \ , \qquad \forall t \in \mathbb{R} \ , \quad x \in \mathbb{R}^d \ .$$

Let us consider equation (1.1) with $L(t) = \sum_{1\le j\le d} A_j(t)D_j + B(t,x)$. Such equation is symmetric-hyperbolic. A basic example is the Maxwell system. An other example is the Dirac equation with a time dependent electro-magnetic field:

$$\mathrm{i}\hbar \frac{\partial \psi(x,t)}{\partial t} = \left(\beta mc^2 + c\left(\sum_{n=1}^{3} \alpha_n(\hbar D_n) \right) + V(t,x) \right) \psi(x,t)$$

where $D_n = \mathrm{i}^{-1}\frac{\partial}{\partial_{x_n}}$, (α_n, β) are the Dirac matrices and $V(t,x)$ is 4×4 Hermitian matrix (the electro-magnetic potential).

Let us introduce the reference operator $H = (-\triangle + |x|^{2k})\mathbb{I}_{\mathbb{C}^N}$, $k \in \mathbb{N}$, and the scale of Hilbert spaces $\mathcal{H}_k^r = D\left((-\triangle + |x|^{2k})^{r/2}\right)$, for any $r \ge 0$. We compute the commutator $[L(t), H]$. If $m \le k+1$ we can check that $[L(t), H]$ is H-bounded and if $m \le k$ then $[L(t), H]$ is $H^{1-\theta}$-bounded with $\theta = \frac{1}{2k}$. So Theorem 1.2 and Theorem 1.5 can be applied to give

Theorem 5.9. *Let $m \le k+1$. Then problem (1.1) is well-posed in the weighted Sobolev spaces $\mathcal{H}_k^r$ for any $r \ge 0$. Moreover if $m \le k$ then we have for any $r \ge 0$,*

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}_k^r)} \le C_r \langle t-s \rangle^{kr} \ . \tag{5.13}$$

Remark 5.10. It is easy to see that the first part of Theorem 5.9 holds true if $A_j(t) = A_j(t,x)$ are smooth in x and satisfy

$$|\partial_x^\alpha A_j(t,x)| \le C_\alpha \langle x \rangle^{(1-|\alpha|)_+} \ , \qquad \forall t \in \mathbb{R} \ .$$

Moreover, in case $|\partial_x^\alpha A_j(t,x)| \le C_\alpha$, $\forall t \in \mathbb{R}$, then also the estimate (5.13) holds true.

5.5. A discrete model example

This model was considered in [3]. We keep our notations which are different from [3].

Let us consider the Hilbert space $\mathcal{H}^0 = \ell^2(\mathbb{Z}^d)$ and its canonical Hilbert base $\{\mathrm{e}_n\}_{n\in\mathbb{Z}^d}$ defined by $\mathrm{e}_n(k) = \delta(n-k)$, $k \in \mathbb{Z}^d$. We consider equation (1.1) with Hamiltonian $L(t) = H_0 + V(t)$ where H_0 is the discrete Laplacian and $V(t)$ is a diagonal operator:

$$H_0 u(n) = \sum_{|k-n|=1} u(k) \ , \qquad V(t)u(n) = \omega_n(t)u(n)$$

(here $|\cdot|$ denotes the sup norm). Assume that $\omega_n(t)$ are real and that there exists $M \ge 0$ such that

$$|\omega_n(t)| \leq C \langle n \rangle^M \quad , \quad \forall t \in \mathbb{R} \ . \tag{5.14}$$

Introduce the reference operator $Hu(n) := \langle n \rangle u(n)$ and the usual scale of Sobolev spaces $\mathcal{H}^r = D(H^{r/2}) \equiv \{\{u(n)\}_{n\in\mathbb{Z}^d} \colon \sum_{n\in\mathbb{Z}^d} \langle n \rangle^r |u(n)|^2 < +\infty\}$.

Let us check that assumptions (H0), (H1), (H3) are satisfied with $\tau = 0$. With (5.14) assumption (H0) and (H1) are satisfied. Now we verify (H3).

Lemma 5.11. *The commutator $[H, H_0]$ is bounded on $\mathcal{H}^r$ for every $r \geq 0$.*

Proof. A direct computation gives

$$\Big([H_0, H]u\Big)(n) = \sum_{|\epsilon|=1,\ \epsilon\in\mathbb{Z}^d} (\langle n+\epsilon \rangle - \langle n \rangle)\, u(n+\epsilon) \ .$$

Thus for any $u, v \in \mathcal{H}^0$ we have

$$\left| \langle v, [H_0, H]u \rangle_{\mathcal{H}^0} \right| = \Big| \sum_{|\epsilon|=1} \sum_{n\in\mathbb{Z}^d} (\langle n+\epsilon \rangle - \langle n \rangle)\, u(n+\epsilon)\, v(n) \Big| \leq 2d \, \|u\|_{\mathcal{H}^0} \, \|v\|_{\mathcal{H}^0} \ ,$$

which shows that $[H_0, H]$ is bounded on $\mathcal{H}^0$.

Now we prove that $[H_0, H]$ is bounded on $\mathcal{H}^r$ for any $r > 0$. An easy computation gives

$$H^r[H_0, H]H^{-r}u = \sum_{m\in\mathbb{Z}^d, |\epsilon|=1} \Big(u(m+\epsilon) \left[\langle m \rangle^r \, (\langle m+\epsilon \rangle - \langle m \rangle) \, \langle m+\epsilon \rangle^{-r} \right] \Big) \, \mathrm{e}_m \tag{5.15}$$

Since

$$\sup_{m,\epsilon\in\mathbb{Z}^d,\ |\epsilon|=1} \left| \langle m \rangle^r \, (\langle m+\epsilon \rangle - \langle m \rangle) \, \langle m+\epsilon \rangle^{-r} \right| \leq C \ ,$$

it results that $[H_0, H]$ is bounded on $\mathcal{H}^r$ for any $r > 0$. $\square$

Thus it follows that $[H_0 + V(t), H]$ is a bounded operator. Applying Theorem 1.5 with $\tau = 0$ we get in particular that the propagator $\mathcal{U}(t, s)$ associated with $L(t)$ is well defined as a bounded operator on $\mathcal{H}^r$ and satisfies

$$\|\mathcal{U}(t, s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_r \, \langle t-s \rangle^{\frac{r}{2}} \quad , \qquad \forall t, s \in \mathbb{R}. \tag{5.16}$$

Comparison with previous result: Estimate (5.16) appeared first in the work of Barbaroux and Joye [3]. Zhao [40] showed that when $d = 1$ there exists a family of functions $\omega_n(t)$ s.t. the $\mathcal{H}^2$-norm of the solution of the Schrödinger equation grows linearly in time when $t \to \infty$, saturating the bound (5.16). In [39], Zhang and Zhao extended this result to general $r > 1$ and a larger family of functions $\omega_n(t)$.

5.6. Pseudodifferential operators on $\mathbb{R}^n$

We consider here equation (1.1) in case $L(t)$ is a time dependent pseudodifferential operator on $\mathbb{R}^n$. A very general Weyl calculus is detailed in the book [19]. We recall some basic facts needed here on some particular cases and some more properties in Appendix E.

Recall that for smooth symbols $A(x,\xi)$, $x,\xi \in \mathbb{R}^n$, one defines the Weyl-quantization $\mathrm{Op}_\hbar^W(A)$ by the formula

$$\left(\mathrm{Op}_\hbar^W(A)u\right)(x) := \frac{1}{(2\pi\hbar)^n} \iint_{y,\xi} e^{\frac{\mathrm{i}}{\hbar}(x-y)\cdot\xi} \, A\left(\frac{x+y}{2},\xi\right) u(y)\, dyd\xi \; . \tag{5.17}$$

This formula is valid for A in the space $\mathcal{S}(\mathbb{R}^{2n})$ of Schwartz functions and one can extend it to functions in more general classes. To introduce the class we are interested in, let us introduce the weight

$$\lambda_{k,\ell}(x,\xi) = (a + |x|^{2\ell} + |\xi|^{2k})^{\frac{1}{2k\ell}} \; .$$

Here the real number $a > 0$ will be chosen large enough.

Definition 5.12. Fix $\nu \in \mathbb{R}$, $k,\ell \in \mathbb{R}_+$. A function $A(x,\xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi, \mathbb{C})$ will be called a *symbol* in the class $\widetilde{S}^\nu_{k,\ell}$ if for every $\alpha,\beta \in \mathbb{N}^n$ there exists a constant $C_{\alpha,\beta} > 0$ s.t.

$$\left|\partial_x^\alpha \partial_\xi^\beta A(x,\xi)\right| \le C_{\alpha,\beta} \; \lambda_{k,\ell}(x,\xi)^{(\nu - k|\alpha| - \ell|\beta|)_+} \; , \tag{5.18}$$

where $r_+ := \max(0,r)$.

The class $\widetilde{S}^\nu_{k,\ell}$ does not contain symbols of strictly negative order. In particular a symbol $A \in \widetilde{S}^\nu_{k,\ell}$ with $\nu < 0$ is not decreasing faster at infinity than a symbol simply in $\widetilde{S}^0_{k,\ell}$. Thus we define also the following class of symbols:

Definition 5.13. Fix $\nu \in \mathbb{R}$, $k,\ell \in \mathbb{R}_+$. A function $A(x,\xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi, \mathbb{C})$ will be called a *symbol* in the class $S^\nu_{k,\ell}$ if for every $\alpha,\beta \in \mathbb{N}^n$ there exists a constant $C_{\alpha,\beta} > 0$ s.t.

$$\left|\partial_x^\alpha \partial_\xi^\beta A(x,\xi)\right| \le C_{\alpha,\beta} \; \lambda_{k,\ell}(x,\xi)^{\nu - k|\alpha| - \ell|\beta|} \; . \tag{5.19}$$

Such classes were introduced in [31,18], where it is proved that $\mathrm{Op}_\hbar^W(A)$ is well defined for $A \in \widetilde{S}^\nu_{k,\ell}$.

Remark 5.14. (i) For $\nu = 2$, $k = \ell = 1$, $\widetilde{S}^2_{1,1}$ is the class of symbols satisfying the *sub-quadratic* growth condition

$$\left|\partial_x^\alpha \partial_\xi^\beta A(x,\xi)\right| \leq C_{\alpha,\beta} \ , \qquad \forall \ |\alpha| + |\beta| \geq 2.$$

(ii) The function $\lambda_{k,\ell}^\nu$ belongs to $S_{k,\ell}^\nu$.
(iii) If $A \in S_{k,\ell}^\nu$, $\nu \geq 0$, then $A \in \widetilde{S}_{k,\ell}^\nu$.

We endow $\widetilde{S}_{k,\ell}^\nu$ with the family of semi-norms defined by

$$p_{\alpha\beta}^\nu(A) := \sup_{x,\xi \in \mathbb{R}^n} \lambda_{k,\ell}(x,\xi)^{-(\nu - k|\alpha| - \ell|\beta|)_+} \left|\partial_x^\alpha \partial_\xi^\beta A(x,\xi)\right| \ , \tag{5.20}$$

and for every integer M we define

$$|A|_{M,\nu} := \sup_{|\alpha|+|\beta| \leq M} p_{\alpha\beta}^\nu(A) \ . \tag{5.21}$$

Correspondingly we endow $S_{k,\ell}^\nu$ with the family of semi-norms defined by

$$p_{\alpha\beta}^\nu(A) := \sup_{x,\xi \in \mathbb{R}^n} \lambda_{k,\ell}(x,\xi)^{-(\nu - k|\alpha| - \ell|\beta|)} \left|\partial_x^\alpha \partial_\xi^\beta A(x,\xi)\right| \ , \tag{5.22}$$

and for every integer M we define (abusing notation)

$$|A|_{M,\nu} := \sup_{|\alpha|+|\beta| \leq M} p_{\alpha\beta}^\nu(A) \ . \tag{5.23}$$

We define now the reference operator H to be

$$H \equiv \widehat{H}_{k,\ell}^{k+\ell} := \mathrm{Op}_\hbar^W(\lambda_{k,\ell}^{k+\ell}) \ .$$

The constant $a > 0$ in the definition of $\lambda_{k,\ell}$ is chosen large enough such that $\widehat{H}_{k,\ell}^{k+\ell}$ is a positive self-adjoint operator in $L^2(\mathbb{R}^n)$. As usual we define the scale of Hilbert spaces $\mathcal{H}^r := D\left(\left(\widehat{H}_{k,\ell}^{k+\ell}\right)^{r/2}\right)$ for every real $r \geq 0$. Formally one has

$$\mathcal{H}^r = \{u \in L^2(\mathbb{R}^n) \mid u \in H^{\frac{(k+\ell)r}{2k}}(\mathbb{R}^n), \ |x|^{\frac{(k+\ell)r}{2\ell}} u \in L^2(\mathbb{R}^n)\} \tag{5.24}$$

equipped with a natural norm of Hilbert space.

Remark 5.15. In the class of sub-quadratic symbols $\widetilde{S}_{1,1}^2$ one has simply that $H = \widehat{H}_{osc} \equiv -\Delta + |x|^2$ (harmonic oscillator) and $\mathcal{H}^r$ are the more classical spaces

$$\mathcal{H}^r(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) \mid x^\alpha \left(\hbar\partial_x\right)^\beta u \in L^2(\mathbb{R}^n), \ |\alpha| + |\beta| \leq r\right\} \ . \tag{5.25}$$

In order to study evolution equations we need to consider time dependent symbols. We give the following

Definition 5.16. Let $\mathcal{I} \subseteq \mathbb{R}$. We say that a time-dependent symbol $A(\cdot) \in C_b^0(\mathcal{I}, \widetilde{S}^{\nu}_{k,\ell})$ iff $A(t) \in \widetilde{S}^{\nu}_{k,\ell}$ for every $t \in \mathcal{I}$ and the map $t \mapsto p^{\nu}_{\alpha\beta}(A(t))$ is continuous and uniformly bounded for every α, β.

We are ready to state the results:

Theorem 5.17. *Fix $k, \ell \in \mathbb{R}_+$ and $\nu \in \mathbb{R}$ with $\nu \leq k + \ell$. Then the following is true:*

(i) Assume that A is a real symbol with $A \in \widetilde{S}^{\nu}_{k,\ell}$. Then $\mathrm{Op}^W_\hbar(A)$ is essentially self-adjoint with core $\mathcal{S}(\mathbb{R}^n)$.
(ii) Assume that $A(\cdot) \in C_b^0(\mathbb{R}, \widetilde{S}^{\nu}_{k,\ell})$. Then the Schrödinger equation (1.1) with $L(t) \equiv \mathrm{Op}^W_\hbar(A(t))$ generates a flow $\mathcal{U}(t,s)$ which fulfills (i)–(iv) of Theorem 1.2.
(iii) If $\nu < k + \ell$, then the flow $\mathcal{U}(t,s)$ fulfills the bound

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C \langle t - s \rangle^{\frac{r\,(k+\ell)}{2\,(k+\ell-\nu)}} .$$

Proof. (i) It follows by the same arguments used to prove item (ii) and Proposition A.2.

(ii) We verify the assumptions of Theorem 1.2 using the symbolic calculus for symbols in the classes $\widetilde{S}^{\nu}_{k,\ell}$. By Remark 5.14, $\lambda^{\nu}_{k,\ell}$ is a symbol in $S^{\nu}_{k,\ell}$ and it is invertible provided a is sufficiently large. By symbolic calculus the operator H is invertible and its inverse $H^{-1} \in \mathrm{Op}^W_\hbar(S^{-(k+\ell)}_{k,\ell})$. It follows easily by symbolic calculus (see Theorem E.1, E.2 and Corollary E.3) that $[A(t), H]H^{-1} \in \mathrm{Op}^W_\hbar(\widetilde{S}^{\nu-(k+\ell)}_{k,\ell})$. Then by Calderon–Vaillancourt theorem (see Theorem E.4) if $\nu \leq k + \ell$ such operator is bounded on the scale of Hilbert spaces (5.24). Theorem 1.2 can be applied.

(iii) One applies Theorem 1.5 remarking that $[A(t), H]H^{-\tau} \in \mathrm{Op}^W_\hbar(\widetilde{S}^{\nu-\tau(k+\ell)}_{k,\ell})$. Then if $\nu < k + \ell$, choosing $\tau = \frac{\nu}{k+\ell}$ one has that $\tau < 1$ and $[A(t), H]H^{-\tau}$ is a bounded operator. □

Remark 5.18. If $A \in \widetilde{S}^2_{1,1}$ then $A(t, x, \xi)$ is a sub-quadratic symbol in (x, ξ) and we recover a result already proved by Tataru [35] using a complex WKB parametrix for the Schrödinger equation.

Example 5.19 *(A balance between position and momentum behavior).* Consider a symbol A of the form

$$A(x, \xi) = f(\xi) + g(x)$$

where the functions f, g are smooth and fulfill

$$|\partial_x^\alpha f(\xi)| \leq C_\alpha \langle \xi \rangle^{(p-|\alpha|)_+} , \quad |\partial_x^\alpha g(x)| \leq C_\alpha \langle x \rangle^{(q-|\alpha|)_+} , \tag{5.26}$$

for some $p, q \in \mathbb{Q}$ such that $\frac{1}{p} + \frac{1}{q} = 1$, with $1 < p < +\infty$. Then $\mathrm{Op}_\hbar^W(A)$ is essentially self-adjoint. Indeed in such case it is possible to find integers k, ℓ such that $p = (k+\ell)/\ell$, $q = (k+\ell)/k$. Then with such k, ℓ, one verifies easily that $A \in \widetilde{S}_{k,\ell}^{k+\ell}$.

Moreover if f, g are time-dependent the operator $A^w(t)$ generates a propagator satisfying (i)–(iii). It satisfies (iv) if furthermore estimates (5.26) are uniform in time $t \in \mathbb{R}$.

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Appendix A. Essentially self-adjointness

In this section we give the proof of essentially self-adjointness which is based to the commutator method of Nelson [26]. The method was further extended by Faris and Lavine [13]. The general principle is related with the Friedrichs smoothing method [15].

We start to recall some standard definitions. Let $\mathcal{H}$ be a complex Hilbert space and $(\cdot, \cdot)_{\mathcal{H}}$ its inner product. Let $\mathcal{K} \subset \mathcal{H}$ be a dense subspace. Let L be a linear operator with domain $D(L) = \mathcal{K}$ and *symmetric*, i.e. verifying

$$(Lu, v)_{\mathcal{H}} = (u, Lv)_{\mathcal{H}} \qquad \text{for every } u, v \in \mathcal{K} \; .$$

We say that $(L, \mathcal{K}, \mathcal{H})$ is *essentially self-adjoint* if L admits a unique self-adjoint extension as an unbounded operator on $\mathcal{H}$. When this is true $\mathcal{K}$ is called a *core* for L. Let $(L, \mathcal{K}, \mathcal{H})$ be a symmetric operator. It is known that the operator $(L, \mathcal{K}, \mathcal{H})$ is *closable*, i.e. it admits at most one closed extension $(L_{\min}, D(L_{\min}), \mathcal{H})$. $L_{\min}$ is the smallest closed extension of L, and we call $(L_{\min}, D(L_{\min}), \mathcal{H})$ the *minimal operator* associated to L.

We denote by $(L^*_{\min}, D(L^*_{\min}), \mathcal{H})$ the adjoint of $(L_{\min}, D(L_{\min}), \mathcal{H})$. Recall that by definition

$$D(L^*_{\min}) = \{u \in \mathcal{H} \; : \; |(u, Lv)_{\mathcal{H}}| \leq C_u \|v\|_{\mathcal{H}} \; , \forall v \in \mathcal{K}\} \; .$$

It is a classical result [32, Proposition 2] that $(L^*_{\min}, D(L^*_{\min}), \mathcal{H})$ is the largest closed extension of L. Denote $L_{\max} := L^*_{\min}$. Then we call $(L_{\max}, D(L_{\max}), \mathcal{H})$ the *maximal operator* associated to L. Thus L is essentially self-adjoint if $L_{\min}$ is self-adjoint. This means that $(L_{\min}, D(L_{\min}), \mathcal{H})$ and $(L_{\max}, D(L_{\max}), \mathcal{H})$ coincide.

Let us introduce a smoothing family of operators $\{R_\varepsilon\}_{\varepsilon \in]0,1]}$ satisfying

$$\|R_\varepsilon\|_{\mathcal{L}(\mathcal{H})} \leq C, \; \forall \varepsilon \in \,]0, 1], \tag{A.1}$$

$$R_\varepsilon \mathcal{H} \subseteq \mathcal{K}, \quad \forall \varepsilon \in]0,1], \tag{A.2}$$

$$\lim_{\varepsilon\to 0} \|R_\varepsilon u - u\|_{\mathcal{H}} = 0, \forall u \in \mathcal{H}. \tag{A.3}$$

Proposition A.1. *Let $(L, \mathcal{K}, \mathcal{H})$ be a symmetric operator. Assume that the commutators $[R_\varepsilon, L] = R_\varepsilon L - LR_\varepsilon$ satisfies*

$$\|[R_\varepsilon, L]u\|_{\mathcal{H}} \le C\|u\|_{\mathcal{H}}, \ \forall u \in \mathcal{K}, \quad \forall \varepsilon \in]0,1], \tag{A.4}$$

$$\lim_{\varepsilon\to 0} \|[R_\varepsilon, L]u\|_{\mathcal{H}} = 0 \ , \forall u \in \mathcal{K} \ . \tag{A.5}$$

Then $(L, \mathcal{K}, \mathcal{H})$ is essentially self-adjoint.

Proof. We have to prove that $D(L_{\max}) \subseteq D(L_{\min})$. Let $u \in D(L_{\max})$. Then by property (A.1), $u_\varepsilon := R_\varepsilon u \in D(L_{\min})$ and $u_\varepsilon \to u$ in $\mathcal{H}$. But we have

$$Lu_\varepsilon = R_\varepsilon Lu + [L, R_\varepsilon]Au \ .$$

So by assumption (A.4) we get that $\lim_{\varepsilon\to 0} Lu_\varepsilon = Lu$ so $u \in D(L_{\min})$. □

The following criterium apply Proposition A.1 and is due to Nelson [26].

Proposition A.2. *Let H be a positive self-adjoint operator in $\mathcal{H}$ with a dense domain $D(H)$.*

Let L be a linear and symmetric operator from $D(H)$ into $\mathcal{H}$.

Assume that the operators $LH^{-\tau}$ ($\tau > 0$) and $H^{-1/2}[H, L]H^{-1/2}$ are bounded on $\mathcal{H}$ then $(L, D(H), \mathcal{H})$ is essentially self-adjoint.

Proof. Let us repeat here the rather simple proof. We have to verify that the assumptions of Proposition A.1 are satisfied with $R_\varepsilon = \mathrm{e}^{-\varepsilon H}$.

First we have, for $u \in D(H^\tau)$,

$$[\mathrm{e}^{-\varepsilon H}, L]u = \mathrm{e}^{-\varepsilon H}Lu - L\mathrm{e}^{-\varepsilon H}u \ .$$

We have $Lu \in \mathcal{H}$ so $\lim_{\varepsilon\to 0} \|\mathrm{e}^{-\varepsilon H}Lu - Lu\|_{\mathcal{H}} = 0$. Writing $L\mathrm{e}^{-\varepsilon H}u = (LH^{-\tau})(\mathrm{e}^{-\varepsilon H}H^\tau u)$ we also have $\lim_{\varepsilon\to 0} \|L\mathrm{e}^{-\varepsilon H}u - Lu\|_{\mathcal{H}} = 0$. So we have proved

$$\lim_{\varepsilon\to 0} \|[\mathrm{e}^{-\varepsilon H}, L]u\|_{\mathcal{H}} = 0, \ \ \forall u \in D(H^\tau). \tag{A.6}$$

Let us estimate now $\|[\mathrm{e}^{-\varepsilon H}, L]\|_{\mathcal{L}(\mathcal{H})}$. We start with the following known formula

$$[\mathrm{e}^{-\varepsilon H}, L] = -\int_0^\varepsilon \mathrm{e}^{-(\varepsilon - s)H}[L, H]\mathrm{e}^{-sH} ds \ . \tag{A.7}$$

Following [26] we have

$$[\mathrm{e}^{-\varepsilon H}, L] = -\int_0^{\varepsilon} \mathrm{e}^{-(\varepsilon-s)H} H^{1/2}(H^{-1/2}[L,H]H^{-1/2})H^{1/2}\mathrm{e}^{-sH} ds \,. \tag{A.8}$$

Using that

$$\|H^{1/2}\mathrm{e}^{-sH}\|_{\mathcal{L}(\mathcal{H})} = \sup_{\lambda\geq 0} \lambda^{1/2}\mathrm{e}^{-s\lambda} \leq Cs^{-1/2}$$

and the beta function computation: $\int_0^{\epsilon}(\epsilon - s)^{-1/2}s^{-1/2}ds = B(1/2,1/2) = \frac{\pi}{2}$ we get

$$\sup_{\epsilon\in]0,1]} \|[\mathrm{e}^{-\varepsilon H}, L]\|_{\mathcal{L}(\mathcal{H})} < +\infty. \quad \square$$

Appendix B. Technical estimates for perturbations smooth in time

In this section we prove some technical estimates which are useful in the proof of Theorem 1.9.

First we state a result about boundedness of the resolvent. In all the section H will be a self-adjoint, positive operator in $\mathcal{H}^0$ fulfilling (Hgap). Let $H_W(t) := H + W(t)$, $W(t)$ a symmetric operator fulfilling $(\mathrm{Vs})_n$.

Lemma B.1. *Assume that W fulfills $(\mathrm{Vs})_n$. Then*

$$R_{n,0}\ \|H^{\nu}(H-z)^{-1}\| \leq \frac{1}{2} \ ,$$

we have for any integer $0 \leq p \leq n$, any real $0 \leq \theta \leq 1$,

$$\sup_{t\in\mathbb{R}} \|H^{p+\theta}(H_W(t)-z)^{-1}H^{-p}\| \leq 2\|H^{\theta}(H-z)^{-1}\| \ . \tag{B.1}$$

Proof. This is a consequence of the resolvent identity:

$$(H_W(t)-z)^{-1} = (H-z)^{-1} - (H_W(t)-z)^{-1}W(t)(H-z)^{-1} \ , \tag{B.2}$$

so we have for $0 \leq \theta \leq 1$, $0 \leq p \leq n$,

$$\begin{aligned} H^{p+\theta}(H_W(t)-z)^{-1}H^{-p} = H^{\theta}(H-z)^{-1} - \left(H^{p+\theta}(H_W(t)-z)^{-1}H^{-p}\right) \\ \times \left(H^{p}W(t)H^{-p-\nu}\right) H^{\nu}(H-z)^{-1} \ . \end{aligned} \tag{B.3}$$

Provided

$$\sup_{t\in\mathbb{R}} \|H^{p}\, W(t)\, H^{-p-\nu}\|\ \|H^{\nu}(H-z)^{-1}\| \leq R_{n,0}\ \|H^{\nu}(H-z)^{-1}\| \leq \frac{1}{2} \ ,$$

estimate (B.1) follows. $\square$

Lemma B.2. *Fix $n \geq 0$. Let $P(t)$ be an operator fulfilling* $(\mathrm{Vs})_n$ *with*

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^\ell P(t) H^{-p-\nu}\| \leq D_{n,\ell}\ , \quad \forall \ell \geq 0\ ,\ 0 \leq p \leq n\ . \tag{B.4}$$

Consider the operator $H+P(t)$. Then, provided $\mathtt{J}$ is sufficiently large, the following holds true:

(i) $H + P(t)$ fulfills $(\widetilde{\mathrm{Hgap}})$ uniformly in time $t \in \mathbb{R}$.
(ii) Let Γ_j be as in (3.7). Any $\lambda \in \Gamma_j$ belongs to the resolvent set of the operator $H+P(t)$. Denote $R_P(t,\lambda) := (H + P(t) - \lambda)^{-1}$. Then for any $\lambda \in \Gamma_j$, $j \geq 1$ one has

$$\sup_{t\in\mathbb{R}} \|H^p R_P(t,\lambda) H^{-p}\| \leq \frac{2}{\mathrm{dist}(\lambda, \sigma(H))}\ , \quad \forall 0 \leq p \leq n \tag{B.5}$$

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^\ell R_P(t,\lambda)\, H^{-p}\| \leq \frac{C_{n,\ell}}{\mathrm{dist}(\lambda,\sigma(H))} \frac{1}{\widetilde{\Delta}_{j-1}^{\delta}}\ , \quad \forall 0 \leq p \leq n\ ,\ \ell \geq 1\ , \tag{B.6}$$

where $C_{n,\ell}$ does not depend on $j, \mathtt{J}$.

(iii) For any $j \geq 1$ define the projector

$$\Pi_j(t) := -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} R_P(t,\lambda)\, d\lambda\ . \tag{B.7}$$

It fulfills

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^{\ell+1} \Pi_j(t)\, H^{-p}\| \leq \frac{C_{n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta}}\ , \qquad \forall 0 \leq p \leq n,\ \ell \geq 0\ ,$$

where $C_{n,\ell}$ does not depend on $j, \mathtt{J}$.

Proof. (i) It follows by Lemma 3.2 provided $\mathtt{J}$ is sufficiently large to fulfill condition (3.9). Thus $\sigma(H + P(t)) \subseteq \bigcup_{j\geq 1} \widetilde{\sigma}_j$ (with $\widetilde{\sigma}_j$ as in (3.10)).

(ii) By the previous item each Γ_j is contained in the resolvent set of $H + P(t)$. To estimate $\|R_P(t,\lambda)\|$ we use Lemma B.1 and Lemma 3.4. Indeed for $\mathtt{J}$ sufficiently large and $\lambda \in \Gamma_j$ we have

$$D_{n,0} \|H^\nu (H - \lambda)^{-1}\| \leq D_{n,0} \frac{\widetilde{\mathtt{C}}_H}{\widetilde{\Delta}_{j-1}^{\delta}} \leq D_{n,0} \frac{\widetilde{\mathtt{C}}_H}{2^{\mathtt{J}\mu\delta}} \leq \frac{1}{2}\ ,$$

hence we can apply Lemma B.1 with $\theta = 0$ to obtain estimate (B.5).

To prove (B.6), use the formula

$$\partial_t^\ell R_P(t,\lambda) = \sum_{k=1}^{\ell} \sum_{\substack{n_1,\ldots,n_k \in \mathbb{N} \\ n_1+\cdots+n_k=\ell}} \binom{\ell}{n_1 \cdots n_k} R_P(t,\lambda)\, (\partial_t^{n_1} P(t))\, R_P(t,\lambda)\, (\partial_t^{n_2} P(t)) \cdots (\partial_t^{n_k} P(t))\, R_P(t,\lambda) \tag{B.8}$$

and take the conjugates with H^p to obtain

$$
\begin{aligned}
H^p\,\partial_t^\ell R_P(t,\lambda)\,H^{-p} = \sum_{k=1}^{\ell} \sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\cdots+n_k=\ell}} & \binom{\ell}{n_1\cdots n_k} (H^p R_P(t,\lambda) H^{-p}) \\
& \times [H^p(\partial_t^{n_1}P(t))\,H^{-p-\nu}]\,[H^{p+\nu}\,R_P(t,\lambda)\,H^{-p}]\cdots \\
& \cdots [H^p\,(\partial_t^{n_k}P(t))\,H^{-p-\nu}]\,[H^{p+\nu}\,R_P(t,\lambda) H^{-p}]\ . \qquad \text{(B.9)}
\end{aligned}
$$

Then using estimate (B.5), Lemma B.1 with $\theta = \nu$, estimates (B.4) and (3.13), we obtain for $\lambda \in \Gamma_j$, $j \geq 1$

$$
\sup_{t\in\mathbb{R}} \|H^p\,\partial_t^\ell R_P(t,\lambda) H^{-p}\| \leq \frac{C_{n,\ell}}{\operatorname{dist}(\lambda,\sigma(H))} \frac{1}{\widetilde{\Delta}_{j-1}^{\delta}}\ , \qquad \forall \ell \geq 1\ ,\ 0 \leq p \leq n\ , \qquad \text{(B.10)}
$$

where the $C_{n,\ell}$ can be chosen independent of j, $\mathtt{J}$. (iii) For $\ell \geq 1$, one has $H^p\,\partial_t^\ell \Pi(t)\,H^{-p} = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} H^p\,\partial_t^\ell R_P(t,\lambda) H^{-p}\,d\lambda$, hence by (B.6)

$$
\sup_{t\in\mathbb{R}} \|H^p\,\partial_t^\ell \Pi(t)\,H^{-p}\| \leq \frac{C_{n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta}} \frac{1}{2\pi} \oint_{\Gamma_j} \frac{d\lambda}{\operatorname{dist}(\lambda,\sigma(H))} \leq \frac{C_{n,\ell}}{\widetilde{\Delta}_{j-1}^{\delta}}
$$

where to pass from the first to the second inequality we used that, deforming the contour Γ_j to two vertical lines passing between the middle of the gaps one has

$$
\begin{aligned}
\frac{1}{2\pi} \oint_{\Gamma_j} \frac{d\lambda}{\operatorname{dist}(\lambda,\sigma(H))} &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{((\widetilde{\Delta}_{j-1}/2)^2 + x^2)^{1/2}} + \frac{1}{((\widetilde{\Delta}_j/2)^2 + x^2)^{1/2}} \right) dx \\
&\leq 2 \quad \square \qquad \text{(B.11)}
\end{aligned}
$$

Lemma B.3. *Fix $n \geq 0$. Let $P(t), Q(t)$ be operators fulfilling* (Vs)$_n$ *with estimates as in* (B.4). *Furthermore assume that $B(\cdot) \in C_b^\infty(\mathbb{R}, \mathcal{L}(\mathcal{H}^{2p}))$, $\forall 0 \leq p \leq n$, fulfilling the estimates*

$$
\sup_{t\in\mathbb{R}} \|H^p\,\partial_t^\ell B(t)\,H^{-p}\| \leq b_{n,\ell}\ , \quad \forall \ell \geq 0,\ 0 \leq p \leq n\ . \qquad \text{(B.12)}
$$

Provided $\mathtt{J}$ is sufficiently large, the operator

$$
K(t) := -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} R_P(t,\lambda)\,B(t)\,R_Q(t,\lambda)\,d\lambda
$$

is well defined and bounded from $\mathcal{H}^{2p}$ to itself, $\forall 0 \leq p \leq n$, and fulfills

$$
\sup_{t\in\mathbb{R}} \|H^p\,\partial_t^\ell K(t)\,H^{-p}\| \leq \frac{C_{n,\ell}}{\widetilde{\Delta}_{j-1}} \sup_{l\leq\ell} b_{n,l}\ , \qquad \forall \ell \geq 0\ ,\ 0 \leq p \leq n\ .
$$

Proof. By Lemma B.2, provided $\mathtt{J}$ is sufficiently large, Γ_j is contained in the resolvent sets of $H + P(t)$ and $H + Q(t)$, thus $K(t)$ is well defined. To estimate it, take first $\ell = 0$. Then by (B.5) and (B.12)

$$\sup_{t\in\mathbb{R}} \|H^p\, K(t)\, H^{-p}\| \leq \frac{1}{2\pi} \oint_{\Gamma_j} \frac{4\, b_{n,0}\, d\lambda}{\mathrm{dist}(\lambda, \sigma(H))^2} \leq \frac{16\, b_{n,0}}{\widetilde{\Delta}_{j-1}} \ ,$$

where once again we deformed the contour as in (B.11). Take now $\ell \geq 1$. By Leibnitz formula we get

$$\partial_t^\ell K(t) = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} R_P(t,\lambda)\ (\partial_t^\ell B(t))\ R_Q(t,\lambda)\, d\lambda \tag{B.13}$$

$$- \sum_{\substack{n_1+n_2=\ell \\ n_1\geq 1}} \binom{\ell}{n_1\, n_2} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_j} (\partial_t^{n_1} R_P(t,\lambda))\ (\partial_t^{n_2} B(t))\ R_Q(t,\lambda)\, d\lambda \tag{B.14}$$

$$- \sum_{\substack{n_2+n_3=\ell \\ n_3\geq 1}} \binom{\ell}{n_2\, n_3} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_j} R_P(t,\lambda)\ (\partial_t^{n_2} B(t))\ (\partial_t^{n_3} R_Q(t,\lambda))\, d\lambda \tag{B.15}$$

$$- \sum_{\substack{n_1+n_2+n_3=\ell \\ n_1,n_3\geq 1}} \binom{\ell}{n_1\, n_2\, n_3} \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_j} (\partial_t^{n_1} R_P(t,\lambda))\ (\partial_t^{n_2} B(t))\ (\partial_t^{n_3} R_Q(t,\lambda))\, d\lambda \tag{B.16}$$

Using (B.12) and (B.5) one finds easily that $\partial_t^\ell K(t)$ fulfills the claimed estimate (see the proof of Lemma C.3 for the details in the case of perturbations analytic in time). □

Appendix C. Technical estimates for perturbations analytic in time

In this section we repeat the estimates of the previous section in case of perturbations analytic in time.

In the following we fix $n \in \mathbb{N} \cup \{0\}$ and $\mathtt{L} \in \mathbb{N}$. Then for any $0 \leq p \leq n$, all constants may depend on n, but *not* on $\mathtt{L}$. Finally we denote by A a constant as in (4.2).

Lemma C.1. *Let P and Q be operators analytic in time fulfilling* $\forall 0 \leq \ell \leq \mathtt{L}$, $\forall 0 \leq p \leq n$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell P(t)\ H^{-p}\| \leq a\, b^{\min(\ell,1)}\, c^{k+\ell}\, \frac{(k+\ell)!}{A(1+\ell)^2} \ , \tag{C.1}$$

and

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell Q(t)\ H^{-p}\| \leq d\, f^{\min(\ell,1)}\, c^{i+\ell}\, \frac{(i+\ell)!}{A(1+\ell)^2} \ , \tag{C.2}$$

for some positive constants $a, b, c, d, f \in \mathbb{R}$ *and* $k, i \in \mathbb{N} \cup \{0\}$. *Then* $\forall 0 \leq \ell \leq \mathtt{L}$, $\forall 0 \leq p \leq n$

$$\sup_{t\in\mathbb{R}} \|H^p \ \partial_t^\ell (PQ)(t) \ H^{-p}\| \leq a\, d\, (b + f + bf)^{\min(\ell,1)} \, c^{k+i+\ell} \, \frac{(k+i+\ell)!}{A(1+\ell)^2} \ .$$

Proof. First consider the case $\ell = 0$. One has $\sup_{t\in\mathbb{R}} \|H^p \ P(t)Q(t) \ H^{-p}\| \leq a\, d\, c^{k+i} \frac{(k+i)!}{A}$, where we used $A \geq 1$. Now take $1 \leq \ell \leq \mathtt{L}$. By Leibnitz formula

$$\partial_t^\ell (PQ) = (\partial_t^\ell P)Q + P(\partial_t^\ell Q) + \sum_{j=1}^{\ell-1} \binom{\ell}{j} (\partial_t^j P)\, (\partial_t^{\ell-j} Q) \ .$$

Using (C.1) and (C.2) we get immediately

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|H^p \ \partial_t^\ell (PQ)(t) \ H^{-p}\| &\leq a\,(b+f)\, d\, c^{k+\ell+i} \, \frac{(k+\ell+i)!}{A(1+\ell)^2} \\ &\quad + a\, d\, b\, f\, c^{k+\ell+i} \, \frac{(k+\ell+i)!}{A^2} \sum_{n=1}^{\ell-1} \binom{\ell}{j} \binom{k+i+\ell}{k+j}^{-1} \\ &\quad \times \frac{1}{(1+j)^2\,(1+\ell-j)^2} \ . \end{aligned}$$

Now use that $\binom{k+i+\ell}{k+j} \geq \binom{\ell}{j}$ and (4.2) to conclude the proof. □

Lemma C.2. *Let* P *be an operator analytic in time fulfilling* $\forall 0 \leq \ell \leq \mathtt{L}$, $\forall 0 \leq p \leq n$

$$\sup_{t\in\mathbb{R}} \|H^p \ \partial_t^\ell P(t) \ H^{-p-\nu}\| \leq a\, b^{\min(\ell,1)} \, \frac{\ell!\, c^\ell}{A(1+\ell)^2} \tag{C.3}$$

for some positive constants $a, b, c \in \mathbb{R}$. *Provided that*

$$2^4 \, \mathtt{C}_H \, a(1+b) \leq 2^{\mathtt{J}\mu\delta} \ , \tag{C.4}$$

the following holds true:

(i) $H + P(t)$ *fulfills* $(\widetilde{\text{Hgap}})$ *uniformly in time* $t \in \mathbb{R}$.
(ii) *Let* Γ_j *be as in* (3.7). *Any* $\lambda \in \Gamma_j$ *belongs to the resolvent set of the operator* $H + P(t)$. *Denote* $R_P(t, \lambda) := (H + P(t) - \lambda)^{-1}$ *Then for any* $\lambda \in \Gamma_j$, $\forall j \in \mathbb{N}$, *estimate* (B.5) *holds and furthermore*

$$\sup_{t\in\mathbb{R}} \|H^p \, \partial_t^\ell R_P(t,\lambda) \, H^{-p}\| \leq \frac{\ell! \ c^\ell}{A(1+\ell)^2} \ \frac{2^3 \, \mathtt{C}_H \, a\, b}{\widetilde{\Delta}_{j-1}^\delta \, \operatorname{dist}(\lambda, \sigma(H))} \ , \qquad \forall 0 \leq p \leq n \ , \ \ 1 \leq \ell \leq \mathtt{L} \ . \tag{C.5}$$

(iii) For any $j \geq 1$ consider the projector (B.7). It fulfills

$$\sup_{t\in\mathbb{R}} \|H^p\, \partial_t^\ell \Pi_j(t)\, H^{-p}\| \leq \frac{\ell!\; c^\ell}{A(1+\ell)^2}\, \frac{2^4\, \mathtt{C}_H\, a\, b}{\widetilde{\Delta}^\delta_{j-1}}\ , \qquad \forall 0 \leq p \leq n,\ 1 \leq \ell \leq \mathtt{L}\ .$$

Proof. (i) See the proof of Lemma B.1(i).

(ii) We prove only estimate (C.5). The other statements are proved as in Lemma B.2(ii). First remark that by Lemma B.1 with $\theta = \nu$ and Lemma 3.4

$$\sup_{t\in\mathbb{R}} \|H^{p+\nu}\ R_P(t,\lambda)\ H^{-p}\| \leq \frac{2\,\mathtt{C}_H}{\widetilde{\Delta}^\delta_{j-1}}\ , \quad \forall \lambda \in \Gamma_j\ , \quad \forall 0 \leq p \leq n \tag{C.6}$$

provided

$$a\,4\,\mathtt{C}_H\ \leq \widetilde{\Delta}^\delta_{j-1}\ .$$

Clearly such condition is implied by (C.4).

Now take $1 \leq \ell \leq \mathtt{L}$. Formula (B.9) and estimates (C.3), (B.5), (C.6) give for any $\lambda \in \Gamma_j$, $\forall j \geq 1$, $\forall 0 \leq p \leq n$

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell R_P(t,\lambda)\ H^{-p}\| \qquad\qquad (C.7)\\
&\leq \ell!\ c^\ell \sum_{k=1}^{\ell} \frac{2}{\operatorname{dist}(\lambda,\sigma(H))} \left(\frac{2\,\mathtt{C}_H\, a\, b}{\widetilde{\Delta}^\delta_{j-1}}\right)^k \frac{1}{A^k} \sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\cdots+n_k=\ell}} \frac{1}{(1+n_1)^2} \cdots \frac{1}{(1+n_k)^2}\\
&\leq \frac{\ell!\ c^\ell}{A(1+\ell)^2}\ \frac{2}{\operatorname{dist}(\lambda,\sigma(H))} \sum_{k=1}^{\ell} \left(\frac{2\,\mathtt{C}_H\, a\, b}{\widetilde{\Delta}^\delta_{j-1}}\right)^k\\
&\leq \frac{\ell!\ c^\ell}{A(1+\ell)^2}\ \frac{2^3\,\mathtt{C}_H\, a\, b}{\widetilde{\Delta}^\delta_{j-1}\operatorname{dist}(\lambda,\sigma(H))}
\end{aligned}$$

where to pass from the third to fourth line we used that by (C.4) $\frac{2\,\mathtt{C}_H\, a\, b}{\widetilde{\Delta}^\delta_{j-1}} \leq \frac{1}{2}$. Thus (C.5) is proved.

(iii) By (C.5) one has $\forall 1 \leq \ell \leq \mathtt{L}$, $\forall 0 \leq p \leq n$

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell \Pi(t)\ H^{-p}\| &\leq \frac{\ell!\ c^\ell}{A(1+\ell)^2}\ \frac{2^3\,\mathtt{C}_H\, a\, b}{\widetilde{\Delta}^\delta_{j-1}}\ \frac{1}{2\pi} \oint_{\Gamma_j} \frac{d\lambda}{\operatorname{dist}(\lambda,\sigma(H))}\\
&\leq \frac{\ell!\ c^\ell}{A(1+\ell)^2}\ \frac{2^4\,\mathtt{C}_H\, a\, b}{\widetilde{\Delta}^\delta_{j-1}}
\end{aligned}$$

where we used also (B.11). □

Lemma C.3. *Let $P(t), Q(t)$ be operators analytic in time fulfilling* $\forall 0 \leq \ell \leq \mathtt{L}$, $\forall 0 \leq p \leq n$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell P(t)\ H^{-p-\nu}\| \ ,\ \sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell Q(t)\ H^{-p-\nu}\| \leq a\, b^{\min(\ell,1)} \frac{\ell!\, c^\ell}{A(1+\ell)^2}\ . \tag{C.8}$$

Assume that (C.4) holds. Furthermore let $B(t)$ be an operator analytic in time fulfilling $\forall 0 \leq \ell \leq \mathtt{L}$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell B(t)\ H^{-p}\| \leq h\, c^\ell \frac{\ell!}{A(1+\ell)^2}\ , \quad \forall 0 \leq p \leq n \tag{C.9}$$

for some positive $h \in \mathbb{R}$. Then the operator

$$K(t) := -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_j} R_P(t,\lambda)\, B(t)\, R_Q(t,\lambda)\, d\lambda$$

is analytic in time, bounded from H^p to H^p $\forall 0 \leq p \leq n$, and fulfills $\forall 0 \leq \ell \leq \mathtt{L}$, $\forall 0 \leq p \leq n$

$$\sup_{t\in\mathbb{R}} \|\partial_t^\ell K(t)\| \leq \frac{\ell!\, c^l}{A(1+\ell)^2} \frac{h\, 2^5}{\widetilde{\Delta}_{j-1}}\ .$$

Proof. First consider the resolvents $R_P(t,\lambda)$, $R_Q(t,\lambda)$. Proceeding as in the proof of Lemma B.2, they are well defined for any $\lambda \in \Gamma_j$, $\forall j \geq 1$, and fulfill estimates (B.5), (C.5). Consider now $K(t)$. For $\ell = 0$ one has

$$\sup_{t\in\mathbb{R}} \|H^p\ K(t)\ H^{-p}\| \leq \frac{h}{A} \frac{1}{2\pi} \oint_{\Gamma_j} \frac{4\, d\lambda}{\mathrm{dist}(\lambda,\sigma(H))^2} \leq \frac{4\, h}{A\, \widetilde{\Delta}_{j-1}}\ , \quad \forall 0 \leq p \leq n\ .$$

For $1 \leq \ell \leq \mathtt{L}$, consider (B.13)–(B.16). We estimate each line. By (C.9), (B.5) one has

$$\sup_{t\in\mathbb{R}} \|H^p\ \text{(B.13)}\ H^{-p}\| \leq \frac{4\, h}{\widetilde{\Delta}_{j-1}} \frac{\ell!\, c^\ell}{A(1+\ell)^2}\ , \quad \forall 0 \leq p \leq n\ .$$

To estimate the second line we use (B.5), (C.5), (C.9) and (C.4) to get $\forall 0 \leq p \leq n$

$$\sup_{t\in\mathbb{R}} \|H^p\ \text{(B.14)}\ H^{-p}\| \leq \frac{2^4\, \mathtt{C}_H\, a\, b\, h}{\widetilde{\Delta}_{j-1}^\delta} \frac{1}{2\pi} \oint_{\Gamma_j} \frac{d\lambda}{\mathrm{dist}(\lambda,\sigma(H))^2} \leq \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{2^6\, \mathtt{C}_H\, h\, a\, b}{\widetilde{\Delta}_{j-1}^{1+\delta}}\ .$$

The third line is estimated exactly as the second one. We pass to the last line. Using (C.5), (C.9) we get $\forall 0 \leq p \leq n$

$$\sup_{t\in\mathbb{R}} \|H^p \text{ (B.16) } H^{-p}\| \le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{2^6\, \mathtt{C}_H^2\, a^2\, b^2\, h}{\widetilde{\Delta}_{j-1}^{2\delta}} \frac{1}{2\pi} \oint_{\Gamma_j} \frac{d\lambda}{\operatorname{dist}(\lambda, \sigma(H))^2}$$

$$\le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{2^8\, h\, (\mathtt{C}_H\, a\, b)^2}{\widetilde{\Delta}_{j-1}^{1+2\delta}} .$$

Altogether we find that for $1 \le \ell \le \mathtt{L}$, $\forall 0 \le p \le n$

$$\sup_{t\in\mathbb{R}} \|H^p\ \partial_t^\ell K(t)\ H^{-p}\| \le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{2^2\, h}{\widetilde{\Delta}_{j-1}} \left(1 + \frac{\mathtt{C}_H\, a\, b\, 2^5}{\widetilde{\Delta}_{j-1}^{\delta}} + \frac{(\mathtt{C}_H\, a\, b)^2\, 2^6}{\widetilde{\Delta}_{j-1}^{2\delta}}\right)$$

$$\le \frac{\ell!\, c^\ell}{A(1+\ell)^2} \frac{h\, 2^5}{\widetilde{\Delta}_{j-1}}$$

where we used again (C.4). □

Appendix D. Proof of Lemma 3.8

We start with an abstract result. Let $H_W(t) := H + W(t)$, H being a self-adjoint positive operator in $\mathcal{H}^0$, $W(t)$ a symmetric operator, H^ν-bounded with $\nu < 1$. We assume that, for a fixed $n \in \mathbb{N}$, we have

$(\mathrm{W})_n$ $H^p\ W(\cdot)\ H^{-p-\nu} \in C_b^0(\mathbb{R}, \mathcal{L}(\mathcal{H}^0))$, $\sup_{t\in\mathbb{R}} \|H^p\ W(t)\ H^{-p-\nu}\| \le D_n$, $\forall 0 \le p \le n$.

Lemma D.1. *Let $n \ge 1$. Assume that W satisfies condition $(\mathrm{W})_n$.*

Define $W_n(t) = (H + W(t))^n - H^n$. Then we have $W_n H^{1-n-\nu} \in \mathcal{L}(\mathcal{H}^0)$. Furthermore there exist positive constants γ_0, γ_1 depending only on H such that

$$\|W_n H^{1-n-\nu}\| \le \gamma_0 \gamma_1^n D_n^{n+1}. \tag{D.1}$$

Finally we have

$$c_n \|\psi\|_{2n} \le \|(H + W(t) + c_0)^n\|_0 \le C_n \|\psi\|_{2n}, \quad \forall \psi \in \mathcal{H}^{2n},\ \forall t \in \mathbb{R}, \tag{D.2}$$

where c_n, C_n depend only on D_n.

Proof. We proceed by induction on n. For $n = 1$ the two side estimate is a classical perturbation result using $(\mathrm{W})_0$. For $n > 1$ we have

$$W_{n+1}(t) = W_n(t)\ H + W_n(t)\ W(t) + H^n\ W(t). \tag{D.3}$$

Let us denote $a_n(t) = \|H^n \ W(t) \ H^{-n-\nu}\|$ and $f_n(t) = \|W_n(t) \ H^{1-n-\nu}\|$. By induction on n, using (D.3), we get

$$f_{p+1}(t) \leq a_p(t) + f_p(t) + \gamma_0 \, a_p(t) \, f_p(t) \tag{D.4}$$

where γ_0 is a constant depending only on H. From (D.4) we get easily (D.1).

Now we can conclude easily to get (D.2) using the interpolation inequality: for $0 \leq s < n$ and $\varepsilon \in]0,1]$ we have:

$$\|H^s \psi\|_0^2 \leq \varepsilon^2 \|H^n \psi\|_0^2 + \varepsilon^{\frac{2s}{s-n}} \|\psi\|_0^2.$$

From (D.1) we have

$$\|W_n(t)\psi\|_0 \leq \|W_n(t) \ H^{1-n-\nu}\| \ \|H^{n+\nu-1}\psi\|_0 \ .$$

Taking $s = n + \nu - 1$ and ε small enough, we get (D.2) where c_n and C_n depend only on D_n. □

Proof of Lemma 3.8. (i) Recall that $H_{ad,m}(t) = H + V(t) - B_m(t)$. We apply Lemma D.1 with $W = V - B_m$. By the assumptions on V and Lemma 3.5, W fulfills $(\mathrm{W})_n$, thus we get (3.22).

(ii) If $\mathtt{J}$ is sufficiently large, by Lemma 3.2 the Hamiltonian $H_{ad,m}(t)$ satisfy $\widetilde{(\mathrm{Hgap})}$ uniformly in $t \in \mathbb{R}$ (see Corollary 3.7). Then writing

$$H_{ad,m}(t) = \sum_{j \geq 1} \Pi_{m,j}(t) \, H_{ad,m}(t) \, \Pi_{m,j}(t) \ ,$$

one gets easily that

$$\begin{aligned} \sum_{j \geq 1} (\lambda_j^- + c_0)^{2p} \|\Pi_{m,j}(t)\psi\|_0^2 &\leq \|(H_{ad,m}(t) + c_0)^p \psi\|_0^2 \\ &\leq \sum_{j \geq 1} (\lambda_j^+ + c_0)^{2p} \|\Pi_{m,j}(t)\psi\|_0^2 \end{aligned} \tag{D.5}$$

and

$$\sum_{j \geq 1} (\lambda_j^+ + c_0)^{2p} \|\Pi_{m,j}(t)\psi\|_0^2 \leq C_p 2^{\mathtt{J}(\mu+1)2p} \|\Lambda_m^p(t)\psi\|_0^2$$
$$\sum_{j \geq 1} (\lambda_j^- + c_0)^{2p} \|\Pi_{m,j}(t)\psi\|_0^2 \geq c_p 2^{\mathtt{J}(\mu+1)2p} \|\Lambda_m^p(t)\psi\|_0^2$$

from which (3.23) follows. □

Appendix E. Some properties of the pseudodifferential calculus

We recall here some fundamental results of symbolic calculus. For the proof see [31,18].

Theorem E.1 *(Symbolic calculus I). Let $A \in \widetilde{S}^{\nu}_{k,\ell}$, $B \in \widetilde{S}^{\mu}_{k,\ell}$ be symbols. Then there exists a unique semi-classical symbol $A\sharp B \in \widetilde{S}^{\nu+\mu}_{k,\ell}$ such that $\mathrm{Op}^W_\hbar(A)\,\mathrm{Op}^W_\hbar(B) = \mathrm{Op}^W_\hbar(A\sharp B)$. $A\sharp B$ is the* Moyal product *of A and B.*

The Moyal product is a bilinear continuous map. More precisely it holds the following: for every α, β, there exists a positive constant $C_{\alpha\beta}$ (independent of A and B) and an integer $M \equiv M(\alpha, \beta) \geq 1$ such that

$$p^{\nu+\mu}_{\alpha\beta}(A\sharp B) \leq C_{\alpha\beta}\, |a|_{M,\nu}\, |b|_{M,\mu} \ .$$

Theorem E.2 *(Symbolic calculus II). Let $A \in \widetilde{S}^{\nu}_{k,\ell}$, $B \in S^{\mu}_{k,\ell}$ be symbols. Then there exists a unique semi-classical symbol $A\sharp B \in \widetilde{S}^{\nu+\mu}_{k,\ell}$ such that $\mathrm{Op}^W_\hbar(A)\,\mathrm{Op}^W_\hbar(B) = \mathrm{Op}^W_\hbar(A\sharp B)$. $A\sharp B$ is the* Moyal product *of A and B.*

Theorem E.2 is useful in case $\mu < 0$, so that the symbol $A\sharp B$ gains in decay at infinity. The symbolic calculus implies the following result on the commutator of two pseudodifferential operators:

Corollary E.3 *(Commutator). Let $A \in \widetilde{S}^{\nu}_{k,\ell}$, $\nu \geq 0$, $B \in \widetilde{S}^{\mu}_{k,\ell}$ be symbols. Then there exists a unique semi-classical symbol $C \in \widetilde{S}^{\nu+\mu-(k+\ell)}_{k,\ell}$ such that $[\mathrm{Op}^W_\hbar(A), \mathrm{Op}^W_\hbar(B)] = \mathrm{Op}^W_\hbar(C)$.*

The second result concerns the boundedness of pseudodifferential operators:

Theorem E.4 *(Calderon–Vaillancourt). Let $A \in \widetilde{S}^{0}_{k,\ell}$ be a symbol. Then there exist constants $C, N > 0$ such that $\mathrm{Op}^W_\hbar(A)$ extends to a linear bounded operator from L^2 to itself, and the following estimate holds:*

$$\|\mathrm{Op}^W_\hbar(A)\|_{\mathcal{L}(L^2)} \leq C\ |A|_{N,0}\ , \quad \forall \hbar \in\,]0,1]. \tag{E.1}$$

Notice that C and N are universal constants, independent on A (see for example [32]).

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