13 Application: flow of Schrödinger equation

In this section we apply methods of pseudodifferential operators to prove several results on the flow of Schrödinger equation with unbounded coefficients.

13.1 Local smoothing effect

Consider the Cauchy problem

$$\begin{cases} \mathrm{i}\dot{u} = Au \\ u(0) = u_0 \end{cases} \tag{13.1}$$

where $A = \mathrm{Op}^w(a)$ has a real symbol $a(x,\xi) \in \mathcal{S}^m$, for instance $-\partial_{x_j}(a_{jk}(x)\partial_{x_k})$, or $-\Delta$, or $\mathrm{i}\partial_x^3$. Since A is selfadjoint, the operator $-\mathrm{i}A$ is skew-adjoint, hence the L^2 norm of the flow is preserved.

We want to prove the so called (local in time) local smoothing estimates, which says, roughly speaking, that the flow generated by A locally in space regularize a bit the solution, provided one integrate over time. The classical smoothing estimate reads

$$\int_0^T \int_{|x|\le R} \left| \langle D\rangle^{\frac{m-1}{2}} u(t,x)\right|^2 \mathrm{d}x\,\mathrm{d}t \le C_{T,R}\|u_0\|_0^2 \tag{13.2}$$

where the positive constant $C_{T,R}$ depends on T, R.
In particular inequality (13.2) implies that if $u_0 \in L^2(\mathbb{R}^d)$ the solution $e^{-\mathrm{i}tA}u_0 \in H^{\frac{m-1}{2}}_{\mathrm{loc}}$ for almost all t. Notice that this gain of derivatives is a pure dispersive phenomenon, which cannot hold in hyperbolic problems.

A very nice method of proof relies on the so called *positive commutator method*, which we now illustrate.
Let us take a bounded operator B, and consider the usual energy estimate, which reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Bu, u\rangle = \left\langle \frac{1}{\mathrm{i}}[B,A]u, u\right\rangle = \langle \mathrm{i}[A,B]u,u\rangle\,,$$

as $-\mathrm{i}A$ is skew-adjoint. Integrating in time from 0 to T and using that B is bounded and the flow is unitary in L^2 we get

$$\int_0^T \langle \mathrm{i}[A,B]u,u\rangle\,\mathrm{d}t \le C\|u_0\|_0^2. \tag{13.3}$$

Now the crucial point: choose B such that $\mathrm{i}[A,B] \ge 0$ is positive.

Let us see some examples.

Case $A = -\Delta$**.** In this case we have the following result, due to Doi [**?**]:

Lemma 13.1 (Doi)**.** *Let λ be even, radially decreasing, non negative, $\lambda \in L^1([0,\infty))$, smooth. Then there exists a real value symbol $b \in \mathcal{S}^0$ and a constant $0 < \beta < 1$ such that*

$$\{|\xi|^2, b\} \ge \beta\lambda(|x|)|\xi| - \frac{1}{\beta}$$

Proof. Define $f(t) := \int_0^t \lambda(r)\mathrm{d}r$ and let

$$\Phi(x) = (f(x_1), \dots, f(x_d))$$

so that $\Phi(x)$ is smooth and bounded. Let

$$\Phi'_{\rm sym}(x) := \frac{1}{2}\left(\partial_{x_j} f_i + \partial_{x_i} f_j\right) = \begin{pmatrix} \lambda(x_1) & 0 & \dots & 0 \\ 0 & \lambda(x_2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda(x_d) \end{pmatrix} \geq \lambda(|x|){\rm Id}$$

as λ is even and decreasing. Define

$$b(x,\xi) := \Phi(x)\cdot \frac{\xi}{\langle \xi \rangle} \in \mathcal{S}^0 .$$

Then

$$\{|\xi|^2, b\} = 2\frac{\xi}{\langle \xi \rangle}\cdot \nabla_x(\Phi(x)\cdot \xi) = \frac{2}{\langle \xi \rangle}\Phi'_{\rm sym}(x)\xi\cdot\xi \geq 2\lambda(|x|)\frac{|\xi|^2}{\langle \xi \rangle}.$$

□

By Doi's lemma, we have that

$$\{a,b\}_{\mathcal{M}} = \{|\xi|^2, b\} + r_0 \geq \lambda(|x|)\,|\xi| - c_0$$

and therefore by Garding inequality

$$\langle {\rm i}[A,B]u,u\rangle \geq \left\langle {\rm Op}\left(\lambda(|x|)\ \langle \xi\rangle^{m-1}\right)u,u\right\rangle - c_0\|u\|_0^2 - C\|u\|_0^2$$

Finally use that

$$\lambda(|x|)\,|\xi| = \sqrt{\lambda(|x|)\,|\xi|}\,\sqrt{\lambda(|x|)\,|\xi|} = \left(\sqrt{\lambda(|x|)\,|\xi|}\right)^*\#\sqrt{\lambda(|x|)\,|\xi|} + \mathcal{S}^0$$

so that

$$\left\langle {\rm Op}\left(\lambda(|x|)\ \langle \xi\rangle^{m-1}\right)u,u\right\rangle = \|{\rm Op}\left(\lambda(|x|)^{\frac12}\,|\xi|^{\frac12}\right)u\|_0^2 - \langle {\rm Op}\,(r)\,u,u\rangle\,, \qquad r\in\mathcal{S}^0$$

Altogether we have proved that

$$\int_0^T \|{\rm Op}\left(\lambda(|x|)^{\frac12}\,|\xi|^{\frac12}\right)u\|_0^2{\rm d}t \leq C_T\|u_0\|_0^2,$$

which is the smoothing effect.

Case $A = -\partial_{x_j}(a_{jk}(x)\partial_{x_k}$**.** The proof is analogous to the previous one. In this case, so get the analogous statement than Lemma 13.2, one needs to require that

(H1) $A(x) = (a_{jk}(x))$ is real, symmetric and positive definite.

(H2) $a_{jk}\in C_b^\infty$ and

$$|\nabla a_{jk}(x)| = o(|x|^{-1}), \qquad x\to\infty$$

(H3) the Hamiltonian flow of $A(x)\xi\cdot\xi$ is non trapped in one direction, i.e. the orbits of the hamiltonian system with hamiltonian $A(x)\xi\cdot\xi$ are unbounded for any initial datum.

Under these conditions, Doi [**?**] proved that Lemma 13.2 is true with $A(x)\xi\cdot\xi$ replacing $|\xi|^2$.

13.2 Cauchy theory for non-selfadjoint perturbation

Consider the Schrödinger equation

$$\mathrm{i}\dot{u} = -\Delta u + b(x) \cdot \partial_x u + c(x)u$$

where $b(x) = (b_1(x), \dots, b_d(x)) \in C^\infty(\mathbb{R}^d, \mathbb{C}^d)$ with

$$|\partial_x^\alpha b_j(x)| \leq C_\alpha \qquad \forall \alpha, \quad \forall j = 1, \dots, d.$$

The difficult is that b might have a real part, and the perturbation $\mathrm{Re}\, b\partial_x$ is bad in an energy estimate.

We assume that $\mathrm{Re}\, b(x)$ decrease when $x \to \infty$, and in particular that

$$|\mathrm{Re}\, b(x)| \leq \frac{C}{\langle x \rangle^2} \tag{13.4}$$

Let us see why.

First we write the operator using Weyl quantization (it is useful because the adjoint are much easier to compute!) We have that

$$\begin{aligned} -\Delta &= \mathrm{Op}^w\left(|\xi|^2\right) \\ b(x)\partial_x = \mathrm{Op}\,(b\mathrm{i}\xi) &= \mathrm{Op}^w\left(\mathrm{i}b(x)\cdot\xi - \frac{1}{2}\nabla_x b\right) \\ c(x) &= \mathrm{Op}^w\,(c) \end{aligned}$$

where we used the change of quantization formula $\mathrm{Op}\,(f) = \mathrm{Op}^w\,(g)$ with

$$g \sim \sum_\alpha \left(-\frac{1}{2}\right)^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha f \ .$$

Hence we rewrite the equation in decreasing order as

$$\mathrm{i}\dot{u} = \mathrm{Op}^w\left(|\xi|^2\right)u + \mathrm{Op}^w\left(\mathrm{i}b(x)\cdot\xi\right)u + \mathrm{Op}^w\left(c_0\right)u, \qquad c_0 \in \mathcal{S}^0. \tag{13.5}$$

Let us make an energy estimate in H^s: if

$$A := \frac{1}{\mathrm{i}}\mathrm{Op}^w\left(|\xi|^2 + \mathrm{i}b(x)\cdot\xi + c_0\right),$$

we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \langle D\rangle^{2s} u, u\right\rangle = \left\langle (A + A^*)\langle D\rangle^{2s} u, u\right\rangle + \left\langle [\langle D\rangle^{2s}, A]u, u\right\rangle. \tag{13.6}$$

We need to bound the two terms in H^s. We start with the second one, which does not create problems. Indeed

$$\begin{aligned} [\langle D\rangle^{2s}, A] &= \frac{1}{\mathrm{i}}[\langle D\rangle^{2s}, \mathrm{Op}^w\left(|\xi|^2 + b(x)\mathrm{i}\xi + c - \frac{1}{2}b_x\right)] \\ &= \frac{1}{\mathrm{i}}[\langle D\rangle^{2s}, \mathrm{Op}^w\left(\mathrm{i}b(x)\cdot\xi + c_0\right)] \in \mathrm{Op}^w\left(S^{2s}\right) \end{aligned}$$

hence $[\langle D\rangle^{2s}, A]$ is good in an energy estimate:

$$\left|\left\langle [\langle D\rangle^{2s}, A]u, u\right\rangle\right| \leq \|[\langle D\rangle^{2s}, A]u\|_{-s}\|u\|_s \leq C_s\|u\|_s^2.$$

The other term in (13.6) is worse. Let us compute it:

$$\begin{aligned} A + A^* &= \frac{1}{\mathrm{i}}\mathrm{Op}^w\left(|\xi|^2 + \mathrm{i}b(x)\cdot\xi + c_0\right) - \frac{1}{\mathrm{i}}\mathrm{Op}^w\left(|\xi|^2 - \mathrm{i}\bar{b}(x)\cdot\xi + \bar{c}_0\right) \\ &= \mathrm{Op}^w\left((b+\bar{b})\cdot\xi + c_1\right), \qquad c_1 \in \mathcal{S}^0 \\ &= \mathrm{Op}^w\left(2(\mathrm{Re}\, b)\cdot\xi\right) + \mathrm{Op}^w\left(c_1\right) \end{aligned}$$

The first term is a problem in an energy estimate!! We have too many derivatives. Indeed we cannot bound

$$\left|\left\langle (A + A^*)\langle D\rangle^{2s} u, u\right\rangle\right| \leq C_s\|u\|_s^2$$

because the term $\mathrm{Op}^w\left(2(\mathrm{Re}\, b)\cdot\xi\right)\langle D\rangle^{2s} \in \mathrm{Op}\left(\mathcal{S}^{2s+1}\right)$ and we cannot close the energy estimate.

We do not surrender: can we make a change of coordinates which "simplify the term"?

So we look for a change of coordinates of the form

$$\varphi = \mathcal{U}(1,0)u,$$

where $\mathcal{U}(1,0)$ is the time 1 flow of the PDE generated by a pseudodifferential operator $X = \mathrm{Op}\left(\chi\right)$

$$\dot{u} = Xu,$$

with $\chi \in \mathcal{S}^0$ to be determined later on. Note that $\mathcal{U}(1,0)$ is invertible and maps $H^s \to H^s$ $\forall s$. Thus if we prove good estimates for φ, they hold also for u.

Then φ fulfills the equation

$$\mathrm{i}\dot{\varphi} = \mathcal{U}(1,0)\,\mathrm{Op}^w\left(|\xi|^2 + \mathrm{i}b(x)\cdot\xi + c_0\right)\mathcal{U}(1,0)^{-1}\varphi.$$

Decompose

$$\begin{aligned} \mathcal{U}(1,0)\,\mathrm{Op}^w\left(|\xi|^2 + \mathrm{i}b(x)\cdot\xi + c_0\right)\mathcal{U}(1,0)^{-1} = &\,\mathcal{U}(1,0)\,\mathrm{Op}^w\left(|\xi|^2\right)\mathcal{U}(1,0)^{-1} \\ &+ \mathcal{U}(1,0)\,\mathrm{Op}^w\left(\mathrm{i}b(x)\cdot\xi\right)\mathcal{U}(1,0)^{-1} \\ &+ \mathcal{U}(1,0)\,\mathrm{Op}^w\left(c_0\right)\mathcal{U}(1,0)^{-1} \end{aligned}$$

We expand each line in decreasing order:

$$\begin{aligned} &\mathcal{U}(1,0)\,\mathrm{Op}^w\left(|\xi|^2\right)\mathcal{U}(1,0)^{-1} = \mathrm{Op}^w\left(|\xi|^2\right) + [X, \mathrm{Op}^w\left(|\xi|^2\right)] + R_0 \\ &\mathcal{U}(1,0)\,\mathrm{Op}^w\left(\mathrm{i}b(x)\cdot\xi\right)\mathcal{U}(1,0)^{-1} = \mathrm{Op}^w\left(\mathrm{i}b(x)\cdot\xi\right) + R_0 \\ &\mathcal{U}(1,0)\,\mathrm{Op}^w\left(c_0\right)\mathcal{U}(1,0)^{-1} = R_0 \end{aligned}$$

where R_0 is a bounded operator from $H^s \to H^s$, $\forall s$. Collecting the terms of the same order we find that

$$\mathrm{i}\dot{\varphi} = H\varphi$$

with

$$\begin{aligned} H :=&\mathrm{Op}^w\left(|\xi|^2\right) \\ &+ [X, \mathrm{Op}^w\left(|\xi|^2\right)] + \mathrm{Op}^w\left(\mathrm{i}\,\mathrm{Re}\, b(x)\cdot\xi\right) - \mathrm{Op}^w\left(\mathrm{Im}\, b(x)\cdot\xi\right) \\ &+ R_0 \end{aligned}$$

Now $\mathrm{Op}^w(\mathrm{Im}\, b(x)\cdot\xi)$ is selfdjoint, so it does not give us problems in an energy estimate. On the contrary, as we have seen, $\mathrm{Op}^w(\mathrm{iRe}\, b(x)\cdot\xi)$ is problematic and we would like to eliminate it.

We distinguish two cases.

Case $d = 1$**.** In case the dimension $d = 1$, we can find X to eliminate the bad part. Namely we look for $X = \mathrm{Op}^w(\chi)$ such that

$$[X, \mathrm{Op}^w(|\xi|^2)] + \mathrm{Op}^w(\mathrm{i}\,\mathrm{Re}\, b(x)\cdot\xi) = Z_0, \qquad Z_0 \in \mathrm{Op}^w(\mathcal{S}^0).$$

By symbolic calculus, it is enough to choose $X = \mathrm{Op}^w(\chi)$ such that

$$\begin{cases} -\mathrm{i}\{\chi, \xi^2\} + \mathrm{iRe}\, b(x)\xi = 0, \\ z_0 = -\mathrm{i}\left(\{\chi,\xi^2\}_M - \{\chi,\xi^2\}\right) \in \mathcal{S}^0, \qquad Z_0 := \mathrm{Op}^w(z_0) \end{cases}$$

The first equation is

$$\{\xi^2, \chi\} + \mathrm{Re}\, b(x)\xi = 0$$

namely

$$2\xi\chi_x + \mathrm{Re}\, b(x)\xi = 0 \Rightarrow 2\chi_x + \mathrm{Re}\, b(x) = 0$$

which is solved by

$$\chi(x) = \int_0^x \mathrm{Re}\, b(y)\mathrm{d}y$$

The assumption (13.4) guarantees that $\chi \in C^\infty \cap L^\infty$. It follows that $\chi \in \mathcal{S}^0$ is a symbol and our construction work!

With this choice, we have that φ fulfills

$$\mathrm{i}\dot\varphi = \left(\mathrm{Op}^w(|\xi|^2) - \mathrm{Op}^w(\mathrm{Im}\, b(x)\cdot\xi) + R_0\right)\varphi$$

hence it fulfills good energy estimates and we can prove that

$$\partial_t\|\varphi(t)\|_s^2 \le C_s\|\varphi\|_s^2 \quad \Rightarrow \quad \|\varphi(t)\|_s \le e^{tC_s}\|\varphi(0)\|_s.$$

Case $d \ge 2$**.** In this case the homological equation is more complicated, being

$$\{|\xi|^2, \chi\} + \mathrm{Re}\, b(x)\cdot\xi = 0$$

and we cannot solve it exactly (it is a PDEs in χ). However, we can impose the weaker condition which is sufficient for our aim. Indeed it is enough to impose a sign condition. This will be done using Doi lemma

Lemma 13.2 (Doi)**.** *Let λ be even, smooth, radially decreasing, non negative, $\lambda \in L^1([0,\infty))$. Then there exists a real value symbol $p \in \mathcal{S}^0$ and a constant $\beta > 0$ such that*

$$\{|\xi|^2, p\} \ge \lambda(|x|)|\xi| - \beta$$

Proof. Define $f(t) := \int_0^t \lambda(r)\mathrm{d}r$ and let

$$\Phi(x) = (f(x_1), \dots, f(x_d))$$

so that $\Phi(x)$ is smooth and bounded. Let

$$\Phi'_{\mathrm{sym}}(x) := \frac{1}{2}\left(\partial_{x_j} f_i + \partial_{x_i} f_j\right) = \begin{pmatrix} \lambda(x_1) & 0 & \dots & 0 \\ 0 & \lambda(x_2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda(x_d) \end{pmatrix} \ge \lambda(|x|)\mathrm{Id}$$

as λ is even and decreasing. Define

$$p(x,\xi) := \Phi(x)\cdot \frac{\xi}{\langle\xi\rangle} \in \mathcal{S}^0.$$

Then

$$\{|\xi|^2, p\} = 2\frac{\xi}{\langle\xi\rangle}\cdot\nabla_x(\Phi(x)\cdot\xi) = \frac{2}{\langle\xi\rangle}\Phi'_{\rm sym}(x)\xi\cdot\xi \geq 2\lambda(|x|)\frac{|\xi|^2}{\langle\xi\rangle} \geq \lambda(|x|)|\xi| - \beta.$$

for some $\beta > 0$. □

We exploit Doi Lemma. We choose $\lambda(t) = \langle t\rangle^{-2}$, and select $p \in \mathcal{S}^0$ so that

$$\{|\xi|^2, p\} \geq \frac{|\xi|}{\langle |x|\rangle^2} - \beta.$$

Then we put

$$\chi := -2p \in \mathcal{S}^0$$

so that in the homological equation

$$\{|\xi|^2,\chi\} + \operatorname{Re} b(x)\cdot\xi = -2\{|\xi|^2, p\} + \operatorname{Re} b(x)\cdot\xi \leq -2\frac{|\xi|}{\langle x\rangle^2} + \beta + \frac{|\xi|}{\langle x\rangle^2} \leq -\frac{|\xi|}{\langle x\rangle^2} + \beta.$$

Thus the homological equation has a sign. Let's see how to use it!

Let us now perform an energy estimate. We have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \langle D\rangle^{2s}\varphi,\varphi\right\rangle = \left\langle (\frac{1}{\mathrm{i}}(H-H^*)\,\langle D\rangle^{2s} u, u\right\rangle + \left\langle [\langle D\rangle^{2s}, \frac{1}{\mathrm{i}}H]\varphi,\varphi\right\rangle.$$

The second term has good estimate, as before. Concerning the first term we have (as χ is real valued)

$$\begin{aligned}\frac{1}{\mathrm{i}}(H-H^*) &= \frac{2}{\mathrm{i}}[X, \mathrm{Op}^w\left(|\xi|^2\right)] + 2\mathrm{Op}^w\left(\operatorname{Re} b(x)\cdot\xi\right)\\ &= 2\mathrm{Op}^w\left(\{|\xi|^2,\chi\} + \operatorname{Re} b(x)\cdot\xi\right) + \mathrm{Op}^w\left(r_0\right)\end{aligned}$$

Hence, againg by symbolic calculus

$$\left\langle (\frac{1}{\mathrm{i}}(H-H^*)\,\langle D\rangle^{2s} u, u\right\rangle = \left\langle \mathrm{Op}^w\left(\left(\{|\xi|^2,\chi\} + \operatorname{Re} b(x)\cdot\xi\right)\langle\xi\rangle^{2s}\right)u,u\right\rangle + \langle \mathrm{Op}^w\left(r_{2s}\right)u,u\rangle$$

Note that the last term fulfill energy estimates, so it is ok. Concerning the first one, we have that its symbol fulfills

$$g(x,\xi) := \left(\{|\xi|^2,\chi\} + \operatorname{Re} b(x)\cdot\xi\right)\langle\xi\rangle^{2s} \leq -\frac{|\xi|\,\langle\xi\rangle^{2s}}{\langle x\rangle^2} + \beta\,\langle\xi\rangle^{2s}$$

and by the strong Garding inequality

$$\langle \mathrm{Op}^w\left(g\right)u,u\rangle \leq -\left\langle \mathrm{Op}\left(\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s}\right)u,u\right\rangle + C\|u\|_s^2.$$

Finally use that

$$\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s} = \left(\sqrt{\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s}}\right)^2 = \left(\sqrt{\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s}}\right)^* \#\sqrt{\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s}} + \mathcal{S}^{2s}$$

so that

$$\left\langle \mathrm{Op}\left(\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s}\right)\varphi,\varphi\right\rangle = \|\mathrm{Op}\left(\sqrt{\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s}}\right)\varphi\|_0^2 + \langle \mathrm{Op}\left(r_{2s}\right)\varphi,\varphi\rangle\,, \qquad r_{2s}\in\mathcal{S}^{2s}$$

In this way the energy estimate becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_s^2 \le -\|\mathrm{Op}\left(\sqrt{\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s}}\right)\varphi\|_0^2 + C\|\varphi\|_s^2$$

namely

$$\|\varphi(t)\|_s^2 + \int_0^t \|\mathrm{Op}\left(\sqrt{\langle x\rangle^{-2}\,|\xi|\,\langle\xi\rangle^{2s}}\right)\varphi(\tau)\|_0^2\mathrm{d}\tau \le \|\varphi(0)\|_s^2 + C\int_0^t \|\varphi(\tau)\|_s^2\mathrm{d}\tau$$

hence we control the H^s norm!!

Case of variable coefficients at highest order We can also consider the case where the PDEs is with variable coefficients at highest order, namely

$$\mathrm{i}\dot{u} = -(\sum_{j,k}\partial_{x_k}a_{jk}(x)\partial_{x_j})u + b(x)\cdot\partial_x u + c(x)u$$

In this case we make the following assumptions

(H1) $a_{j,k}(x), b(x), c(x)\in\mathcal{S}^0$ (namely smooth, bounded with bounded derivatives), and $(a_{j,k}(x))_{j,k}$ symmetric and real.

(H2) ellipticity at highest order

$$\delta^{-1}|\xi|^2 \le \sum_{a_{j,k}(x)\xi_j\xi_k} \le \delta|\xi|^2$$

(H3) asymptotic flatness at infinity

$$|\partial_x^\alpha(a_{jk}(x)-1)| \le \frac{C}{\langle x\rangle^2}, \qquad \forall j,k,\quad \forall\alpha\in\mathbb{N}^d$$

Also in this case we can prove that the solution exists in H^s.
Case $d=1$**.** Let us put in dimension 1 first. Then

$$\partial_x(a(x)\partial_x)u = a(x)\partial_{xx}u + a_x(x)\partial_x u$$

so we write the equation as

$$\mathrm{i}\dot{u} = -a(x)\partial_{xx}u + (b(x)+a_x)\partial_x u + c(x)u$$

Let us put the coefficient at highest order to constant coefficient. This can be done with Egorov theorem, but in this case it is easier to perform a direct computation: let us make the change of variables

$$u(x) = v(x+\beta(x))$$

with $\beta(x)$ to be determined in such a way that $y = x + \beta(x)$ is a diffeomorphism. Then

$$\partial_x u = (\partial_x v)(x + \beta(x))\,(1 + \beta_x(x))$$
$$\partial_{xx} u = (\partial_{xx} v)(x + \beta(x))\,(1 + \beta_x(x))^2 + (\partial_x v)(x + \beta(x))\,\beta_{xx}(x)$$

So writing the direct and inverse diffeomorphism

$$y = x + \beta(x), \qquad x = y + \breve{\beta}(y)$$

we find for v the equation

$$\mathrm{i}\dot{v} = -\left[a(1+\beta_x)^2\right]_{y+\breve{\beta}(y)} \partial_{yy} v + \left[\beta_{xx} + (1+\beta_x)(b + a_x)\right]_{y+\breve{\beta}(y)} \partial_y v + c_{y+\breve{\beta}(y)} v$$

Now we look for β so that

$$a(1+\beta_x)^2 = 1$$

As the ellipticity assumption in this case gives $a(x) \geq \delta^{-1} > 0$, we have

$$\beta_x = \sqrt{\frac{1}{a}} - 1 = \frac{1-a}{a+\sqrt{a}}$$

and integrating

$$\beta(x) = \int_0^x \frac{1 - a(y)}{a(y) + \sqrt{a(y)}} \mathrm{d}y$$

Now assumption (H2) gives that $|a(x) - 1| = O(\langle x \rangle^{-2})$, hence $\beta(x)$ is bounded. Moreover as

$$1 + \beta_x = \sqrt{\frac{1}{a}} \geq \delta^{-1/2} > 0$$

the function $x + \beta(x)$ is a good diffeomorphism.

With this choice we have

$$\mathrm{i}\dot{v} = -\partial_{yy} v + \tilde{b}(y) \partial_y v + \tilde{c} v$$

with

$$\tilde{b}(y) := \left[\beta_{xx} + (1+\beta_x)(b + a_x)\right]_{y+\breve{\beta}(y)}$$

Then one verifies, using the properties of $\beta(x)$ and the assumptions, that the new coefficient $\tilde{b}$ fulfills

$$\left|\mathrm{Re}\, \tilde{b}(y)\right| \lesssim \langle y \rangle^{-2}.$$

Hence we are in the previous case.