# 12 Conjugation of pseudodifferential operators by flows

Let $a(t) \in C^0(\mathbb{R}, \mathcal{S}^m)$. We know from the previous section that if $a$ is real valued and $m \leq 1$, then

$$\begin{cases} \mathrm{i}\partial_t u = \mathrm{Op}\,(a(t))\, u \\ u(t_0) = u_0 \in H^s \end{cases} \tag{12.1}$$

has a unitary propagator $\mathcal{U}(t, t_0)$ bounded $H^r \to H^r$. We want to understand how pseudodifferential operators are transformed by the propagator $\mathcal{U}(t, t_0)$. First we have the following result.

**Lemma 12.1.** *Assume that* $-\mathrm{i}a$ *fulfills (H1)–(H2) of Theorem 11.3, and denote by* $\mathcal{U}(t, t_0)$ *the propagator of* (12.1). *Let* $b \in \mathcal{S}^\rho$, $\rho \in \mathbb{R}$ *and set*

$$B_t := \mathcal{U}(t, 0)\,\mathrm{Op}\,(b)\,\mathcal{U}(t, 0)^{-1}.$$

*Then* $B_t$ *fulfills the* Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t} B_t = -\mathrm{i}[\mathrm{Op}\,(a(t))\,, B_t]\ . \tag{12.2}$$

*Proof.* Since

$$\partial_t \mathcal{U}(t, 0) = -\mathrm{i}\mathrm{Op}\,(a(t))\,\mathcal{U}(t, 0)$$

and

$$\partial_t \left(\mathcal{U}(t, 0)^{-1}\right) = -\mathcal{U}(t, 0)^{-1} \left(\partial_t \mathcal{U}(t, 0)\right) \mathcal{U}(t, 0)^{-1} = +\mathrm{i}\,\mathcal{U}(t, 0)^{-1}\,\mathrm{Op}\,(a(t))\,,$$

we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} B_t &= -\mathrm{i}\mathrm{Op}\,(a(t))\,\mathcal{U}(t, 0)\mathrm{Op}\,(b)\,\mathcal{U}(t, 0)^{-1} + \mathrm{i}\,\mathcal{U}(t, 0)\mathrm{Op}\,(b)\,\mathcal{U}(t, 0)^{-1}\mathrm{Op}\,(a(t)) \\ &= -\mathrm{i}\mathrm{Op}\,(a(t))\, B_t + \mathrm{i}B_t\mathrm{Op}\,(a(t)) = -\mathrm{i}[\mathrm{Op}\,(a(t))\,, B_t]. \end{aligned}$$

□

There are two conceptually distinct cases to treat: the first one is when $a \in \mathcal{S}^m$ with $m < 1$, the second one when $m = 1$. The difference is that in the first case the commutator in (12.2) reduces the order of pseudodifferential operators, while in the second case the order is preserved. So we treat the two cases differently.

## 12.1 The case $m < 1$

In this section we deal with $m < 1$, and only in the the autonomous case, namely we assume that the symbol $a \in \mathcal{S}^m$ does not depend on time (this is the case we will encounter in the applications).

In this case the flow of (12.1) is a one parameter group of transformations $U(t)$ fulfilling:

$$U(t + s) = U(t)U(s)\ , \qquad U(0) = \mathbb{1}, \quad U(-t) = U(t)^{-1}. \tag{12.3}$$

Remark that $\mathrm{Op}\,(a)$ commutes with its propagator, namely

$$\mathrm{Op}\,(a)\, U(t) = U(t)\mathrm{Op}\,(a)\,. \tag{12.4}$$

This follows since, as a direct computation gives, $\frac{\mathrm{d}}{\mathrm{d}t}\left(U(-t)\,\mathrm{Op}\,(a)\ U(t)\right) = 0$ which implies $U(-t)\mathrm{Op}\,(a)\, U(t) = \mathrm{Op}\,(a)$.

In this case we can obtain an expansion of $B_t$ in pseudodifferential operators of decreasing order, plus an arbitrary regularizing remainder.

**Lemma 12.2.** *Assume that $a \in \mathcal{S}^m$, $m < 1$, and $b \in \mathcal{S}^\rho$, $\rho \in \mathbb{R}$ are time independent. Then for every $N \geq 0$ one has*

$$U(t)\,\mathrm{Op}\,(b)\;U(-t) = \sum_{\ell=0}^{N} \frac{(-\mathrm{i})^\ell t^\ell}{\ell!} \mathrm{ad}^\ell_{\mathrm{Op}(a)}(\mathrm{Op}\,(b)) + R_N \tag{12.5}$$

*where*

$$\mathrm{ad}_X(A) = [X, A], \tag{12.6}$$

*and $R_N$ is given by*

$$R_N = \frac{t^{N+1}\,(-\mathrm{i})^{N+1}}{N!} \int_0^1 (1-\tau)^{N+1}\, U(\tau t)\, \mathrm{ad}^{N+1}_{\mathrm{Op}(a)}(\mathrm{Op}\,(b))\, U(-\tau t) \mathrm{d}\tau\ . \tag{12.7}$$

*The expansion* (12.5) *is in decreasing order, indeed*

$$\mathrm{ad}^\ell_{\mathrm{Op}(a)}(\mathrm{Op}\,(b)) \in \mathrm{Op}\left(\mathcal{S}^{\rho-\ell(1-m)}\right)\ , \qquad R_N \colon \mathcal{H}^r \to \mathcal{H}^{r-\rho+(N+1)(1-m)} \tag{12.8}$$

*Proof.* We know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(U(t)\,\mathrm{Op}\,(b)\;U(-t)\right) = -\mathrm{i}\;[\mathrm{Op}\,(a)\,, U(t)\,\mathrm{Op}\,(b)\;U(-t)] = U(t)\,(-\mathrm{i}[\mathrm{Op}\,(a), \mathrm{Op}\,(b)])\,U(-t) \tag{12.9}$$

which evaluated at $t = 0$ gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(U(t)\,\mathrm{Op}\,(b)\;U(-t)\right)\right|_{t=0} = -\mathrm{i}\,\mathrm{ad}_{\mathrm{Op}(a)}(\mathrm{Op}\,(b))\ .$$

Then iterating (12.9) we obtain

$$\left.\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}\left(U(t)\,\mathrm{Op}\,(b)\;U(-t)\right)\right|_{t=0} = (-\mathrm{i})^\ell\,\mathrm{ad}^\ell_{\mathrm{Op}(a)}(\mathrm{Op}\,(b))$$

and the Taylor formula gives us (12.5) and (12.7).

Now by symbolic calculus we have

$$\mathrm{ad}_{\mathrm{Op}(a)}(\mathrm{Op}\,(b)) \in \mathrm{Op}\left(\mathcal{S}^{\rho-(1-m)}\right) \quad \Rightarrow \mathrm{ad}^\ell_{\mathrm{Op}(a)}(\mathrm{Op}\,(b)) \in \mathrm{Op}\left(\mathcal{S}^{\rho-\ell(1-m)}\right)$$

Concerning $R_N$, it is sufficient to use (12.7) and the fact that $U(t)$ is bounded from $H^r \to H^r$ for any $r \in \mathbb{R}$. □

The condition $m < 1$ is fundamental, since it allows us to get an expansion in decreasing order of pseudodifferential operators.

## 12.2 The case $m = 1$

The second case is when $m = 1$. This is conceptually different from the previous one, because the expansion (12.5) is not anymore in decreasing order, so we must proceed differently.

How to deal with this case is the content of the Egorov theorem, which we state and prove for nonautonomous classical symbols. Here nonautonomous means symbols depending explicitly on time.

Classical symbols means the following:

**Definition 12.3.** *We say that $a \in \mathcal{S}^m_{cl}$ is a classical symbol if $a \sim \sum_{j\geq 0} a_{m-j}$ where $a_{m-j}(x,\xi)$ is positively homogeneous of degree $m-j$ in $\xi$ and localized in $x$, namely there exists $R>0$ such that $a_{m-j}(x,\xi)=0$ for any $j$ if $|x|\geq R$.*

The main result is the following one. We follow the presentation of [Tay81].

**Theorem 12.4** (Egorov)**.** *Assume that $a(t,x,\xi)\in \mathcal{S}^1_{cl}$, $a(t,x,\xi)\sim \sum_j a_{1-j}$ with $a_1$ real valued. Let $\mathcal{U}(t,0)$ be the flow of $\mathrm{i}\partial_t u = \mathrm{Op}\,(a(t))\,u$. Then for any $b\in \mathcal{S}^\rho$, $\rho\in\mathbb{R}$, the operator*

$$B_t = \mathcal{U}(t,0)\,\mathrm{Op}\,(b)\,\mathcal{U}(t,0)^{-1} \tag{12.10}$$

*is a pseudodifferential operator in $\mathrm{Op}\,(\mathcal{S}^\rho)$ whose principal symbol is given by*

$$b_t(x,\xi) = b((\phi^{t,0}_{a_1})^{-1}(x,\xi)) \tag{12.11}$$

*where $\phi^{t,0}_{a_1}(x,\xi)$ is the time $t$-flow of the time dependent Hamiltonian system having $a_1$ as Hamiltonian and $(x,\xi)$ as initial datum:*

$$\begin{cases} \dot x = \partial_\xi a_1(t,x,\xi) \\ \dot \xi = -\partial_x a_1(t,x,\xi)\end{cases}, \qquad (x(0),\xi(0)) = (x,\xi) \tag{12.12}$$

*Proof.* Recall that $B_t$ fulfills the Heisenberg equation (12.2). We will construct an approximate solution $Q_t$ of the Heisenberg equation (12.2) and then show that $B_t - Q_t$ is a smoothing operator. So we are looking for $Q_t = \mathrm{Op}\,(q(t))$, $q\in\mathcal{S}^\rho_{cl}$ solving

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_t = -\mathrm{i}\,[\mathrm{Op}\,(a)\,, Q_t] + R(t), \qquad Q_0 = B_0 \equiv \mathrm{Op}\,(b) \tag{12.13}$$

where $R(t)$ is a smooth family of operators in $\mathrm{Op}\,(\mathcal{S}^{-\infty})$. We do this by constructing $q(t,x,\xi)\in C^\infty(\mathbb{R},\mathcal{S}^\rho_{cl})$ as an asymptotic series

$$q(t,x,\xi)\sim \sum_{j\geq 0} q_{\rho-j}(t,x,\xi), \qquad q_{\rho-j}\in C^\infty(\mathbb{R},\mathcal{S}^{\rho-j})$$

with initial datum

$$q_\rho(0,x,\xi) = b(x,\xi), \qquad q_{\rho-j}(0,x,\xi) = 0, \quad \forall j\geq 1. \tag{12.14}$$

We determine $q_{\rho-j}$ recursively, starting from $q_\rho$. At the level of symbols, the Heisenberg equation (12.13) reads

$$\partial_t q(t,x,\xi) = -\{a(t), q(t)\}_{\mathcal{M}} + r(t,x,\xi) \tag{12.15}$$

and we solve this equation order by order. As

$$-\{a(t),q(t)\}_{\mathcal{M}} = -\{a(t),q(t)\} + \mathcal{S}^{\rho-1} = -\{a_1(t),q_\rho(t)\} + \mathcal{S}^{\rho-1},$$

at principal order (12.13) is given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} q_\rho(t,x,\xi) = -\{a_1(t,x,\xi), q_\rho(t,x,\xi)\} \\ q_\rho(0,x,\xi) = b(x,\xi)\end{cases} \tag{12.16}$$

Namely $q_\rho$ is a solution of the transport equation (recall the normalization of Poisson bracket in (4.7))

$$\frac{\mathrm{d}}{\mathrm{d}t} q_\rho + \partial_\xi a_1\cdot\partial_x q_\rho - \partial_x a_1\cdot\partial_\xi q_\rho = 0. \tag{12.17}$$

Any solution of a transport equation is constant along its characteristics, which for (12.17) are given by

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = \partial_\xi a_1(t,x,\xi), \qquad \frac{\mathrm{d}\xi}{\mathrm{d}s} = -\partial_x a_1(t,x,\xi). \tag{12.18}$$

Denote by $\phi^{t,0}(x,\xi)$ the flow of (12.12), which is the flow along system (12.18) at time $t$ with initial datum $(x,\xi)$ at time 0. Note that it is a *non-autonomous flow*, as such it enjoys the following properties

$$\phi^{t,s} \circ \phi^{s,\tau} = \phi^{t,\tau}, \qquad (\phi^{t,0})^{-1} = \phi^{0,t}. \tag{12.19}$$

To find the solution of a transport equation, one employs the method of characteristic. This reduces basically to the following consideration: assume that $q_\rho(t,x,\xi)$ is a solution, then the function

$$s \mapsto \mathsf{q}(s,x,\xi) := q_\rho(s, \phi^{s,0}(x,\xi)) \tag{12.20}$$

is constant. Indeed denoting $(x(s),\xi(s)) = \phi^{s,0}(x,\xi)$ one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathsf{q}(s,x,\xi) &= \partial_s q_\rho(s,x(s),\xi(s)) + (\partial_x q_\rho)(s,x(s),\xi(s))\frac{\mathrm{d}x(s)}{\mathrm{d}s} + (\partial_\xi q_\rho)(s,x(s),\xi(s))\frac{\mathrm{d}\xi(s)}{\mathrm{d}s} \\ &\overset{(12.18)}{=} (\partial_t q_\rho + \{a_1, q_\rho\})\,|_{(s,x(s),\xi(s))} \overset{(12.16)}{=} 0\end{aligned}$$

It follows that

$$q_\rho(t,\phi^{t,0}(x,\xi)) = q_\rho(0,\phi^{0,0}(x,\xi)) = q_\rho(0,x,\xi) = b(x,\xi) \tag{12.21}$$

and using (12.19) we obtain

$$q_\rho(t,x,\xi) = b((\phi^{t,0})^{-1}(x,\xi)) \tag{12.22}$$

which is exactly (12.11). It follows from Lemma 12.5 below that (12.22) defines a symbol.
So far we have obtained $q_\rho \in C^\infty(\mathbb{R}, \mathcal{S}^\rho)$ so that

$$\partial_t q_\rho(t,x,\xi) = -\{a(t), q_\rho(t)\}_{\mathcal{M}} + r_{\rho-1} \tag{12.23}$$

Then we proceed recursively: assume that we know already $q_\rho, \ldots, q_{\rho-N+1}$ so that

$$\partial_t\left(q_\rho + \ldots + q_{\rho-N+1}\right) = -\{a(t), q_\rho + \ldots + q_{\rho-N+1}\}_{\mathcal{M}} + r_{\rho-N}, \tag{12.24}$$

and let us compute $q_{\rho-N} \in C^\infty(\mathbb{R}, \mathcal{S}^{\rho-N})$ so that

$$\partial_t\left(q_\rho + \ldots + q_{\rho-N}\right) = -\{a(t), q_\rho + \ldots + q_{\rho-N}\}_{\mathcal{M}} + r_{\rho-N-1}. \tag{12.25}$$

This is achieved provided $q_{\rho-N}$ solves

$$\partial_t q_{\rho-N} = -\{a(t), q_{\rho-N}\}_{\mathcal{M}} + r_{\rho-N} + \mathcal{S}^{\rho-N-1}.$$

As

$$\{a, q_{\rho-N}\}_{\mathcal{M}} = \sum_{j\geq 0}\{a_{1-j}, q_{\rho-N}\}_{\mathcal{M}} = \sum_j \underbrace{\{a_{1-j}, q_{\rho-N}\}}_{\in\mathcal{S}^{\rho-(N+j)}} + \underbrace{r(a_{1-j}, q_{\rho-N})}_{\in\mathcal{S}^{\rho-(N+j+1)}},$$

we get what we want provided that

$$\frac{\mathrm{d}}{\mathrm{d}t}q_{\rho-N} = -\{a_1, q_{\rho-N}\} + b_{\rho-N}, \qquad q_{\rho-N}(0,x,\xi) = 0, \tag{12.26}$$

but this is a forced version of (12.16), which is solved by Duhamel formula. We have thus solved (12.13) up to arbitrary order.

Now we show that $B_t - Q_t$ is a smoothing operator. We show that we show that for arbitrary $s, s' \in \mathbb{R}$, one has that for any $f \in H^s$

$$\mathcal{U}(t,0)\mathrm{Op}\,(b)\,\mathcal{U}(t,0)^{-1}f - Q_t f \in H^{s'}$$

or equivalently, that

$$\mathcal{U}(t,0)\mathrm{Op}\,(b)\,f - Q_t\mathcal{U}(t,0)f \in H^{s'}$$

Denote $v(t) = \mathcal{U}(t,0)\mathrm{Op}\,(b)\,f$ and $w(t) = Q_t\mathcal{U}(t,0)f$. Then $v(t)$ solves

$$\mathrm{i}\partial_t v = \mathrm{Op}\,(b)\,v, \qquad v(0) = \mathrm{Op}\,(b)\,f \tag{12.27}$$

while, by (12.13), $w(t)$ solves

$$\mathrm{i}\partial_t w = \mathrm{Op}\,(a)\,w + R(t)\mathcal{U}(t,0)w, \qquad w(0) = \mathrm{Op}\,(b)\,f \tag{12.28}$$

Hence taking the difference

$$\mathrm{i}\partial_t(v-w) = \mathrm{Op}\,(a)\,(v-w) + R(t)\mathcal{U}(t,0)w, \qquad v(0) - w(0) = 0 \tag{12.29}$$

Hence Duhamel formula gives us

$$v(t) - w(t) = \int_0^t \mathcal{U}(t,0)^{-1}\mathcal{U}(s,0)R(s)\mathcal{U}(s,0)w(s)\mathrm{d}s \in H^{s'}$$

since $R(t)$ is a smoothing operator. ☐

A crucial ingredient in the proof of Egorov theorem is that (12.11) is still a symbol. This is also the property that one has to verify in different setups, and it might fail according to the pseudodifferential classes that are employed.

**Lemma 12.5.** *Let $a_1(t,x,\xi) \in \mathcal{S}^1$ be positive homogeneous of degree 1 is $\xi$ and localized in $x$. Denote by $\phi^{t,s}_{a_1}(x,\xi)$ the solution of the time dependent Hamltonian eqution*

$$\begin{cases} \dot{x} = \partial_\xi a_1(t,x,\xi) \\ \dot{\xi} = -\partial_x a_1(t,x,\xi) \end{cases}, \qquad (x(s),\xi(s)) = (x,\xi) \tag{12.30}$$

*Then the following holds true:*

*(i) For any $t,s \in \mathbb{R}$, the flow $\phi^{t,s}_{a_1}(x,\xi)$ exists and moreover, denoting $\phi^{t,s}_{a_1}(x,\xi) = (\mathbf{x}^{t,s}(x,\xi), \boldsymbol{\xi}^{t,s}(x,\xi))$, one has that*

$$\mathbf{x}^{t,s}(x,\lambda\xi) = \mathbf{x}^{t,s}(x,\xi), \qquad \boldsymbol{\xi}^{t,s}(x,\lambda\xi) = \lambda\boldsymbol{\xi}^{t,s}(x,\xi), \qquad \forall\lambda > 0, \qquad \forall t,s \in \mathbb{R} \tag{12.31}$$

*and moreover $\mathbf{x}^{t,s}(x,\xi) - x$ and $\boldsymbol{\xi}^{t,s}(x,\xi)$ are localized in $x$.*

*(ii) For any $t,s \in \mathbb{R}$, one has $\mathbf{x}^{t,s}(x,\xi) - x \in \mathcal{S}^0$, $\boldsymbol{\xi}^{t,s}(x,\xi) \in \mathcal{S}^1$ (meaning each component of the flow is in the class).*

*(iii) If $b \in \mathcal{S}^\rho$, $\rho \in \mathbb{R}$, then for any $t,s$ in $\mathbb{R}$ one has*

$$\widetilde{b}(x,\xi) := b(\mathbf{x}^{t,s}(x,\xi), \boldsymbol{\xi}^{t,s}(x,\xi)) \in \mathcal{S}^\rho$$

*Proof.* (*i*) The flow exists globally since the vector field grows at most linearly. To check the homogeneity of the flow, denote for any $\lambda > 0$,

$$\mathbf{x}_\lambda^{t,s}(x,\xi) := \mathbf{x}^{t,s}(x,\lambda\xi) \ , \qquad \boldsymbol{\xi}_\lambda^{t,s}(x,\xi) := \lambda^{-1}\boldsymbol{\xi}^{t,s}(x,\lambda\xi)$$

Now we have that $\partial_\xi a_1$ is positive homogeneous of degree 0 in $\xi$, and $\partial_x a_1$ is positive homogeneous of degree 1 in $\xi$, thus we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_\lambda^{t,s}(x,\xi) &= \partial_\xi a_1(t,\mathbf{x}^{t,s}(x,\lambda\xi),\boldsymbol{\xi}^{t,s}(x,\lambda\xi)) = \partial_\xi a_1(t,\mathbf{x}^{t,s}(x,\lambda\xi),\lambda^{-1}\boldsymbol{\xi}^{t,s}(x,\lambda\xi)) \\
&= \partial_\xi a_1(t,\mathbf{x}_\lambda^{t,s}(x,\xi),\boldsymbol{\xi}_\lambda^{t,s}(x,\xi)) \\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}_\lambda^{t,s}(x,\xi) &= -\lambda^{-1}\partial_x a_1(t,\mathbf{x}^{t,s}(x,\lambda\xi),\boldsymbol{\xi}^{t,s}(x,\lambda\xi)) = -\partial_x a_1(t,\mathbf{x}^{t,s}(x,\lambda\xi),\lambda^{-1}\boldsymbol{\xi}^{t,s}(x,\lambda\xi)) \\
&= -\partial_x a_1(t,\mathbf{x}_\lambda^{t,s}(x,\xi),\boldsymbol{\xi}_\lambda^{t,s}(x,\xi))
\end{aligned}$$

and moreover

$$\begin{aligned}
\mathbf{x}_\lambda^{s,s}(x,\xi) &:= \mathbf{x}^{s,s}(0,x,\lambda\xi) = x \ , \\
\boldsymbol{\xi}_\lambda^{s,s}(x,\xi) &:= \lambda^{-1}\boldsymbol{\xi}^{s,s}(x,\lambda\xi) = \lambda^{-1}\lambda\xi = \xi
\end{aligned}$$

It follows that $\big(\mathbf{x}_\lambda^{t,s}(x,\xi),\boldsymbol{\xi}_\lambda^{t,s}(x,\xi)\big)$ solves the Cauchy problem (12.30), thus by unicity it coincides with $\phi_{a_1}^{t,s}(x,\xi)$. This gives (12.31).

To prove that $\mathbf{x}^{t,s}(x,\xi) - x$ and $\boldsymbol{\xi}^{t,s}(x,\xi)$ are localized in $x$ it is sufficient to exploit the fact that $a_1(t,x,\xi)$ is localized in $x$, thus the flow equals the identity for $|x| \geq R$.

(*ii*) The smooth function $\boldsymbol{\xi}^{t,s}(x,\xi)$ is localized in $x$ and positively homogeneous of degree 1 in $\xi$. By adapting the arguments of Section 3.1 it follows that it is symbol in $\mathcal{S}^1$ (up to a smooth cutoff around the origin in $\xi$. An analogous argument gives $\mathbf{x}^{t,s}(x,\xi) - x \in \mathcal{S}^0$.

(*iii*) It follows by a direct computation, essentially exploiting the symbolic properties of the symbols. First we have

$$\left|\widetilde{b}(x,\xi)\right| \preceq \langle \boldsymbol{\xi}^{t,s}(x,\xi)\rangle \preceq \langle\xi\rangle$$

Similarly

$$\begin{aligned}
\partial_{x_j}\widetilde{b}(x,\xi) &= \sum_\ell \frac{\partial b}{\partial x_\ell}\left(\mathbf{x}^{t,s}(x,\xi),\boldsymbol{\xi}^{t,s}(x,\xi)\right)\frac{\partial \mathbf{x}_\ell^{t,s}(x,\xi)}{\partial x_j} + \frac{\partial b}{\partial \xi_\ell}\left(\mathbf{x}^{t,s}(x,\xi),\boldsymbol{\xi}^{t,s}(x,\xi)\right)\frac{\partial \boldsymbol{\xi}_\ell^{t,s}(x,\xi)}{\partial x_j} \\
\partial_{\xi_j}\widetilde{b}(x,\xi) &= \sum_\ell \frac{\partial b}{\partial x_\ell}\left(\mathbf{x}^{t,s}(x,\xi),\boldsymbol{\xi}^{t,s}(x,\xi)\right)\frac{\partial \mathbf{x}_\ell^{t,s}(x,\xi)}{\partial \xi_j} + \frac{\partial b}{\partial \xi_\ell}\left(\mathbf{x}^{t,s}(x,\xi),\boldsymbol{\xi}^{t,s}(x,\xi)\right)\frac{\partial \boldsymbol{\xi}_\ell^{t,s}(x,\xi)}{\partial \xi_j}
\end{aligned}$$

Then use the properties of $b$ and the flow to bound the derivatives. $\square$

**Remark 12.6.** *In case a does not depend on time, then the propagator fulfills* (12.3)*. In particular*

$$\begin{aligned}
U(-t)\mathrm{Op}\,(b)\,U(t) = U(-t)\mathrm{Op}\,(b)\,U(-t)^{-1} &= \mathrm{Op}\left(b\circ(\phi_a^{-t})^{-1}\right) + \mathcal{S}^{\rho-1} \\
&= \mathrm{Op}\left(b\circ\phi_a^{t}\right) + \mathcal{S}^{\rho-1}
\end{aligned}$$

## 12.3 Application: near to identity diffeomorphism

We have a PDE of the form

$$\partial_t u = \mathrm{Op}\,(a)\, u, \quad x \in \mathbb{R} \tag{12.32}$$

where $a \in \mathcal{S}^m$, $m \in \mathbb{R}$. We want to make a change of variables in the form of a diffeomorphism of $\mathbb{R}$, by need to perform the change of variables

$$v(x) = u(x + \beta(x))$$

where $\beta : \mathbb{R}^n \to \mathbb{R}^n$ is small, namely

$$\|\widetilde{\beta}\|_{C^2} \leq 1/2. \tag{12.33}$$

We want to understand the equation fulfilled by $v$. Denoting

$$[\mathcal{B}u](x) = u(x + \beta(x))$$

we put $v = \mathcal{B}u$, $u = \mathcal{B}^{-1}v$ and thus (12.32) becomes

$$\partial_t v = \mathcal{B}\mathrm{Op}\,(a)\,\mathcal{B}^{-1}v$$

We would like to know if $\mathcal{B}\mathrm{Op}\,(a)\,\mathcal{B}^{-1}$ has an expansion in pseudodifferential operator and which is the principal symbol.

Egorov theorem can help us if we are able to realize $\mathcal{B}$ as the time 1-flow of a certain PDE: namely, can we find $\mathcal{U}(t,0)$ propagator of a PDEs (to determine) such that $\mathcal{B}u = \mathcal{U}(1,0)u$? In such a case we can apply Egorov theorem and conclude that $\mathcal{B}\mathrm{Op}\,(a)\,\mathcal{B}^{-1}$ is still pseudodifferential of the same order as $\mathrm{Op}\,(a)$ and we can compute its principal symbol.

The first step is to write the diffeomorphism $x + \beta(x)$ as the time 1-flow of a family of vector fields. This is done in the following way: for $t \in [0,1]$ we define the family of diffeomorphism

$$\phi^{t,0}(x) := x + t\beta(x). \tag{12.34}$$

One has clearly

$$\phi^{0,0} = \mathbb{1}, \qquad \phi^{1,0} = x + \beta(x).$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi^{t,0}(x) = \beta(x).$$

We denote by $(\phi^{t,0})^{-1}(y)$ the inverse diffeomorphism; explicitly one has

$$y = x + t\beta(x) \Longleftrightarrow x = y + \breve{\beta}(t,y).$$

The second step is to define

$$[\mathcal{U}(t,0)u](x) := u(\phi^{t,0}(x)) = u(x + t\beta(x))$$

so that $\mathcal{U}(0,0) = \mathbb{1}$ and $\mathcal{U}(1,0) = \mathcal{B}$. Now we ask if $\mathcal{U}(t,0)$ is the propagator of a certain PDE.

We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(t,0)u = \frac{\mathrm{d}}{\mathrm{d}t}u(\phi^{t,0}(x)) = (\partial_x u)(\phi^{t,0}(x))\,\beta(x). \tag{12.35}$$

It is still not the flow of a PDE, since we have $(\partial_x u)(\phi^{t,0}(x))$ and would like to have $\partial_x[\mathcal{U}(t,0)u](x) \equiv \partial_x\left(u(\phi^{t,0}(x))\right)$. But by the chain rule

$$\partial_x\left[u(\phi^{t,0}(x))\right] = (\partial_x u)(\phi^t(x))\partial_x\phi^{t,0}(x) = (\partial_x u)(\phi^t(x))\,(1+t\beta_x(x))$$

from which we have

$$(\partial_x u)(\phi^t(x)) = \frac{1}{1+t\beta_x(x)}\partial_x\left[u(\phi^{t,0}(x))\right]$$

Inserting in (12.35) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(t,0)u = \frac{\beta(x)}{1+t\beta_x(x)}\partial_x\mathcal{U}(t,0)u$$

Hence $\mathcal{U}(t,0)u$ is the solution of the PDE

$$\partial_t\varphi = b(t,x)\partial_x\varphi \tag{12.36}$$

where

$$b(t,x) := \frac{\beta(x)}{1+t\beta_x(x)}. \tag{12.37}$$

We rewrite it as a Schrödinger equation

$$\mathrm{i}\partial_t\varphi = \mathrm{Op}\left(-b(t,x)\cdot\xi\right)\varphi \tag{12.38}$$

But this is the perfect situation in order to apply Egorov theorem! Indeed we have realized $\mathcal{U}(t,0)$ as the time $t$ flow of a PDE whose generator is a pseudodifferential operator.

At this point we are ready to apply Egorov theorem: we have $b(t,x)\xi \in \mathcal{S}^1$, thus we know that

$$\begin{aligned}&\mathcal{B}\mathrm{Op}\,(a)\,\mathcal{B}^{-1} = \mathcal{U}(1,0)\,\mathrm{Op}\,(a)\,\mathcal{U}(1,0)^{-1} = \mathrm{Op}\,(\widetilde{a}_\rho) + \mathcal{S}^{\rho-1}\\ &\widetilde{a}_\rho(x,\xi) = a((\gamma^{1,0})^{-1}(x,\xi))\end{aligned}$$

where $\gamma^{t,0}(x,\xi)$ is the flow of the system with Hamiltonian $-b(t,x)\cdot\xi$, namely

$$\begin{cases}\dot{x} = \nabla_\xi(-b(t,x)\xi) = -b(t,x)\\ \dot{\xi} = -\nabla_x(-b(t,x)\xi) = b_x(t,x)\xi\end{cases}, \qquad (x(0),\xi(0)) = (x,\xi) \tag{12.39}$$

and $(\gamma^{t,0})^{-1}(x,\xi)$ is the inverse flow.

We claim that

$$\begin{aligned}\gamma^{t,0}(y,\eta) &= \left((\phi^{t,0})^{-1}(y),\ [\mathrm{d}\phi^{t,0}_{|_{(\phi^{t,0})^{-1}(y)}}]^T\eta\right)\\ &= \left(y+\breve{\beta}(t,y),\ (1+t\beta_x)|_{x=y+\breve{\beta}(t,y)}\eta\right)\end{aligned} \tag{12.40}$$

where $\phi^{t,0}(x,\xi)$ is the flow (12.34). Indeed start from the identity

$$\phi^{t,0}\left((\phi^{t,0})^{-1}(y)\right) = y,$$

take the time derivative and obtain

$$(\frac{\mathrm{d}}{\mathrm{d}t}\phi^{t,0})\left((\phi^{t,0})^{-1}(y)\right) + \left[D\phi^{t,0}\left((\phi^{t,0})^{-1}(y)\right)\right]\frac{\mathrm{d}}{\mathrm{d}t}(\phi^{t,0})^{-1}(y) = 0$$

Now recall that $\frac{\mathrm{d}}{\mathrm{d}t}\phi^{t,0}(x) = \beta(x)$, and $D\phi^{t,0}(x) = (1 + t\beta_x(x))$, thus letting $y^t(y) := (\phi^{t,0})^{-1}(y)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} y^t(y) = - \left.\frac{\beta(x)}{1 + t\beta_x(x)}\right|_{x=y^t(y)}.$$

In other words, $y^t(y)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t} y^t(y) = -b(t, y^t(y)), \qquad y^0(y) = y$$

so it solves the first of (12.39).

We verify now that $\eta^t(y, \eta) := (1 + t\beta_x(y + \breve{\beta}(t, y))\eta$ solves the second equation (12.39). Let us compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\eta^t(y,\eta) &= \left.\left(\beta_x(x) + t\beta_{xx}(x)\frac{\mathrm{d}}{\mathrm{d}t}(y + \breve{\beta}(t,y))\right)\right|_{x=y+\breve{\beta}(t,y)}\eta \\
&= \left.\left(\beta_x(x) + t\beta_{xx}(x)\frac{-\beta(x)}{1 + t\beta_x(x)}\right)\right|_{x=y+\breve{\beta}(t,y)}\eta \\
&= \left.\left(\frac{\beta_x(x)}{1 + t\beta_x(x)} - \frac{t\beta_{xx}(x)\beta(x)}{(1 + t\beta_x(x))^2}\right)\right|_{x=y+\breve{\beta}(t,y)} (1 + t\beta_x(x))|_{x=y+\breve{\beta}(t,y)}\eta \\
&= \left.\left(\partial_x \frac{\beta(x)}{1 + t\beta_x(x)}\right)\right|_{x=y+\breve{\beta}(t,y)} \eta^t(y,\eta) \\
&= b_x(t, y^t(y))\, \eta^t(y, \eta)
\end{aligned}$$

hence it solves the second equation!

It follows from (12.40) that

$$\begin{aligned}
(\gamma^{t,0})^{-1}(x,\xi) &= \left(\phi^{t,0}(x),\ [\mathrm{d}\phi^{t,0}_x]^{-1}\xi\right) \\
&= \left(x + \beta(x), (1 + \breve{\beta}_y)|_{x+t\beta(x)}\xi\right)
\end{aligned}$$

hence we have

$$\widetilde{a}_\rho(x,\xi) = a\left(x + \beta(x), (1 + \breve{\beta}_y)|_{x+t\beta(x)}\xi\right). \tag{12.41}$$