7. Periodic Pseudodifferential Operators

In this chapter we present a systematic theory of periodic pseudodifferential operators. In next chapters the pseudodifferential structure of periodic integral operators will be extensively used by constructing fast solvers for integral equations.

From the point of view of the theory of pseudodifferential operators in $\mathbb{R}^n$ (see, e.g.,[KN65],[See69],[Hör85],[Tay81], [Tre82]), periodic pseudodifferential operators present a rather special case of those on a submanifold of $\mathbb{R}^2$ – the unit circle. Unfortunately, the local theory using the manifold charts is impractical. Due to Agranovich [Agr79],[Agr85],[Agr94], we have an equivalent global definition of periodic pseudodifferential operators on the basis of Fourier series and the so called symbol function of the operator; see also [Els85],[SW87],[McL91],[TV98],[Vai99],[Tur00].

We do not assume any acquaintance of the reader with the theory of pseudodifferential operators in $\mathbb{R}^n$. In the contrary, this chapter could be used as a helpful bridge to more general and complicated theories.

7.1 Prolongation of a Function Defined on $\mathbb{Z}$

Here we present some preliminaries that we will use analyzing definitions of periodic pseudodifferential operators. Introduce the space $\mathcal{S}(\mathbb{R})$ of all C^∞-smooth functions which are rapidly decreasing at infinity: for a $f \in \mathcal{S}(\mathbb{R})$ we have

$$\sup_{x\in\mathbb{R}} |x^k f^{(l)}(x)| < \infty \qquad \forall\, k, l \in \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi x\xi} f(x)\, dx \qquad (\xi \in \mathbb{R}), \qquad f \in \mathcal{S}(\mathbb{R}),$$

$$(\mathcal{F}^{-1}g)(x) = \tilde{g}(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi x\xi} g(\xi)\, d\xi \qquad (x \in \mathbb{R}), \qquad g \in \mathcal{S}(\mathbb{R}).$$

A fundamental fact is that both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are one-to-one mappings from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$, and $\mathcal{F}^{-1}\mathcal{F}f = \mathcal{F}\mathcal{F}^{-1}f = f$ for a $f \in \mathcal{S}(\mathbb{R})$. A proof can be found e.g. in [Yos65].

In Chapter 5 we introduced a function $\theta \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}\theta \subset [-\frac{2}{3}, \frac{2}{3}]$ and

$$\sum_{k\in\mathbb{Z}} \theta(x+k) = 1 \quad (x \in \mathbb{R}).$$

For $x \in [0,1]$ only two terms of this series do not vanish, and we have

$$\theta(x) + \theta(x-1) = 1 \quad (0 \le x \le 1).$$

Now we will use the Fourier transform of this function for interpolation purposes. Denote $\varphi = \mathcal{F}\theta$. Clearly $\theta \in \mathcal{S}(\mathbb{R})$, therefore $\varphi \in \mathcal{S}(\mathbb{R})$. It follows from the properties of θ that

$$\varphi(k) = \delta_{0k} = \begin{cases} 1, & k = 0, \\ 0, & 0 \ne k \in \mathbb{Z}. \end{cases} \tag{7.1}$$

Indeed,

$$\begin{aligned}\varphi(k) &= (\mathcal{F}\theta)(k) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi xk}\,\theta(x)\,dx = \left(\int_{-1}^{0} + \int_{0}^{1}\right)\theta(x)\,\mathrm{e}^{-\mathrm{i}k2\pi x}\,dx \\ &= \int_0^1 \big[\theta(x-1) + \theta(x)\big]\,\mathrm{e}^{-\mathrm{i}k2\pi x}\,dx = \int_0^1 \mathrm{e}^{-\mathrm{i}k2\pi x}\,dx = \delta_{0k}.\end{aligned}$$

Further, we prove that for any $l \in \mathbb{N}$, there is a function $\varphi_l \in \mathcal{S}(\mathbb{R})$ such that

$$\varphi^{(l)}(\xi) = \Delta^l \varphi_l(\xi) \qquad (\xi \in \mathbb{R}). \tag{7.2}$$

We use the notations

$$\Delta\psi(\xi) = \psi(\xi+1) - \psi(\xi), \quad \overline{\Delta}\psi = \psi(\xi) - \psi(\xi-1) \quad (\xi \in \mathbb{R}),$$

and call to attention the formula of summation by parts,

$$\sum_{k\in\mathbb{Z}} [\Delta\psi(k)]\chi(k) = -\sum_{k\in\mathbb{Z}} \psi(k)\overline{\Delta}\chi(k),$$

which holds for all functions $\psi, \chi : \mathbb{Z} \to \mathbb{C}$ such that one of the two series converges and $\psi(\pm N)\chi(\pm N - 1) \to 0$ as $N \to \infty$ (then also the other series converges).

Instead of (7.2), we prove an equivalent relation: there is a function $\varphi_l \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}^{-1}\varphi^{(l)} = \mathcal{F}^{-1}\Delta_l\varphi^{(l)}$, i.e.

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi x\xi}\,\varphi^{(l)}(\xi)\,d\xi = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi x\xi}\,\Delta^l\varphi_l(\xi)\,d\xi \qquad (x \in \mathbb{R}).$$

Integrating by parts in the left hand side and using shifts in the right hand side we rewrite the condition in the form

$$(-\,\mathrm{i}2\pi x)^l\tilde{\varphi}(x) = (\mathrm{e}^{-\,\mathrm{i}2\pi x}-1)^l\tilde{\varphi}_l(x)\,, \qquad x \in \mathbb{R},$$

resulting to the formula

$$\tilde{\varphi}_l(x) = \left(\frac{-\,\mathrm{i}2\pi x}{\mathrm{e}^{-\,\mathrm{i}2\pi x}-1}\right)^l \theta(x)\,.$$

Note that $\tilde{\varphi}_l \in \mathcal{S}(\mathbb{R})$ and consequently also $\varphi_l = \mathcal{F}\tilde{\varphi}_l \in \mathcal{S}(\mathbb{R})$. Indeed, the factor $-\,\mathrm{i}2\pi x/(\mathrm{e}^{-\,\mathrm{i}2\pi x}-1)$ is C^∞-smooth in a neighborhood of $x = 0$, and the singularities at $0 \neq k \in \mathbb{Z}$ cannot influence since θ is supported on $[-\frac{2}{3}, \frac{2}{3}]$. We see that the $\varphi_l \in \mathcal{S}(\mathbb{R})$ satisfying (7.2) exists and is unique.

Property (7.1) allows us to use the function φ for prolongation of functions $\sigma : \mathbb{Z} \to \mathbb{C}$ up to functions $p\sigma : \mathbb{R} \to \mathbb{C}$ setting

$$(p\sigma)(\xi) = \sum_{k\in\mathbb{Z}} \sigma(k)\varphi(\xi - k) \qquad (\xi \in \mathbb{R})\,. \tag{7.3}$$

Clearly, $(p\sigma)(k) = \sigma(k)$ $(k \in \mathbb{Z})$, provided that the series in the right hand converges (locally) uniformly.

Lemma 7.1.1. *Let the function $\sigma : \mathbb{Z} \to \mathbb{C}$ satisfy with an $\alpha \in \mathbb{R}$ the inequalities*

$$|\Delta^l\sigma(k)| \le c_l\,\underline{k}^{\alpha-l} \qquad (k \in \mathbb{Z}\,,\ l \in \mathbb{N}_0)\,. \tag{7.4}$$

Then $p\sigma \in C^\infty(\mathbb{R})$ and

$$\left|\left(\frac{d}{d\xi}\right)^l (p\sigma)(\xi)\right| \le c'_l c_l\,|\xi|^{\alpha-l} \quad (|\xi| \ge 1\,,\ l \in \mathbb{N}_0) \tag{7.5}$$

where the constant c'_l is independent of the function σ.

Proof. According to (7.2), the formal differentiation of (7.3) under the sum yields

$$\left(\frac{d}{d\xi}\right)^l (p\sigma)(\xi) = \sum_{k\in\mathbb{Z}} \sigma(k)\Delta_\xi^l\varphi_l(\xi - k).$$

Notice that $\Delta_\xi\varphi_l(\xi - k) = \varphi_l(\xi - k + 1) - \varphi_l(\xi - k) = -\overline{\Delta}_k\varphi_l(\xi - k)$. Using the summation by parts we rewrite

$$\left(\frac{d}{d\xi}\right)^l (p\sigma)(\xi) = (-1)^l\sum_{k\in\mathbb{Z}} \sigma(k)\overline{\Delta}_k^l\varphi_l(\xi - k) = \sum_{k\in\mathbb{Z}} [\Delta^l\sigma(k)]\,\varphi_l(\xi - k)\,.$$

Since $\varphi_l \in \mathcal{S}(\mathbb{R})$, (7.4) implies the uniform convergence of the last series, as well as the series (7.3) itself, and this justifies the formal differentiation. Thus $p\sigma \in C^\infty(\mathbb{R})$ and

$$\left|\left(\frac{d}{d\xi}\right)^l (p\sigma)(\xi)\right| \le c_l \sum_{k\in\mathbf{Z}} \underline{k}^{\alpha-l}|\varphi_l(\xi-k)|$$

$$\le c_l\left[c_{l,r}\sum_{\substack{k\in\mathbf{Z}\\|k-\xi|\ge 1}} \underline{k}^{\alpha-l}|\xi-k|^{-r} + \sum_{|k-\xi|<1}\underline{k}^{\alpha-l}\max_{x\in\mathbb{R}}|\varphi_l(x)|\right] \quad (|\xi|\ge 1)$$

where $r > 0$ is arbitrary (we fix it sufficiently large). Clearly, for $|k-\xi| < 1$, $|\xi| \ge 1$ we have $\frac{1}{2} \le \underline{k}/|\xi| \le 2$ which implies $\underline{k}^{\alpha-l} \le 2^{|\alpha-l|}|\xi|^{\alpha-l}$. We have to prove that, with a sufficiently large r, also

$$\sum_{k\in\mathbf{Z}:|k-\xi|\ge 1} \underline{k}^{\alpha-l}|\xi-k|^{-r} \le c_{\alpha-l,r}|\xi|^{\alpha-l}.$$

It is more convenient to prove an equivalent inequality

$$\sum_{k\in\mathbf{Z}:|k-\xi|\ge 1} \underline{k}^{\beta}|\xi|^{-\beta}|\xi-k|^{-r} \le c_{\beta,r} \qquad (|\xi|\ge 1,\ r>|\beta|+1). \tag{7.6}$$

If $\beta \ge 0$ we estimate $\underline{k}^\beta \le (|k-\xi|+|\xi|)^\beta \le 2^\beta(|k-\xi|^\beta+|\xi|^\beta)$; if $\beta < 0$ we estimate $|\xi|^{-\beta} \le 2^{-\beta}(|\xi-k|^{-\beta}+\underline{k}^{-\beta})$, and this easily results to (7.6). Indeed, in the case $\beta \ge 0$,

$$\sum_{k\in\mathbf{Z}:|k-\xi|\ge 1} \underline{k}^{\beta}|\xi|^{-\beta}|\xi-k|^{-r}$$
$$\le 2^\beta\left[\sum_{k\in\mathbf{Z}:|k-\xi|\ge 1}|\xi|^{-\beta}|k-\xi|^{-r+\beta} + \sum_{k\in\mathbf{Z}:|k-\xi|\ge 1}|k-\xi|^{-r}\right]$$
$$\le 2^\beta[|\xi|^{-\beta}c_{r-\beta}+c_r] \le 2^\beta(c_{r-\beta}+c_r) \qquad (|\xi|\ge 1),$$

and in the case $\beta < 0$,

$$\sum_{k\in\mathbf{Z}:|k-\xi|\ge 1} \underline{k}^{\beta}|\xi|^{-\beta}|\xi-k|^{-r} \le$$
$$2^{|\beta|}\left[\sum_{k\in\mathbf{Z}:|k-\xi|\ge 1}\underline{k}^\beta|\xi-k|^{-r+|\beta|} + \sum_{k\in\mathbf{Z}:|k-\xi|\ge 1}|\xi-k|^{-r}\right] \le 2^{|\beta|}\Big(c_{r-|\beta|}+c_r\Big).$$

□

7.2 Two Definitions of PPDO and Their Equivalence

Let us begin with the observation that any operator $A \in \mathcal{L}(H^\lambda, H^\mu)$ can be represented in the form

$$(Au)(t) = \sum_{n\in\mathbf{Z}}\sigma(t,n)\hat{u}(n)\,\mathrm{e}^{in2\pi t}, \quad \sigma(t,n) = \sigma_A(t,n) = \mathrm{e}_{-n}(t)(A\,\mathrm{e}_n)(t)$$

where $e_n(t) = e^{in2\pi t}$. For $u \in H^\lambda$, the series converges in H^μ. Indeed, using the Fourier representation of u we find

$$\begin{aligned}(Au)(t) &= \Big(A\sum_{n\in\mathbb{Z}}\hat{u}(n)\,e_n\Big)(t) = \sum_{n\in\mathbb{Z}}\hat{u}(n)(A\,e_n)(t) \\ &= \sum_{n\in\mathbb{Z}}\hat{u}(n)\,e^{-in2\pi t}(A\,e_n)(t)\,e^{in2\pi t} = \sum_{n\in\mathbb{Z}}\sigma(t,n)\hat{u}(n)\,e^{in2\pi t}.\end{aligned}$$

The function $\sigma(t,n)$ ($t \in \mathbb{R}$, $n \in \mathbb{Z}$) is called the *symbol* of the operator A. Clearly $\sigma(t,n)$ is 1-periodic in t.

Definition 7.2.1. *A linear operator A defined by*

$$(Au)(t) = \sum_{n\in\mathbb{Z}}\sigma(t,n)\hat{u}(n)\,e^{in2\pi t} \tag{7.7}$$

is called periodic pseudodifferential operator *(PPDO) of order $\le \alpha$ if its symbol $\sigma(t,n)$ is 1-periodic and C^∞-smooth in t and satisfies the inequalities*

$$\left|\left(\frac{\partial}{\partial t}\right)^k \Delta_n^l\sigma(t,n)\right| \le c_{k,l}\,\underline{n}^{\alpha-l} \qquad (t\in\mathbb{R},\ n\in\mathbb{Z},\ k,l\in\mathbb{N}_0). \tag{7.8}$$

Here the subindex n in $\Delta_n^l\sigma(t,n)$ indicates that the differences are taken with respect to the variable n. The set of all symbols satisfying (7.8) will be denoted by Σ^α, and the set of all PPDOs of order $\le \alpha$ will be denoted by $\mathrm{Op}\,\Sigma^\alpha$ and $\mathrm{Op}(\sigma)$ denotes the PPDO corresponding to σ, i.e. $A = \mathrm{Op}(\sigma)$ is given by (7.7). We present also a second definition which occurs to be equivalent to the first one.

Definition 7.2.2. *A linear operator A is called PPDO of order $\le \alpha$ if its symbol $\sigma(t,n)$ is the restriction to $\mathbb{R}\times\mathbb{Z}$ of a function $\sigma(t,\xi)$ which is C^∞-smooth on $\mathbb{R}\times\mathbb{R}$, 1-periodic in t and satisfies the inequalities*

$$\left|\left(\frac{\partial}{\partial t}\right)^k\left(\frac{\partial}{\partial \xi}\right)^l\sigma(t,\xi)\right| \le c_{k,l}(1+|\xi|)^{\alpha-l} \qquad (t\in\mathbb{R},\ \xi\in\mathbb{R},\ k,l\in\mathbb{N}_0). \tag{7.9}$$

Theorem 7.2.1. *Definitions 7.2.1 and 7.2.2 are equivalent.*

Proof. If $\sigma(t,\xi)$ satisfies (7.9) then its restriction $\sigma(t,n)$ satisfies (7.8) since

$$\Delta_n^l\sigma(t,n) = \Delta_\xi^l\sigma(t,\xi)\Big|_{\xi=n} = \left(\frac{\partial}{\partial\xi}\right)^l\sigma(t,\xi)\Big|_{\xi=n+\varepsilon l} \quad \text{with an} \quad \varepsilon\in(0,1).$$

Conversely, if $\sigma(t,n)$ satisfies (7.8) then, due to Lemma 7.1.1, its prolongation $(p\sigma)(t,\xi)$ satisfies (7.9). □

Definitions 7.2.1 and 7.2.2 originate from [Agr79],[Agr85]. Also the equivalence of the definitions is mentioned there. A general theory of pseudodifferential operators in a region $\Omega \subseteq \mathbb{R}^n$ can be found e.g. in [Hör85], [Tay81] or [Tre82]. Definition 7.2.1 is more convenient in some applications. On the other hand, Definition 7.2.2 is closer to the definition of a pseudodifferential operator in $\mathbb{R}^n$ enabling a more straightforward transfer of many fundamental results, e.g. concerning the symbol analysis of PPDOs, see Sections 7.4–7.8.

Let us shortly discuss first examples of PPDOs. Consider a differential operator

$$(Au)(t) = \sum_{j=0}^{m} a_j(t) u^{(j)}(t)$$

with 1-periodic C^∞-smooth coefficients $a_j(t)$. We have

$$(Au)(t) = A \sum_{n\in\mathbb{Z}} \hat{u}(n)\, e^{in2\pi t} = \sum_{n\in\mathbb{Z}} \hat{u}(n) \sum_{j=0}^{m} a_j(t)(in2\pi)^j\, e^{in2\pi t},$$

$$\sigma(t,n) = \sum_{j=0}^{m} a_j(t)(2\pi i\, n)^j .$$

Condition (7.8) is fulfilled with $\alpha = m$. Thus, a differential operator of order m belongs to $\operatorname{Op}\Sigma^m$. Formally we may write $A = \sigma(t, D)$ where $D = \frac{1}{2\pi i}\frac{\partial}{\partial t}$. The notation $A = \sigma(t, D)$ is often used also for general operators defined by (7.7).

As a second example, let us consider the integral operator

$$(Au)(t) = a(t) \int_0^1 \kappa(t-s) u(s)\, ds \tag{7.10}$$

where $a(t)$ is a 1-periodic C^∞-smooth function and κ is a 1-periodic function or distribution such that, with an $\alpha \in \mathbb{R}$,

$$|\Delta^l \hat{\kappa}(n)| \le c_l\, \underline{n}^{\alpha-l} \qquad (n \in \mathbb{Z}\,,\ l \in \mathbb{N}_0)\,. \tag{7.11}$$

Using Theorem 5.5.1 we find that

$$(Au)(t) = \sum_{n\in\mathbb{Z}} \hat{u}(n) a(t) \int_0^1 \kappa(t-s)\, e^{in2\pi s}\, ds = \sum_{n\in\mathbb{Z}} a(t)\hat{\kappa}(n)\hat{u}(n)\, e^{in2\pi t},$$

$$\sigma(t,n) = a(t)\hat{\kappa}(n)\,.$$

Due to (7.11), $\sigma(t,n)$ satisfies (7.8), and $A \in \operatorname{Op}\Sigma^\alpha$. In Section 7.6 we will see that a class of more general integral operators can be interpreted as PPDOs.

Exercises

Exercise 7.2.1. Extend the examples of PPDOs to integro-differential operators proving that $A_j \in \operatorname{Op}\Sigma^{\alpha_j}$ $(j = 1,2)$ implies $A_1 + A_2 \in \operatorname{Op}\Sigma^{\max(\alpha_1,\alpha_2)}$.

Exercise 7.2.2. Check that the IOs presented in Sections 5.6–5.10 are PPDOs. (You must check (7.11) for those IOs.)

7.3 Boundedness of a PPDO

We are ready to prove that any $A \in \mathrm{Op}\,\Sigma^\alpha$ is bounded from H^λ to $H^{\lambda-\alpha}$ for all $\lambda \in \mathbb{R}$. This assertion does not need the condition (7.8) to a full extent, it is sufficient if (7.8) holds only for $l = 0$:

Theorem 7.3.1. *Assume that*

$$\left|\left(\frac{\partial}{\partial t}\right)^k \sigma(t,n)\right| \le c_k \underline{n}^\alpha \qquad (t \in \mathbb{R},\ n \in \mathbb{Z},\ k \in \mathbb{N}_0). \tag{7.12}$$

Then for the operator A defined by (7.7) we have $A \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ for any $\lambda \in \mathbb{R}$.

Proof. Using the Fourier expansions $\sigma(t,n) = \sum_{m\in\mathbb{Z}} \hat\sigma(m,n)\,\mathrm{e}^{\mathrm{i}m2\pi t}$ $(n \in \mathbb{Z})$ we represent A in the form

$$\begin{aligned}(Au)(t) &= \sum_{n\in\mathbb{Z}} \sigma(t,n)\hat u(n)\,\mathrm{e}^{\mathrm{i}n2\pi t} = \sum_{m,n\in\mathbb{Z}} \hat\sigma(m,n)\hat u(n)\,\mathrm{e}^{\mathrm{i}(m+n)2\pi t}\\ &= \sum_{k\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}} \hat\sigma(k-n,n)\hat u(n)\right]\mathrm{e}^{\mathrm{i}k2\pi t}.\end{aligned}$$

Thus

$$\|Au\|_{\lambda-\alpha} = \left\{\sum_{k\in\mathbb{Z}} \underline{k}^{2(\lambda-\alpha)}\left|\sum_{n\in\mathbb{Z}} \hat\sigma(k-n,n)\hat u(n)\right|^2\right\}^{1/2}.$$

Integrating by parts and using condition (7.12) we find that for $m \ne 0$

$$\hat\sigma(m,n) = \int_0^1 \sigma(t,n)\,\mathrm{e}^{-\mathrm{i}m2\pi t}\,dt = \left(\frac{1}{\mathrm{i}m2\pi}\right)^r \int_0^1 \frac{\partial^r\sigma}{\partial t^r}(t,n)\,\mathrm{e}^{-\mathrm{i}m2\pi t}\,dt,$$

$$|\hat\sigma(m,n)| \le c_r\,\underline{m}^{-r}\underline{n}^\alpha \qquad (m,\, n \in \mathbb{Z}) \qquad \forall r \in \mathbb{N}_0. \tag{7.13}$$

The estimate of $\|Au\|_{\lambda-\alpha}$ can be continued as follows:

$$\begin{aligned}\|Au\|_{\lambda-\alpha} &\le c_r\left\{\sum_{k\in\mathbb{Z}} \underline{k}^{2(\lambda-\alpha)}\left[\sum_{n\in\mathbb{Z}} (\underline{k-n})^{-r}\underline{n}^\alpha|\hat u(n)|\right]^2\right\}^{1/2}\\ &= c_r\left\{\sum_{k\in\mathbb{Z}} \underline{k}^{2(\lambda-\alpha)}\left|(\widehat{bv})(k)\right|^2\right\}^{1/2} = c_r\,\|bv\|_{\lambda-\alpha}\end{aligned}$$

where the functions b and v are defined by their Fourier coefficients

$$\hat{b}(n) = \underline{n}^{-r}, \quad \hat{v}(n) = \underline{n}^{\alpha}|\hat{u}(n)| \qquad (n \in \mathbb{Z}).$$

The norm $||bv||_{\lambda-\alpha}$ can be estimated with the help of Lemma 5.13.1 obtaining

$$||Au||_{\lambda-\alpha} \le c\,||b||_{\max(|\lambda-\alpha|,\nu)}||v||_{\lambda-\alpha} = c\,||b||_{\max(|\lambda-\alpha|,\nu)}||u||_{\lambda} \quad (\nu > \tfrac{1}{2})$$

with a constant $c = c(r, \lambda, \alpha, \nu)$. This proves the boundedness of A considered as an operator from H^{λ} to $H^{\lambda-\alpha}$. Note that the norm $||b||_{\mu}$ is finite if r is taken sufficiently large ($r > \mu + \frac{1}{2}$). □

7.4 Asymptotic Expansion of the Symbol

Definition 7.4.1. *Let $\sigma \in \Sigma^{\alpha_0}$, $\sigma_j \in \Sigma^{\alpha_j}$ ($j = 0, 1, 2, \ldots$), where $\alpha_0 > \alpha_1 > \alpha_2 > \ldots$, $\alpha_j \to -\infty$, and*

$$\sigma - \sum_{j=0}^{N-1} \sigma_j \in \Sigma^{\alpha_N} \qquad \text{for all } N \in \mathbb{N}. \tag{7.14}$$

Then the series $\sum_{j=0}^{\infty} \sigma_j$ is called an asymptotic expansion *of σ (we write $\sigma \sim \sum_{j=0}^{\infty} \sigma_j$) , and σ_0 is called the* principal symbol.

Notice that the series $\sum \sigma_j$ itself must not converge.

Lemma 7.4.1. *Let $\sigma_j \in \Sigma^{\alpha_j}$ ($j = 0, 1, 2, \ldots$), $\alpha_j \to -\infty$ monotonically. Then there exists a symbol $\sigma \in \Sigma^{\alpha_0}$ such that $\sigma \sim \sum_{j=0}^{\infty} \sigma_j$.*

Proof (outlines). We define σ by the formula

$$\sigma(t, n) = \sum_{j=0}^{\infty} \varphi(\varepsilon_j n)\sigma_j(t, n) \qquad (t \in \mathbb{R},\ n \in \mathbb{Z}) \tag{7.15}$$

where $\varphi \in C^{\infty}(\mathbb{R})$ satisfies $\varphi(\xi) = 0$ for $|\xi| \le 1/2$ and $\varphi(\xi) = 1$ for $|\xi| \ge 1$, and $\varepsilon_j > 0$ are chosen so small that

$$\left|\left(\frac{\partial}{\partial t}\right)^k \Delta^l\left(\varphi(\varepsilon_j n)\sigma_j(t, n)\right)\right| \le 2^{-j}\underline{n}^{\alpha_j - l + 1} \text{ for } k + l \le j,\ t \in \mathbb{R},\ n \in \mathbb{Z}.$$

It is easy to check that such ε_j exist, the series (7.15) and the series which we obtain applying $\left(\frac{\partial}{\partial t}\right)^k \Delta^l$ under the sum converge uniformly, and σ satisfies (7.14). □

Notice that σ is non-uniquely defined by its asymptotic expansion $\sum \sigma_j$. Indeed, if $\sigma \sim \sum \sigma_j$ then also $\sigma + \sigma_{-\infty} \sim \sum \sigma_j$ with any $\sigma_{-\infty} \in \Sigma^{-\infty}$ where $\Sigma^{-\infty}$ consists of symbols which satisfy

$$\left| \left(\frac{\partial}{\partial t} \right)^k \sigma_{-\infty}(t,n) \right| \leq c_{k,r} \underline{n}^{-r} \quad (n \in \mathbb{Z}\,,\ t \in \mathbb{R},\ k \in \mathbb{N}_0) \text{ with any } r > 0.$$

Conversely, if σ and $\tilde{\sigma}$ have the same asymptotic expansion then $\sigma - \tilde{\sigma} \in \Sigma^{-\infty}$. Thus a symbol can be reconstructed from its asymptotic expansion with the accuracy of an additive function of the class $\Sigma^{-\infty}$.

To an asymptotic expansion $\sigma \sim \sum_{j=0}^{\infty} \sigma_j$ of symbols , there corresponds the *asymptotic expansion* $A \sim \sum_{j=0}^{\infty} A_j$ of corresponding operators $A = \mathrm{Op}(\sigma)$, $A_j = \mathrm{Op}(\sigma_j)$. Relation (7.14) implies that

$$A - \sum_{j=0}^{N-1} A_j \in \mathrm{Op}\,\Sigma^{\alpha_N} \qquad (N = 1, 2, \dots)$$

(but nothing can be said about the smallness of this difference in operator norms).

An operator $A_{-\infty}$ with a symbol $\sigma_{-\infty} \in \Sigma^{-\infty}$ is infinitely smoothing (is bounded from any H^λ to any H^μ, $\lambda, \mu \in \mathbb{R}$). It can be represented as the integral operator

$$(A_{-\infty}u)(t) = \int_0^1 a(t,s)u(s)\,ds$$

with the 1-biperiodic C^∞-smooth kernel

$$a(t,s) = \sum_{n \in \mathbb{Z}} \sigma_{-\infty}(t,n)\, e^{in2\pi(t-s)}\,.$$

Thus, an asymptotic expansion of the symbol of a PPDO enables to reconstruct the operator with the accuracy of an infinitely smoothing additive operator.

Exercise 7.4.1. Present a detailed proof of Lemma 7.4.1.

7.5 Amplitudes

Amplitude is a natural extension of the concept of symbol and it is a useful tool in the symbol analysis.

Definition 7.5.1. *A function $a(t,s,n)$ $(t, s \in \mathbb{R}, n \in \mathbb{Z})$ is called an* amplitude *of order $\leq \alpha$ if it is C^∞-smooth and 1-periodic with respect to t and s, and*

$$\left|\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial s}\right)^k \Delta_n^l a(t,s,n)\right| \le c_{j,k,l}\underline{n}^{\alpha-l} \qquad (t,s\in\mathbb{R},\ n\in\mathbb{Z},\ j,k,l\in\mathbb{N}_0). \tag{7.16}$$

We write $a \in \mathcal{A}^\alpha$ in this case.

Using Lemma 7.1.1, such an amplitude can be extended up to a C^∞-smooth function $a : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{C}$ which is 1-periodic with respect to t and s and satisfies

$$\left|\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial s}\right)^k\left(\frac{\partial}{\partial \xi}\right)^l a(t,s,\xi)\right| \le c'_{j,k,l}(1+|\xi|)^{\alpha-l} \ (t,s,\xi\in\mathbb{R},\ j,k,l\in\mathbb{N}_0). \tag{7.17}$$

Conversely, if a satisfies (7.17) then its restriction to $\mathbb{R}\times\mathbb{R}\times\mathbb{Z}$ satisfies (7.16).

For $a \in \mathcal{A}^\alpha$ we define the amplitude operator $A = \mathrm{Op}(a)$ by

$$(Au)(t) = \int_0^1 \sum_{n\in\mathbb{Z}} a(t,s,n)\, e^{in2\pi(t-s)}\, u(s)\, ds. \tag{7.18}$$

For $\alpha < -1$, the kernel $\mathcal{K}(t,s) = \sum_{n\in\mathbb{Z}} a(t,s,n)\, e^{in2\pi(t-s)}$ is a continuous function. For $\alpha \ge -1$ and a smooth function u, (7.18) is to be interpreted as a result of a formal integration by parts, being an abbreviation of

$$(Au)(t) = \int_0^1 \sum_{0\ne n\in\mathbb{Z}} \frac{e^{in2\pi(t-s)}}{(2\pi\, in)^q}\left(\frac{\partial}{\partial s}\right)^q \left[a(t,s,n)u(s)\right] ds + \int_0^1 a(t,s,0)u(s)\, ds.$$

Due to (7.16), the kernels

$$\mathcal{K}_{p,q}(t,s) = \sum_{0\ne n\in\mathbb{Z}} \frac{e^{in2\pi(t-s)}}{(2\pi\, in)^q}\left(\frac{\partial}{\partial s}\right)^p a(t,s,n)\,,\ \ 0\le p\le q\,,$$

are continuous for $q > \alpha+1$. A consequence of this interpretation is that the order of summation and integration in (7.18) may be changed.

Clearly, symbols $\sigma(t,n)$ are the amplitudes which are independent of the argument s, and $A = \mathrm{Op}(\sigma)$ has the representation (7.18):

$$\begin{aligned}(Au)(t) &= \sum_{n\in\mathbb{Z}} \sigma(t,n)\hat{u}(n)\, e^{in2\pi t} = \sum_{n\in\mathbb{Z}} \sigma(t,n)\, e^{in2\pi t}\int_0^1 u(s)\, e^{in2\pi s}\, ds \\ &= \int_0^1 \sum_{n\in\mathbb{Z}} \sigma(t,n)\, e^{in2\pi(t-s)}\, u(s)\, ds.\end{aligned}$$

Here, for $\sigma \in \Sigma^\alpha$ with $\alpha < -1$ and $u \in H^0$, the change of the order of integration and summation is legitimate. For $\alpha \ge -1$ and smooth u, again the formal integration by parts should be incorporated into understanding of the representation.

Surprisingly, the sets $\operatorname{Op}\Sigma^\alpha$ and $\operatorname{Op}\mathcal{A}^\alpha$ coincide although $\Sigma^\alpha \subset \mathcal{A}^\alpha$ properly.

Theorem 7.5.1. *For every amplitude $a \in \mathcal{A}^\alpha$ there exists a unique symbol $\sigma \in \Sigma^\alpha$ of the same order such that $\operatorname{Op}(a) = \operatorname{Op}(\sigma)$, and σ has the asymptotic expansions*

$$\sigma(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \Delta_n^j \partial_s^{(j)} a(t,s,n)\Big|_{s=t}, \tag{7.19}$$

$$\sigma(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \partial_s^j a(t,s,\xi)\Big|_{s=t} \tag{7.20}$$

where

$$\partial_s = \frac{1}{2\pi i}\frac{\partial}{\partial s}, \qquad \partial_s^{(0)} = I, \qquad \partial_s^{(j)} = \prod_{l=0}^{j-1}(\partial_s - lI) \quad \text{for } j \geq 1. \tag{7.21}$$

We precede some preliminaries to the proof. Denoting

$$\xi^{(0)} = 1, \qquad \xi^{(j)} = \prod_{l=0}^{j-1}(\xi - l) = \xi(\xi-1)\dots(\xi - l + 1), \quad j \geq 1, \tag{7.22}$$

we can represent

$$\xi^{(k)} = \sum_{j=0}^{k} \alpha_j^{(k)} \xi^j, \qquad \xi^k = \sum_{j=0}^{k} \beta_j^{(k)} \xi^{(j)} \quad (\xi \in \mathbb{R})$$

with (see Exercise 7.5.1)

$$\alpha_j^{(k)} = \frac{1}{j!}\left(\frac{\partial}{\partial \xi}\right)^j \xi^{(k)}\Big|_{\xi=0}, \quad \beta_j^{(k)} = \frac{1}{j!}\Delta^j \xi^k\Big|_{\xi=0} \quad (0 \leq j \leq k) \tag{7.23}$$

called the *Stirling numbers* of the first and second kind, respectively. Comparing (7.21) and (7.22) we see that

$$\partial_s^{(k)} = \sum_{j=0}^{k} \alpha_j^{(k)} \partial_s^j, \qquad \partial_s^k = \sum_{j=0}^{k} \beta_j^{(k)} \partial_s^{(j)}. \tag{7.24}$$

Lemma 7.5.1 (Markov). *For $1 \leq j \leq N-1$, $\varphi \in C^N(\mathbb{R})$, $\xi \in \mathbb{R}$, the formula*

$$\frac{1}{j!}\Delta^j \varphi(\xi) = \sum_{k=j}^{N-1} \beta_j^{(k)} \frac{1}{k!} \varphi^{(k)}(\xi) + \int_0^j b_{j,N}(\eta)\, \varphi^{(N)}(\xi + \eta)\, d\eta \tag{7.25}$$

holds true with

$$b_{j,N}(\eta) = \frac{1}{j!(N-1)!}\left[\Delta_\xi^j(\xi-\eta)_+^{N-1}\right]\Big|_{\xi=0}$$

where

$$(\xi-\eta)_+ = \begin{cases} \xi-\eta, & \xi-\eta \ge 0, \\ 0, & \xi-\eta < 0. \end{cases}$$

Proof. Denote by S the shift operator, $(S\varphi)(\xi) = \varphi(\xi+1)$. Then

$$\Delta^j\varphi(\xi) = (S-I)^j\varphi(\xi) = \sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}S^l\varphi(\xi) = \sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}\varphi(\xi+l).$$

Applying the Taylor formula ($0 \le l \le j$),

$$\begin{aligned}\varphi(\xi+l) &= \sum_{k=0}^{N-1}\frac{1}{k!}\varphi^{(k)}(\xi)l^k + \frac{1}{(N-1)!}\int_0^l (l-\eta)^{N-1}\varphi^{(N)}(\xi+\eta)\,d\eta \\ &= \sum_{k=0}^{N-1}\frac{1}{k!}\varphi^{(k)}(\xi)l^k + \frac{1}{(N-1)!}\int_0^j (l-\eta)_+^{N-1}\varphi^{(N)}(\xi+\eta)\,d\xi\end{aligned}$$

we obtain

$$\begin{aligned}\Delta^j\varphi(\xi) &= \sum_{k=0}^{N-1}\frac{1}{k!}\Big[\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}l^k\Big]\varphi^{(k)}(\xi) \\ &+ \frac{1}{(N-1)!}\int_0^l\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}(l-\eta)_+^{N-1}\varphi^{(N)}(\xi+\eta)d\eta.\end{aligned}$$

Since

$$\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}l^k = \Delta^j\xi^k\big|_{\xi=0} = \begin{cases} j!\beta_j^{(k)}, & k \ge j, \\ 0, & k < j, \end{cases}$$

$$\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}(l-\eta)_+^{N-1} = \left[\Delta_\xi^j(\xi-\eta)_+^{N-1}\right]_{\xi=0},$$

this results to (7.25). □

Formula (7.25) holds trivially also for $j = 0$. Notice that $\beta_0^{(k)} = \delta_{0k}$.

Exercises

Exercise 7.5.1. Check that $\Delta\xi^{(j)} = j\xi^{(j-1)}$, $j \geq 1$. Using this establish the formula (7.23) for the Stirling number $\beta_j^{(k)}$ $(0 \leq j \leq k)$.

Exercise 7.5.2. For $1 \leq j \leq N-1$, prove that $b_{j,N}(\eta) \geq 0$ $(0 \leq \eta \leq j)$. Using this prove that there is an $\eta \in (0, j)$ such that

$$\frac{1}{j!}\Delta^j\varphi(\xi) = \sum_{k=j}^{N-1} \beta_j^{(k)} \frac{1}{k!}\varphi^{(k)}(\xi) + \beta_j^{(N)} \frac{1}{N!}\varphi^{(N)}(\xi+\eta) \quad \text{(Markov formula)}.$$

Exercise 7.5.3. Prove that, for $1 \leq j \leq N-1$, there is an $\eta \in (0, N)$ such that

$$\frac{1}{j!}\varphi^{(j)}(\xi) = \sum_{k=j}^{N-1} \alpha_j^{(k)} \frac{1}{k!}\Delta^k\varphi(\xi) + \alpha_j^{(N)} \frac{1}{N!}\varphi^{(N)}(\xi+\eta) \quad \text{(Markov formula)}.$$

Proof (of Theorem 7.5.1). Let $a \in \mathcal{A}^\alpha$. The operator $A = \mathrm{Op}(a)$ is defined by (7.18), and clearly $A \in \mathcal{L}(H^q, H^0)$ where $\mathbb{N} \ni q > \alpha + 1$. Consequently, the symbol of A is well defined, unique and given by

$$\sigma(t,n) = \mathrm{e}^{-\mathrm{i}n2\pi t}(A\,\mathrm{e}_n)(t) = \mathrm{e}^{-\mathrm{i}n2\pi t}\int_0^1 \sum_{m\in\mathbb{Z}} a(t,s,m)\,\mathrm{e}^{\mathrm{i}m2\pi(t-s)}\,\mathrm{e}^{\mathrm{i}n2\pi s}\,ds$$

where $\mathrm{e}_n(t) = \mathrm{e}^{\mathrm{i}n2\pi t}$. Remembering the interpretation of (7.18), we may change the order of summation and integration:

$$\begin{aligned}\sigma(t,n) &= \sum_{m\in\mathbb{Z}} \int_0^1 a(t,s,m)\,\mathrm{e}^{-\mathrm{i}(m-n)2\pi s}\,ds\,\mathrm{e}^{\mathrm{i}(m-n)2\pi t} \\ &= \sum_{m\in\mathbb{Z}} \hat{a}_2(t, m-n, m)\,\mathrm{e}^{\mathrm{i}(m-n)2\pi t} \\ &= \sum_{m\in\mathbb{Z}} \hat{a}_2(t, m, n+m)\,\mathrm{e}^{\mathrm{i}m2\pi t}\end{aligned}$$

where $\hat{a}_2(t,m,\xi) = \int_0^1 a(t,s,\xi)\,\mathrm{e}^{-\mathrm{i}m2\pi s}\,ds$ are the Fourier coefficients of $a(t,s,\xi)$ as the function of the second argument s; integration by parts and (7.17) yield with any $r > 0$ the estimates

$$\left|\left(\frac{\partial}{\partial t}\right)^p \left(\frac{\partial}{\partial \xi}\right)^q \hat{a}_2(t,m,\xi)\right| \leq c_{p,q,r}\,\underline{m}^{-r}(1+|\xi|)^{\alpha-q}. \qquad (7.26)$$

There is a natural way to extend $\sigma(t,n)$ from $\mathbb{R}\times\mathbb{Z}$ to $\mathbb{R}\times\mathbb{R}$ setting

$$\sigma(t,\xi) = \sum_{m\in\mathbb{Z}} \hat{a}_2(t,m,\xi+m)\,\mathrm{e}^{\mathrm{i}m2\pi t}.$$

By the Taylor formula,

$$\hat{a}_2(t,m,\xi+m) = \sum_{j=0}^{N-1} \frac{1}{j!}\left(\frac{\partial}{\partial\xi}\right)^j \hat{a}_2(t,m,\xi)m^j + R_N(t,m,\xi),$$

$$R_N(t,m,\xi) = \frac{1}{(N-1)!}\int_0^m (m-\eta)^{N-1}\left(\frac{\partial}{\partial\xi}\right)^N \hat{a}_2(t,m,\xi+\eta)\,d\eta,$$

and it follows from (7.26) that

$$\left|\left(\frac{\partial}{\partial t}\right)^p\left(\frac{\partial}{\partial\xi}\right)^l R_N(t,m,\xi)\right| \le \frac{c_{p,l+N,r}}{(N-1)!}\underline{m}^{-r}\int_0^{|m|}(|m|-\eta)^{N-1}(1+|\xi\pm\eta|)^{\alpha-l-N}\,d\eta$$

where "+" and "−" in $|\xi\pm\eta|$ correspond to $m>0$ and $m<0$ respectively. It is easy to check that the following *Peetre's inequality* holds true:

$$(1+|\xi\pm\eta|)^\lambda \le (1+|\xi|)^\lambda(1+|\eta|)^{|\lambda|} \qquad \text{for} \quad \xi,\,\eta,\,\lambda\in\mathbb{R}.$$

Since $|\eta|\le|m|$ in the integral above, we obtain with any $r>0$

$$\left|\left(\frac{\partial}{\partial t}\right)^p\left(\frac{\partial}{\partial\xi}\right)^l R_N(t,m,\xi)\right| \le c_{p,l,N,r}\,\underline{m}^{-r+|\alpha|+l+2N}(1+|\xi|)^{\alpha-l-N}. \quad (7.27)$$

Further,

$$\sum_{m\in\mathbb{Z}} \hat{a}_2(t,m,\xi)m^j\,\mathrm{e}^{\mathrm{i}m2\pi t} = \partial_s^j \sum_{m\in\mathbb{Z}} \hat{a}_2(t,m,\xi)\,\mathrm{e}^{\mathrm{i}m2\pi s}\Big|_{s=t} = \partial_s^j a(t,s,\xi)\Big|_{s=t}.$$

Denoting

$$\sigma_j(t,\xi) = \frac{1}{j!}\left(\frac{\partial}{\partial\xi}\right)^j \partial_s^j a(t,s,\xi)\Big|_{s=t},$$

we obtain

$$\sigma(t,\xi) - \sum_{j=0}^{N-1}\sigma_j(t,\xi) = \sum_{m\in\mathbb{Z}} R_N(t,m,\xi)\,\mathrm{e}^{\mathrm{i}m2\pi t},$$

and by Leibniz rule,

$$\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^l \Big[\sigma(t,\xi) - \sum_{j=0}^{N-1} \sigma_j(t,\xi)\Big]$$
$$= \sum_{p=0}^{k} \binom{k}{p} \sum_{m\in\mathbf{Z}} \left[\left(\frac{\partial}{\partial t}\right)^p \left(\frac{\partial}{\partial \xi}\right)^l R_N(t,m,\xi)\right] (2\pi \mathrm{i}\, m)^{k-p}\, \mathrm{e}^{\mathrm{i}m2\pi t}\,.$$

With the help of (7.27) we get the estimate

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^l \Big[\sigma(t,\xi) - \sum_{j=0}^{N-1} \sigma_j(t,\xi)\Big]\right| \le c_{k,l,N}(1+|\xi|)^{\alpha-N-l}$$

for all $t,\xi \in \mathbb{R}$, $k,l \in \mathbb{N}_0$, $N \in \mathbb{N}$. This means that

$$\sigma - \sum_{j=0}^{N-1} \sigma_j \in \Sigma^{\alpha-N} \qquad (N \in \mathbb{N}). \tag{7.28}$$

Due (7.17), $\sigma_j \in \Sigma^{\alpha-j}$ $(j \in \mathbb{N})$. Consequently $\sigma \in \Sigma^\alpha$ and $\sigma \sim \sum_{j=0}^{\infty} \sigma_j$. The last is the asymptotic expansion (7.20). Further, denoting

$$\tilde\sigma_j(t,\xi) = \frac{1}{j!} \Delta_\xi^j \partial_s^{(j)} a(t,s,\xi)\Big|_{s=t}\,,$$

we prove that

$$\sum_{j=0}^{N-1} (\tilde\sigma_j - \sigma_j) \in \Sigma^{\alpha-N} \qquad (n \in \mathbb{N}). \tag{7.29}$$

Together with (7.28) this implies $\sigma \sim \sum_{j=0}^{\infty} \tilde\sigma_j$ and the asymptotic expansion (7.19). Due to (7.25),

$$\sum_{j=0}^{N-1} \frac{1}{j!} \Delta_\xi^j\, \partial_s^{(j)} a(t,s,\xi) = \sum_{j=0}^{N-1} \sum_{k=j}^{N-1} \beta_j^{(k)} \frac{1}{k!} \left(\frac{\partial}{\partial \xi}\right)^k \partial_s^{(j)} a(t,s,\xi)$$
$$+ \sum_{j=0}^{N-1} \int_0^j b_{j,N}(\eta)\, \partial_s^{(j)} \left(\frac{\partial}{\partial \xi}\right)^N a(t,s,\xi+\eta) d\eta.$$

Changing the order of summation and using (7.24) we see that

$$\sum_{j=0}^{N-1} \sum_{k=j}^{N-1} \beta_j^{(k)} \frac{1}{k!} \left(\frac{\partial}{\partial \xi}\right)^k \partial_s^{(j)} = \sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{\partial}{\partial \xi}\right)^k \sum_{j=0}^{k} \beta_j^{(k)} \partial_s^{(j)} = \sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{\partial}{\partial \xi}\right)^k \partial_s^k.$$

Consequently,

$$\sum_{j=0}^{N-1}\left[\tilde{\sigma}_j(t,\xi)-\sigma_j(t,\xi)\right]=\sum_{j=1}^{N-1}\int_0^1 b_{j,N}(\eta)\left[\partial_s^{(j)}\partial_\xi^N a(t,s,\xi+\eta)\right]_{s=t}d\eta.$$

Using (7.17) we now easily estimate

$$\left|\left(\frac{\partial}{\partial t}\right)^k\left(\frac{\partial}{\partial \xi}\right)^l\sum_{j=0}^{N-1}\left[\tilde{\sigma}_j(t,\xi)-\sigma_j(t,\xi)\right]\right|\le c_{k,l,N}(1+|\xi|)^{\alpha-N-l}\quad (k,l\in\mathbb{N}_0)$$

and this means that (7.29) holds true. □

A consequence of Theorems 7.3.1 and 7.5.1 is that:

$$a\in\mathcal{A}^\alpha \text{ implies } \mathrm{Op}(a)\in\mathcal{L}(H^\lambda,H^{\lambda-\alpha}) \text{ for any } \lambda\in\mathbb{R}.$$

7.6 Asymptotic Expansion of Integral Operators

Consider an integral operator

$$(Au)(t)=\int_0^1 a(t,s)\kappa(t-s)u(s)\,ds \tag{7.30}$$

where $\kappa(t)$ is an 1-periodic function or distribution and $a(t,s)$ is C^∞-smooth 1-biperiodic function.

Theorem 7.6.1. *Assume that*

$$|\Delta^l\hat{\kappa}(n)|\le c_l\underline{n}^{\alpha-l}\qquad (n\in\mathbb{Z}\,,\ l\in\mathbb{N}_0). \tag{7.31}$$

Then the periodic integral operator A defined in (7.30) is a periodic pseudodifferential operator of order $\le\alpha$, and its symbol has the asymptotic expansion

$$\sigma(t,n)\sim\sum_{j=0}^{\infty}\frac{1}{j!}\Delta_n^j\hat{\kappa}(n)\,\partial_s^{(j)}a(t,s)\Big|_{s=t}. \tag{7.32}$$

If $\hat{\kappa}(n)$ is extended up to a C^∞-smooth function $\hat{\kappa}(\xi)$, $\xi\in\mathbb{R}$, satisfying

$$\left|\left(\frac{\partial}{\partial\xi}\right)^l\hat{\kappa}(\xi)\right|\le c_l(1+|\xi|)^{\alpha-l}\qquad(\xi\in\mathbb{R},\ l\in\mathbb{N}_0), \tag{7.33}$$

then the extended symbol has the asymptotic expansion

$$\sigma(t,\xi)\sim\sum_{j=0}^{\infty}\frac{1}{j!}\hat{\kappa}^{(j)}(\xi)\,\partial_s^j a(t,s)\Big|_{s=t} \tag{7.34}$$

(the definitions of ∂_s^j and $\partial_s^{(j)}$ are given in (7.21)).

Proof. Representing $\kappa(t) = \sum_{n\in\mathbb{Z}} \hat{\kappa}(n)\, e^{in2\pi t}$, we have

$$(Au)(t) = \int_0^1 a(t,s) \sum_{n\in\mathbb{Z}} \hat{\kappa}(n)\, e^{in2\pi(t-s)}\, u(s)\, ds.$$

This is (7.18) with $a(t,s,n) = a(t,s)\hat{\kappa}(n)$ which clearly satisfies (7.16). Thus $A \in \mathrm{Op}\,\mathcal{A}^\alpha$, and by Theorem 7.5.1, $A \in \mathrm{Op}\,\Sigma^\alpha$. The asymptotic expansion (7.19) immediately yields (7.32), and (7.20) yields (7.34). □

In the case of (nonintegrable) distribution κ and sufficiently smooth u, we obtain an interpretation of the integral operator (7.30) as an amplitude operator, cf. Section 7.5, the interpretation of (7.18).

According to (7.32), we have $A - \sum_{j=0}^{N-1} A_j \in \mathrm{Op}\,\Sigma^{\alpha-N}$ where

$$(A_j u)(t) = \frac{1}{j!} a_j(t) \sum_{n\in\mathbb{Z}} \hat{u}(n)[\Delta_n^j \hat{\kappa}(n)]\, e^{in2\pi t}, \qquad a_j(t) = \partial_s^{(j)} a(t,s)\Big|_{s=t}. \tag{7.35}$$

Similarly, according to (7.34), we have $A - \sum_{j=0}^{N-1} A_j \in \mathrm{Op}\,\Sigma^{\alpha-N}$ where

$$(A_j u)(t) = \frac{1}{j!} a_j(t) \sum_{n\in\mathbb{Z}} \hat{u}(n)\hat{\kappa}^{(j)}(n)\, e^{in2\pi t}, \qquad a_j(t) = \partial_s^j a(t,s)\Big|_{s=t}. \tag{7.36}$$

We see that $\sum_{j=0}^{N-1} A_j u$ is easily computable in both cases, and later this will be exploited designing numerical methods for periodic integral equations. Let us discuss some examples in more details.

7.6.1 Operator $(Au)(t) = \int_0^1 a(t,s) \log|\sin \pi(t-s)| u(s)\, ds$

We know the Fourier coefficients of $\kappa(t) = \log|\sin \pi t|$:

$$\hat{\kappa}(0) = -\log 2, \qquad \hat{\kappa}(n) = -\frac{1}{2}|n|^{-1} \quad (0 \neq n \in \mathbb{Z}).$$

Clearly (7.31) is fulfilled with $\alpha = -1$. We introduce an extension $\hat{\kappa} \in C^\infty(\mathbb{R})$ such that $\hat{\kappa}^{(j)}(\xi) = -\frac{1}{2}|\xi|^{-1}$ for $|\xi| \geq 1$ and $\hat{\kappa}^{(j)}(0) = 0$ $(j \geq 1)$. Then

$$\hat{\kappa}^{(j)}(\xi) = (-1)^{j+1}\, \tfrac{1}{2} j!\, |\xi|^{-j-1} (\mathrm{sign}\,(\xi))^j \qquad \text{for } |\xi| \geq 1,$$
$$\hat{\kappa}^{(j)}(n) = (-1)^{j+1}\, \tfrac{1}{2} j!\, |n|^{-j-1} (\mathrm{sign}\,(n))^j \qquad (0 \neq n \in \mathbb{Z}).$$

Thus $A - \sum_{j=0}^{N-1} A_j \in \mathrm{Op}\,\Sigma^{-1-N}$ where

$$(A_j u)(t) = (-1)^{j+1}\, \tfrac{1}{2} a_j(t) \sum_{0\neq n\in\mathbb{Z}} \hat{u}(n)|n|^{-j-1}(\mathrm{sign}\,(n))^j\, e^{in2\pi t},$$
$$a_j(t) = \partial_s^j a(t,s)\Big|_{s=t}.$$

For $j = 0$, we have omitted the term $(A_{00}u)(t) := -(\log 2)a_0(t)\hat{u}(0)$ which corresponds to $\hat{\kappa}(0) = -\log 2$. Since $A_{00} \in \operatorname{Op} \Sigma^{-\infty}$, it does not influence on the order of the asymptotic approximation. Moreover, asymptotic approximations will be used only for u containing high frequencies, i.e. for u of the form $u = \sum_{|n|\geq q} \hat{u}(n)\, e^{in2\pi t}$ with a $q \gg 0$.

7.6.2 Operator $(Au)(t) = i\int_0^1 a(t,s) \cot \pi(t-s)u(s)\, ds$

As we know, the Fourier coefficients of the distribution $\kappa(t) = i\cot \pi t$ are given by

$$\hat{\kappa}(n) = \operatorname{sign}(n) = \begin{cases} -1, & n < 0, \\ 0, & n = 0, \\ 1, & n > 0. \end{cases}$$

Clearly (7.31) is fulfilled with $\alpha = 0$. The extension $\hat{\kappa} \in C^\infty(\mathbb{R})$ can be constructed so that $\hat{\kappa}(\xi) = -1$ for $\xi \leq -1$, $\hat{\kappa}(\xi) = 1$ for $\xi \geq 1$ and $\hat{\kappa}^{(j)}(0) = 0$ ($j \in \mathbb{N}_0$). Now

$$\hat{\kappa}^{(j)}(n) = 0 \quad (n \in \mathbb{Z}) \quad \text{for } j \geq 1.$$

This means that the asymptotic expansions (7.32) and (7.34) contain only one term: $\sigma(t,s) \sim \hat{\kappa}(n)a(t,t)$. Respectively, $A - A_0 \in \operatorname{Op} \Sigma^{-\infty}$ where $(A_0u)(t) = a(t,t)(H_0u)(t)$,

$$(H_0u)(t) = \sum_{0 \neq n \in \mathbb{Z}} \hat{u}(n) \operatorname{sign} e^{in2\pi t} = i\int_0^1 \cot \pi(t-s)u(s)\, ds.$$

Exercises

Exercise 7.6.1. Assume (7.31) and denote $B_N = A - \sum_{j=0}^{N-1} A_j$ where operators A_j are defined by (7.35). Prove that

$$(B_Nu)(t) = \int_0^1 a_N(t,s)\kappa_N(t-s)u(s)\, ds$$

where a_N is a C^∞-smooth biperiodic function and

$$\kappa_N(t) = (e^{-i2\pi t} - 1)^N \kappa(t), \quad \hat{\kappa}_N(n) = \Delta^N \hat{\kappa}(n) \quad (n \in \mathbb{N}).$$

Exercise 7.6.2. Present asymptotic expansions of the integral operator (7.30) with $\kappa(t) = \sin^2(\pi t) \log|\sin(\pi t)|$.

Exercise 7.6.3. Present asymptotic expansions of the operator (7.30) with $\kappa(t) = 1/\sin^2 \pi t$, cf. Section 5.10.

7.7 The Symbol of Dual and Adjoint Operators

For $A \in \mathcal{L}(H^\lambda, H^\mu)$, the dual operator $A' \in \mathcal{L}(H^{-\mu}, H^{-\lambda})$ and the adjoint operator $A^* \in \mathcal{L}(H^{-\mu}, H^{-\lambda})$ are defined by relations

$$\langle Au\,,\, v\rangle = \langle u\,,\, A'v\rangle \quad \text{for all } \; u \in H^\lambda\,,\; v \in H^{-\mu}\,,$$
$$(Au\,,\, v)_0 = (u\,,\, A^*v)_0 \quad \text{for all } \; u \in H^\lambda\,,\; v \in H^{-\mu}.$$

Recall that for u, $v \in H^0$,

$$\langle u\,,\, v\rangle = \sum_{n\in\mathbf{Z}} \hat{u}(n)\hat{v}(-n) = \int_0^1 u(t)v(t)\,dt\,,$$
$$(u\,,\, v)_0 = \sum_{n\in\mathbf{Z}} \hat{u}(n)\overline{\hat{v}(n)} = \int_0^1 u(t)\overline{v(t)}\,dt.$$

Theorem 7.7.1. *If $A \in \operatorname{Op}\Sigma^\alpha$ then $A' \in \operatorname{Op}\Sigma^\alpha$, $A^* \in \operatorname{Op}\Sigma^\alpha$, and the following asymptotic expansions hold true:*

$$\sigma_{A'}(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!}\partial_t^{(j)} \Delta_n^j \sigma_A(t,-n)\,, \quad \sigma_{A'}(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!}\partial_t^j \left(\frac{\partial}{\partial\xi}\right)^j \sigma_A(t,-\xi)\,,$$
$$\sigma_{A^*}(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!}\partial_t^{(j)} \Delta_n^j \overline{\sigma_A(t,n)}\,, \quad \sigma_{A^*}(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!}\partial_t^j \left(\frac{\partial}{\partial\xi}\right)^j \overline{\sigma_A(t,\xi)}.$$

Proof. For u, $v \in C_1^\infty(\mathbb{R})$, we have

$$\begin{aligned}\langle Au\,,\, v\rangle &= \int_0^1 (Au)(t)v(t)\,dt \\ &= \int_0^1 \left(\int_0^1 \sum_{n\in\mathbf{Z}} \sigma(t,n)\, \mathrm{e}^{\mathrm{i}n2\pi(t-s)}\, u(s)\,ds\right) v(t)\,dt \\ &= \int_0^1 u(s)\left(\int_0^1 \sum_{n\in\mathbf{Z}} \sigma(t,n)\, \mathrm{e}^{\mathrm{i}n2\pi(t-s)}\, v(t)\,dt\right) ds.\end{aligned}$$

We see that

$$(A'v)(s) = \int_0^1 \sum_{n\in\mathbf{Z}} \sigma(t,n)\, \mathrm{e}^{\mathrm{i}n2\pi(t-s)}\, v(t)\,dt\,,$$

or changing the roles of t and s,

$$\begin{aligned}(A'v)(t) &= \int_0^1 \sum_{n\in\mathbf{Z}} \sigma(s,n)\, \mathrm{e}^{\mathrm{i}n2\pi(s-t)}\, v(s)\,ds \\ &= \int_0^1 \sum_{n\in\mathbf{Z}} \sigma(s,-n)\, \mathrm{e}^{\mathrm{i}n2\pi(t-s)}\, v(s)\,ds.\end{aligned}$$

Clearly, $a(t,s,n) = \sigma(s,-n)$ is an amplitude of class $\mathcal{A}^\alpha$, and we obtained that $A' = \mathrm{Op}(a)$ (cf. (7.18)). Now all assertions of the theorem concerning A' immediately follow from Theorem 7.5.1. For A^* the proof is similar. □

Exercise 7.7.1. Present the proof of Theorem 7.7.1 for A^*.

7.8 The Symbol of the Composition of PPDOs

It occurs that the composition (product) of two PPDOs is again a PPDO. Thus the set of all PPDOs occurs to be not only a vector space but also an algebra.

Theorem 7.8.1. *Let $A \in \mathrm{Op}\,\Sigma^\alpha$ and $B \in \mathrm{Op}\,\Sigma^\beta$. Then $BA \in \mathrm{Op}\,\Sigma^{\alpha+\beta}$, and its symbol has the asymptotic expansions*

$$\sigma_{BA}(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!}\left[\Delta_n^j \sigma_B(t,n)\right] \partial_t^{(j)} \sigma_A(t,n)\,, \tag{7.37}$$

$$\sigma_{BA}(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!}\left[\left(\frac{\partial}{\partial\xi}\right)^j \sigma_B(t,\xi)\right] \partial_t^j \sigma_A(t,\xi)\,; \tag{7.38}$$

recall that $\partial_t = \frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial t}$, $\partial_t^{(j)} = \prod_{l=0}^{j-1}(\partial_t - lI)$. *Moreover,* $BA - AB \in \mathrm{Op}\,\Sigma^{\alpha+\beta-1}$.

Proof. We have (cf. first lines of the proof of Theorem 7.3.1)

$$(Au)(t) = \sum_{m,n\in\mathbb{Z}} \hat{\sigma}_A(m,n)\hat{u}(n)\,\mathrm{e}^{\mathrm{i}(m+n)2\pi t}\,.$$

Applying the operator B to this representation and using

$$B\,\mathrm{e}_{m+n} = \sigma_B(t,m+n)\,\mathrm{e}^{\mathrm{i}(m+n)2\pi t}$$

we obtain

$$\begin{aligned}(BAu)(t) &= \sum_{m,n\in\mathbb{Z}} \sigma_B(t,m+n)\hat{\sigma}_A(m,n)\hat{u}(n)\,\mathrm{e}^{\mathrm{i}(m+n)2\pi t} \\ &= \sum_{n\in\mathbb{Z}}\left[\sum_{m\in\mathbb{Z}} \sigma_B(t,m+n)\hat{\sigma}_A(m,n)\,\mathrm{e}^{\mathrm{i}m2\pi t}\right]\hat{u}(n)\,\mathrm{e}^{\mathrm{i}n2\pi t}\,.\end{aligned}$$

This means that

$$\sigma_{BA}(t,n) = \sum_{m\in\mathbb{Z}} \sigma_B(t,m+n)\hat{\sigma}_A(m,n)\,\mathrm{e}^{\mathrm{i}m2\pi t} \qquad (t\in\mathbb{R},\ n\in\mathbb{Z})\,.$$

Using the extended symbols $\sigma_A(t,\xi)$, $\sigma_B(t,\xi)$, we define extended symbol $\sigma_{BA}(t,\xi)$ by

$$\sigma_{BA}(t,\xi) = \sum_{m\in\mathbb{Z}} \sigma_B(t,\xi+m)\hat{\sigma}_A(m,\xi)\, e^{im2\pi t}\,.$$

The last series as well as the series below converge locally uniformly since $\sigma_A \in \operatorname{Op}\Sigma^\alpha$, $\sigma_B \in \operatorname{Op}\Sigma^\beta$ imply the inequalities

$$\left|\left(\frac{\partial}{\partial\xi}\right)^l \hat{\sigma}_A(m,\xi)\right| \le c_{l,r}\underline{m}^{-r}(1+|\xi|)^{\alpha-l} \qquad \text{with any } r>0\,, \tag{7.39}$$

$$\left|\left(\frac{\partial}{\partial t}\right)^p \left(\frac{\partial}{\partial\xi}\right)^q \sigma_B(t,\xi)\right| \le c_{p,q}(1+|\xi|)^{\beta-q}. \tag{7.40}$$

Using the Taylor formula

$$\sigma_B(t,\xi+m) = \sum_{j=0}^{N-1} \frac{1}{j!}\left(\frac{\partial}{\partial\xi}\right)^j \sigma_B(t,\xi)m^j + R_N(t,\xi,m)\,,$$

$$R_N(t,\xi,m) = \frac{1}{(N-1)!}\int_0^m (m-\eta)^{N-1}\left(\frac{\partial}{\partial\xi}\right)^N \sigma_B(t,\xi+\eta)d\eta\,,$$

we rewrite $\sigma_{BA}(t,\xi)$ in the form

$$\begin{aligned}\sigma_{BA}(t,\xi) &= \sum_{j=0}^{N-1}\frac{1}{j!}\left[\left(\frac{\partial}{\partial\xi}\right)^j \sigma_B(t,\xi)\right]\sum_{m\in\mathbb{Z}}\hat{\sigma}_A(m,\xi)m^j\, e^{im2\pi t}\\ &\quad + \sum_{m\in\mathbb{Z}} R_N(t,\xi,m)\hat{\sigma}_A(m,\xi)\, e^{im2\pi t}\,.\end{aligned}$$

Here

$$\sum_{m\in\mathbb{Z}}\hat{\sigma}_A(m,\xi)m^j\, e^{im2\pi t} = \partial_t^j \sum_{m\in\mathbb{Z}}\hat{\sigma}_A(m,\xi)\, e^{im2\pi t} = \partial_t^j\sigma_A(t,\xi).$$

So we have obtained

$$\begin{aligned}\sigma_{BA}(t,\xi) &- \sum_{j=0}^{N-1}\frac{1}{j!}\left[\left(\frac{\partial}{\partial\xi}\right)^j \sigma_B(t,\xi)\right]\partial_t^j\sigma_A(t,\xi)\\ &= \sum_{m\in\mathbb{Z}} R_N(t,\xi,m)\hat{\sigma}_A(m,\xi)\, e^{im2\pi t}\,,\end{aligned}$$

and to prove (7.38) and $\sigma_{BA} \in \Sigma^{\alpha+\beta}$, we have to show that

$$\left|\left(\frac{\partial}{\partial t}\right)^k\left(\frac{\partial}{\partial\xi}\right)^l \sum_{m\in\mathbb{Z}} R_N(t,\xi,m)\hat{\sigma}_A(m,\xi)\, e^{im2\pi t}\right| \le c_{k,l,N}(1+|\xi|)^{\alpha+\beta-l-N} \tag{7.41}$$

for all $t, \xi \in \mathbb{R}$, $k, l \in \mathbb{N}_0$, $N \in \mathbb{N}$. Estimates (7.39) and (7.40) allow to differentiate the series in (7.41) under the sum. By the Leibniz rule,

$$\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^l \sum_{m\in\mathbb{Z}} R_N(t,\xi,m)\hat{\sigma}_A(m,\xi)\,\mathrm{e}^{\mathrm{i}m2\pi t}$$

$$= \sum_{p=0}^{k}\binom{k}{p}\sum_{q=0}^{l}\binom{l}{q}\sum_{m\in\mathbb{Z}}\left(\frac{\partial}{\partial t}\right)^p\left[\left(\frac{\partial}{\partial \xi}\right)^q R_N(t,\xi,m)\right]$$

$$\cdot\left(\frac{\partial}{\partial \xi}\right)^{l-q}\hat{\sigma}_A(m,\xi)(2\pi\mathrm{i}\,m)^{k-p}\,\mathrm{e}^{\mathrm{i}m2\pi t}\,.$$

It follows from (7.40) that (cf. the proof of (7.27))

$$\left|\left(\frac{\partial}{\partial t}\right)^p\left(\frac{\partial}{\partial \xi}\right)^q R_N(t,\xi,m)\right| \le c_{p,q,N}\underline{m}^{|\beta|+q+2N}(1+|\xi|)^{\beta-N-q}.$$

Together with (7.39) we now get (7.41). Hence $\sigma_{BA} \in \Sigma^{\alpha+\beta}$ and (7.38) holds true. To prove (7.37) it suffices to show that for any $N \in \mathbb{N}$, the difference

$$\sum_{j=0}^{N-1}\frac{1}{j!}\left[\Delta_\xi^j\sigma_B(t,\xi)\right]\sigma_t^{(j)}\sigma_A(t,\xi) - \sum_{j=0}^{N-1}\frac{1}{j!}\left[\left(\frac{\partial}{\partial\xi}\right)^j\sigma_B(t,\xi)\right]\partial_t^j\sigma_A(t,\xi) \quad (7.42)$$

belongs to $\Sigma^{\alpha+\beta-N}$. This can be done in a similar way as we established (7.29) in the proof of Theorem 7.5.1. The details are left to reader as an exercise.

Finally, the asymptotic expansions of $\sigma_{BA}(t,\xi)$ and $\sigma_{AB}(t,\xi)$ have the same principal term $\sigma_B(t,\xi)\sigma_A(t,\xi)$. Therefore $\sigma_{BA-AB} = \sigma_{BA} - \sigma_{AB} \in \Sigma^{\alpha+\beta-1}$ and $BA - AB \in \operatorname{Op}\Sigma^{\alpha+\beta-1}$. □

Exercises

Exercise 7.8.1. Prove that, under conditions of Theorem 7.8.1, the function defined in (7.42) belongs to $\Sigma^{\alpha+\beta-N}$.

Exercise 7.8.2. Prove the following difference version of the Taylor formula:

$$\varphi(\xi+m) = \sum_{j=0}^{N-1}\frac{1}{j!}\Delta^j\varphi(\xi)m^{(j)} + R_N(\xi,m) \quad \text{for } \xi\in\mathbb{R},\ m\in\mathbb{Z},\ \varphi\in C(\mathbb{R}),$$

$$R_N(\xi,m) = \begin{cases} \dfrac{1}{(N-1)!}\displaystyle\sum_{k=1}^{m}(m-k)^{(N-1)}\Delta^N\varphi(\xi+k-1)\,, & m\ge 1\,,\\ 0\,, & m=0\,,\\ -\dfrac{1}{(N-1)!}\displaystyle\sum_{k=m}^{-1}(m-k-1)^{(N-1)}\Delta^N\varphi(\xi+k)\,, & m\le -1. \end{cases}$$

Notice that in case $m \geq 1$ we have

$$(m-k)^{(N-1)} = 0 \quad \text{for} \quad k = m, m-1, \ldots, \max\{1, m-N+2\}$$

and consequently $R_N(\xi, m) = 0$ for $1 \leq m \leq N-1$.
Hints. Transform $R_N(\xi, \eta)$ with the help of partial summations

$$\sum_{k=1}^{m} \psi(k)\Delta\varphi(k) = -\sum_{k=1}^{m} \left[\overline{\Delta}\psi(k)\right]\varphi(k) - \psi(0)\varphi(1) + \psi(m)\varphi(m+1),$$

$$\sum_{k=m}^{-1} \psi(k)\Delta\varphi(k) = -\sum_{k=m}^{-1} \left[\overline{\Delta}\psi(k)\right]\varphi(k) + \psi(-1)\varphi(0) - \psi(m-1)\varphi(m)$$

where $m \geq 1$ in the first formula and $m \leq -1$ in the second one. Notice also that

$$\overline{\Delta}_k \psi(m-k) = -\Delta_m \psi(m-k),$$
$$\Delta_m (m-k)^{(l)} = l(m-k)^{(l-1)}, \quad l \geq 1.$$

Exercise 7.8.3. Present a direct proof of asymptotic expansion (7.37) on the basis of the difference version of the Taylor formula (Exercise 7.8.2).

7.9 Pseudolocality

For a differential operator A, $\operatorname{supp}(Au) \subset \operatorname{supp} u$. For PPDOs this is not true in general, but a similar result holds for the singular supports. The *singular support* of $u \in \mathcal{D}_1'$, *singsuppu*, is defined as the complement of the maximal open set in $\mathbb{R}$ where u is C^∞-smooth. For instance, $\mathbb{Z}$ is the singular support of the function $u = \log|\sin \pi t|$.

Theorem 7.9.1. *For any $u \in \mathcal{D}_1'(\mathbb{R})$ and $A \in \operatorname{Op} \Sigma^\alpha$ of an arbitrary order $\alpha \in \mathbb{R}$,*

$$\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp} u. \tag{7.43}$$

Proof. Let $A \in \operatorname{Op} \Sigma^\alpha$. First we prove (7.43) for $u \in H^q$ with $\mathbb{N} \ni q > \alpha + 1$. We represent

$$\begin{aligned}
(Au)(t) &= \sum_{n\in\mathbb{Z}} \sigma(t,n)\hat{u}(n)\, e^{in2\pi t} = \sum_{n\in\mathbb{Z}} \sigma(t,n) \int_0^1 u(s)\, e^{-in2\pi s}\, ds\, e^{in2\pi t} \\
&= \sigma(t,0)\hat{u}(0) + \sum_{0\neq n\in\mathbb{Z}} \frac{\sigma(t,n)}{(2\pi i\, n)^q} \int_0^1 u^{(q)}(s)\, e^{-in2\pi s}\, ds\, e^{in2\pi t} \\
&= \sigma(t,0)\hat{u}(0) + \int_0^1 \sum_{n\in\mathbb{Z}} \frac{\sigma(t,n)}{(2\pi i\, \tilde{n})^q}\, e^{in2\pi(t-s)}\, u^{(q)}(s)\, ds
\end{aligned}$$