11 Flow generation

In the section we aim to use pseudodifferential operators to generate a flow. We shall take symbols $a(t,x,\xi)$ depending on time, which we might regard as an external parameter. We will view these symbols as continuous maps from $\mathbb{R} \to \mathcal{S}^m$, in the sense that the map

$$t \mapsto \wp_k^m(\partial_t^\ell a)$$

is continuous $\forall k \in \mathbb{N}$ and $\forall 0 \leq \ell \leq N$. In this case we will write $a \in C^N(\mathbb{R}, \mathcal{S}^m)$. We write $C^\infty(\mathbb{R}, \mathcal{S}^m) = \bigcap_N C^N(\mathbb{R}, \mathcal{S}^m)$.

The problem is then the following: consider the Cauchy problem

$$\begin{cases} \partial_t u = \mathrm{Op}\,(a(t))\,u \\ u(t_0) = f \in H^s. \end{cases} \tag{11.1}$$

We want to discuss the following questions:

(Q1) Can we give conditions on the symbol $a(t,x,\xi)$ which guarantees the existence of a propagator $\mathcal{U}(t,s)$, namely a linear, bounded operator so that $\mathcal{U}(t,t_0)f$ solves (11.1)? Moreover, is $\mathcal{U}(t,t_0)$ bounded as an operator from $H^s \to H^s$?

(Q2) Let $b \in \mathcal{S}^{m'}$: can we say that

$$\mathcal{U}(t,t_0)^{-1}\,\mathrm{Op}\,(b)\,\mathcal{U}(t,t_0)$$

is a pseudodifferential operator? Which is its symbol?

We will first investigate the problem of existence of the flow. In the next section we will study how pseudodifferential operators transform under flow conjugation.

The key step of the argument is to provide energy estimates. We give a result in a quite general setup.

Lemma 11.1 (Formal energy estimate)**.** *Let $\mathcal{H}$ be Hilbert space with scalar product $\langle \cdot,\cdot \rangle$, $\mathcal{X}$ a dense subset of $\mathcal{H}$, and consider the abstract Cauchy problem*

$$\begin{cases} \dot{u} = A(t)u \\ u(0) = u_0 \in \mathcal{X} \end{cases}$$

Assume that $u(t) \in C(\mathbb{R}, \mathcal{X})$ is a solution of the Cauchy problem. Then, for any operator $B(t)\colon X \to H$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle B(t)u,u\rangle = \langle (A+A^*)Bu,u\rangle + \langle [B,A]u,u\rangle + \left\langle \dot{B}u,u\right\rangle \tag{11.2}$$

Proof. Just compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle Bu,u\rangle &= \left\langle \dot{B}u,u\right\rangle + \langle B\dot{u},u\rangle + \langle Bu,\dot{u}\rangle \\ &= \left\langle \dot{B}u,u\right\rangle + \langle BAu,u\rangle + \langle Bu,Au\rangle \\ &= \left\langle \dot{B}u,u\right\rangle + \langle ABu,u\rangle + \langle [B,A]u,u\rangle + \langle A^*Bu,u\rangle \\ &= \left\langle \dot{B}u,u\right\rangle + \langle (A+A^*)Bu,u\rangle + \langle [B,A]u,u\rangle \end{aligned}$$

□

The following corollary is often useful:

Corollary 11.2. *In the same assumptions of Lemma 11.1, assume furthermore that A is skew-selfadjoint (on its domain), i.e. $A^* = -A$. Then the flow is unitary, i.e. the L^2 norm of the solution is preserved:*

$$\|u(t)\|_0 = \|u_0\|_0$$

Proof. Choose $B = \mathbb{1}$ in the formal energy estimate to deduce that $\partial_t \langle u, u \rangle = 0$. $\square$

11.1 Existence of flow for hyperbolic equation

Let us begin with the following result:

Theorem 11.3 (Flow generation for hyperbolic systems)**.** *Consider the Cauchy problem* (11.1)*. Assume that*

(H1) for some $m \leq 1$, $a(t, x, \xi) \in C(\mathbb{R}, \mathcal{S}^m)$ and $t \mapsto a(t, x, \xi)$ bounded in $\mathcal{S}^m$.

(H2) $a^* + a \in C(\mathbb{R}, \mathcal{S}^0)$.

Then (11.1) *has a unique propagator $\mathcal{U}(t, t_0)$ such that $t \mapsto \mathcal{U}(t, t_0)f$ is the unique global solution of* (11.1) *in H^r. Moreover one has*

(i) Group property*: for any $t, \tau, t_0 \in \mathbb{R}$*

$$\mathcal{U}(t, \tau)\mathcal{U}(\tau, t_0) = \mathcal{U}(t, t_0), \qquad \mathcal{U}(t, t) = \mathbb{1}, \qquad \mathcal{U}(t, t_0)^{-1} = \mathcal{U}(t_0, t)$$

(ii) Continuity*: $\mathcal{U}(t, t_0)$ is bounded as a linear operator from from H^r to H^r for any $r \in \mathbb{R}$.*

(iii) Derivatives*: One has*

$$\partial_t \mathcal{U}(t, t_0) = \mathrm{Op}\,(a(t))\,\mathcal{U}(t, t_0)\ , \qquad \partial_\tau \mathcal{U}(t, \tau) = -\mathcal{U}(t, \tau)\mathrm{Op}\,(a(\tau))$$

(iv) Unitarity*: if $\mathrm{Op}\,(a(t))$ is anti-selfadjoint, i.e. $\mathrm{Op}\,(a(t))^* = -\mathrm{Op}\,(a(t))$ $\forall t$, then $\mathcal{U}(t, t_0)$ is unitary in L^2.*

Proof. The strategy is the following: we regularize (11.1) using a family of smoothing operators J_ϵ, construct a family of propagators $\mathcal{U}_\epsilon(t, s)$ and then pass to the limit $\epsilon \to 0$. The crucial step is to prove *energy estimates*, which means estimates of $\|\mathcal{U}_\epsilon(t, t_0)\|_{H^r \to H^r}$ uniform in ϵ, so that they holds also for the limiting object.

Step 1: regularization. Let $\varphi \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, $\varphi(0) = 1$ and put

$$J_\epsilon := \varphi(\epsilon D_x) \equiv \mathrm{Op}\,(\varphi(\epsilon\xi))\,. \tag{11.3}$$

Since $\varphi \in \mathcal{S}$, J_ϵ is a smoothing operator for any $\epsilon \in (0, 1)$ and fulfills

$$\|J_\epsilon u\|_{H^r} \leq \sup_{\xi \in \mathbb{R}^d} |\varphi(\epsilon\xi)|\ \|u\|_{H^r} \leq C\,\|u\|_{H^r} \quad , \quad \forall r \in \mathbb{R}, \qquad \forall \epsilon \in (0, 1). \tag{11.4}$$

Moreover

$$\|J_\epsilon u\|_{H^r} \leq \sup_{\xi \in \mathbb{R}^d} |\langle\xi\rangle\ \varphi(\epsilon\xi)|\ \|u\|_{H^{r-1}} \leq \frac{C}{\epsilon}\,\|u\|_{H^{r-1}} \quad , \quad \forall r \in \mathbb{R}. \tag{11.5}$$

In particular $\forall \epsilon \in (0,1)$ the operator $J_\epsilon \mathrm{Op}\,(a(t))\, J_\epsilon$ is bounded $H^r \to H^r$ for any r, but its norm is not bounded uniformly in ϵ, as

$$\|J_\epsilon \mathrm{Op}\,(a(t))\, J_\epsilon u\|_{H^r} \le \frac{C}{\epsilon}\|u\|_{H^r}.$$

However we will prove that the norm of the propagator is bounded uniformly in ϵ.

First of all consider the regularized Cauchy problem

$$\partial_t u_\epsilon = J_\epsilon \mathrm{Op}\,(a(t))\, J_\epsilon u_\epsilon, \qquad u_\epsilon(t_0) = f \in H^r; \tag{11.6}$$

it can be seen as a Banach-space value ordinary differential equation, to which the Picard iteration method applies. Thus, given $f \in H^r$, we can solve (11.6) and produce a unique solution $u_\epsilon(t) \in H^r$. Hence we get a propagator $\mathcal{U}_\epsilon(t, t_0)$, defined for all times t, t_0 in $\mathbb{R}$ fulfilling (i)–(iv) of the theorem.

Step 2: energy estimates. The key part of the proof is to show uniform energy estimates, namely that for any t, t_0 in $\mathbb{R}$, there exists $C_r(t, t_0) > 0$ such that

$$\|\mathcal{U}_\epsilon(t, t_0) f\|_{H^r} \le C_r(t, t_0)\|f\|_{H^r}\ , \qquad \forall \epsilon \in (0,1)\ . \tag{11.7}$$

To prove (11.7) we do an energy estimate: namely we consider the solution $u_\epsilon(t) = \mathcal{U}_\epsilon(t, s) f$ of (11.6) and control its Sobolev norm

$$\| \langle D\rangle^r u_\epsilon(t)\|_{L^2}^2 = \left\langle \langle D\rangle^{2r} u_\epsilon, u_\epsilon\right\rangle$$

By the formal energy estimate (11.2), with $B := \langle D\rangle^{2r}$ and $A := J_\epsilon \mathrm{Op}\,(a(t))\, J_\epsilon$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\| \langle D\rangle^r u_\epsilon\|_{L^2}^2 &= \left\langle \left(J_\epsilon \mathrm{Op}\,(a)\, J_\epsilon + J_\epsilon^* \mathrm{Op}\,(a)^*\, J_\epsilon^*\right) \langle D\rangle^{2r} u_\epsilon, u_\epsilon\right\rangle \\ &\quad + \left\langle \left[\langle D\rangle^{2r}, J_\epsilon \mathrm{Op}\,(a)\, J_\epsilon\right] u_\epsilon, u_\epsilon\right\rangle\end{aligned}$$

Now use that J_ϵ is a Fourier multiplier with real symbol, hence it is self-adjoint and commutes with $\langle D\rangle^{2r}$. We get therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\| \langle D\rangle^r u_\epsilon\|_{L^2}^2 &= \left\langle J_\epsilon \left(\mathrm{Op}\,(a) + \mathrm{Op}\,(a)^*\right) J_\epsilon \langle D\rangle^{2r} u_\epsilon, u_\epsilon\right\rangle \\ &\quad + \left\langle J_\epsilon \left[\langle D\rangle^{2r}, \mathrm{Op}\,(a)\right] J_\epsilon u_\epsilon, u_\epsilon\right\rangle\end{aligned}$$

Now the assumptions are clear: by symbolic calculus

$$(\mathrm{H1}) \quad \Longrightarrow \quad \left[\langle D\rangle^{2r}, \mathrm{Op}\,(a)\right] \in \mathrm{Op}\left(\mathcal{S}^{2r+m-1}\right) \subset \mathrm{Op}\left(\mathcal{S}^{2r}\right)$$

while

$$(\mathrm{H2}) \quad \Longrightarrow \quad \mathrm{Op}\,(a + a^*) \in \mathrm{Op}\left(\mathcal{S}^0\right)$$

Hence Sobolev continuity of pseudodifferential operators and estimate (11.4) give immediately: for any $r, t \in \mathbb{R}$, there exists $C_r(t)$ such that

$$\begin{aligned}&\|J_\epsilon \left(\mathrm{Op}\,(a) + \mathrm{Op}\,(a)^*\right) J_\epsilon \langle D\rangle^{2r} u_\epsilon\|_{H^{-r}} \le C_r(t)\|u_\epsilon\|_{H^r}, \qquad \forall \epsilon \in (0,1), \\ &\|J_\epsilon \left[\langle D\rangle^{2r}, \mathrm{Op}\,(a)\right] J_\epsilon u_\epsilon\|_{H^{-r}} \le C_r(t)\|u_\epsilon\|_{H^r}, \qquad \forall \epsilon \in (0,1)\end{aligned}$$

(remark that the constant $C_r(t)$ depends only on the seminorms of a!). Then by duality we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\| \langle D\rangle^r u_\epsilon\|_{L^2}^2 \leq C_r(t)\|u_\epsilon\|_{H^r}^2.$$

In particular Gronwall inequality gives for the propagator

$$\|u_\epsilon(t)\|_{H^r} \leq \widetilde{C}_r(t,t_0)\,\|u_\epsilon(0)\|_{H^r} \leq \widetilde{C}_r(t,t_0)\,\|f\|_{H^r}, \qquad \forall \epsilon \in (0,1). \tag{11.8}$$

This proves the boundedness of the regularized propagator.

Step 3: passage to the limit. We know that the solution (11.8), $\{u_\epsilon \colon 0 \leq \epsilon \leq 1\}$ is a bounded set of $C([t_0,t],H^r)$. Since $\dot{u}_\epsilon = J_\epsilon \mathrm{Op}\,(a)\,J_\epsilon u_\epsilon$, it follows that

$$\|\dot{u}_\epsilon\|_{H^{r-1}} \leq \|\mathrm{Op}\,(a)\,J_\epsilon u_\epsilon\|_{H^{r-1}} \leq \|u_\epsilon\|_{H^r} \leq \widetilde{C}_r(t,t_0)\,\|f\|_{H^r}, \qquad \forall \epsilon \in (0,1)$$

so $\{\dot{u}_\epsilon \colon 0 \leq \epsilon \leq 1\}$ is a bounded set of $C([t_0,t],H^{r-1})$. Hence $\{u_\epsilon \colon 0 \leq \epsilon \leq 1\}$ is a bounded subset of $C^1([t_0,t],H^{r-1})$. Furthermore, for each $\tau \in [t_0,t]$, $\{u_\epsilon(\tau) \colon 0 \leq \epsilon \leq 1\}$, being a bounded subset of H^r, is a relatively compact subset of H^{r-1}. Hence, by Ascoli's theorem[3], there is a sequence $\epsilon_n \to 0$ such that u_{ϵ_n} converges, in $C([t_0,t],H^{r-1})$, to a limit we call $u_\infty \in C([t_0,t],H^{r-1})$ which fulfills also the original equation.

Step 4: construction of solution. Now we want to construct a solution of the original Cauchy problem (11.1). So let $u_{j0} \in H^{r+4}$, with $u_{j0} \to u_0 \in H^r$. By the previous construction, the propagator $\mathcal{U}(t,t_0)u_{j0} \in C([t_0,t],H^{r+2})$ and solves the equation. Then we can apply the energy inequality and deduce that

$$\|\mathcal{U}(t,t_0)u_{j0}\|_{H^r} \leq C\|u_{j0}\|_{H^r};$$

as the propagator is linear one deduces that $\{\mathcal{U}(t,t_0)u_{j0}\}$ is a Cauchy sequence is $C([t_0,t],H^r)$. The limit solves our Cauchy problem.
For the uniqueness apply again the energy inequality. □

Application: we will often apply Theorem 11.3 to the Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t u = \mathrm{Op}\,(a(t))\,u \\ u(t_0) = u_0 \in H^r \end{cases} \tag{11.9}$$

We are in the setup of Theorem 11.3 substituting $a \rightsquigarrow -\mathrm{i}a$. In this case assumptions (H1) – (H2) become

(H1) $a \in C(\mathbb{R},\mathcal{S}^m)$, $m \leq 1$

(H2) If $a = a_m + a_{m-1}$, with $a_m \in \mathcal{S}^m$, $a_{m-1} \in \mathcal{S}^{m-1}$, then a_m is real.

Indeed to fulfill assumption (H2) one needs that $(-\mathrm{i}a)^* - \mathrm{i}a \in \mathcal{S}^0$, hence

$$(-\mathrm{i}a)^* - \mathrm{i}a = \overline{(-\mathrm{i}a_m)} - \mathrm{i}a_m + \mathcal{S}^{m-1} = \mathrm{i}(\overline{a}_m - a_m) + \mathcal{S}^{m-1} \in \mathcal{S}^0$$

namely one needs a_m to be real valued.

[3]Let X be a compact Hausdorff space. Then a subset $\mathcal{F}$ of $C(X)$ is relatively compact in the topology induced by the uniform norm if and only if it is equicontinuous and pointwise bounded.

11.2 Existence of flow for elliptic symbols

Consider now the case of Schrödinger equations with elliptic, real valued symbols

$$\begin{cases} \mathrm{i}\dot{u} = \mathrm{Op}^w\left(a(t,x,\xi)\right)u \\ u(t_0) = u_0 \in H^r. \end{cases} \tag{11.10}$$

We assume that the symbol $a(t,x,\xi)$ fulfills the following assumptions

(H1) There exists $m \leq 2$ such that $t \mapsto \partial_t^\ell a(t,x,\xi) \in \mathcal{S}^m$ is bounded, for any $\ell \in \mathbb{N}$.

(H2) $a(t,x,\xi)$ is real valued.

(H3) $a(t,x,\xi)$ is elliptic, i.e. $\exists R, C > 0$ such that

$$a(t,x,\xi) \geq C_a \langle\xi\rangle^m, \qquad \forall|\xi| \geq R, \qquad \forall t \in \mathbb{R}. \tag{11.11}$$

The difficulty is that the operator is time dependent and variable coefficients.

An example of symbols that fulfills the assumptions (H1)–(H3) is the Schrödinger equation

$$\mathrm{i}\dot{u} = -\left(\sum_{j,k} \partial_{x_k} a_{j,k}(t,x)\partial_{x_j}\right)u$$

whose symbol is the second order variable coefficients laplacian

$$a(t,x,\xi) := \sum_{j,k} a_{j,k}(t,x)\xi_j\xi_k.$$

Then assumptions (H1)–(H3) amounts to require that $t \mapsto a_{j,k}(t,x)$ to be real, uniformly bounded from above and below (together with its derivatives), e.g.

$$c_1 \leq a_{j,k}(t,x) \leq c_2, \qquad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^d$$

Then we have the following result:

Theorem 11.4. *Consider the Cauchy problem* (11.10)*. Assume (H1)–(H3). Then there exists a solution global in time, and the propagator $\mathcal{U}(t,t_0)$ is bounded as a map $H^s \to H^s$ $\forall s \in \mathbb{R}$.*

Proof. The proof is similar to the previous case. The key step is to obtain energy estimate. However, a crucial step in the previous proof was that $[\langle D\rangle^r, \mathrm{Op}(a)] \in \mathrm{Op}(\mathcal{S}^r)$ for any $a \in \mathcal{S}^1$ (we used this to obtain energy estimates). However now $a \in \mathcal{S}^2$, so $[\langle D\rangle^r, \mathrm{Op}(a)] \in \mathrm{Op}\left(\mathcal{S}^{r+1}\right)$ and it is not clear how to close the energy estimate. The idea here is to exploit the ellipticity of $a(t,x,\xi)$. Put

$$B = \mathrm{Op}^w(b), \qquad b(t,x,\xi) := \left(a(t,x,\xi)\right)^{\frac{2s}{m}}$$

Then by symbolic calculus $t \mapsto \partial_t^\ell b(t,x,\xi) \in \mathcal{S}^{2s}$ is uniformly bounded $\forall \ell \in \mathbb{N}$, it is real valued, and it is elliptic of order $2s$, i.e.

$$b(t,x,\xi) \geq C_b \langle\xi\rangle^{2s}, \qquad \forall|\xi| \geq R, \qquad \forall t \in \mathbb{R}.$$

By Garding inequality

$$\|u\|_s^2 \leq \langle \mathrm{Op}^w(b)u, u\rangle + C_0\|u\|_0^2 \tag{11.12}$$

thus, if we control $\langle \mathrm{Op}^w(b)\, u, u\rangle$, we control also $\|u\|_s$ (as the L^2 norm is constant along the flow).

Let us check the energy estimate. By the usual computation (11.2)

$$\partial_t \langle \mathrm{Op}^w(b)\, u, u\rangle = \left\langle \mathrm{Op}^w\left(\dot{b}\right) u, u\right\rangle + \frac{1}{\mathrm{i}} \langle [\mathrm{Op}^w(b), \mathrm{Op}^w(a)]u, u\rangle$$

where we used that, as $a(t,x,\xi)$ is real, $\mathrm{Op}^w(a) - \mathrm{Op}^w(a)^* = \mathrm{Op}^w(a) - \mathrm{Op}^w(\overline{a}) = 0$.
We estimate the two terms: first

$$\left|\left\langle \mathrm{Op}^w\left(\dot{b}\right) u, u\right\rangle\right| \leq \|\mathrm{Op}^w\left(\dot{b}\right) u\|_{-s}\ \|u\|_s \leq \wp_K^{2s}(b)\|u\|_s^2 \leq C_1\|u\|_s^2. \tag{11.13}$$

Now by symbolic calculus

$$[\mathrm{Op}^w(b), \mathrm{Op}^w(a)] = \frac{1}{\mathrm{i}}\mathrm{Op}^w(\{b,a\}) + \mathrm{Op}^w(r), \qquad r \in \mathcal{S}^{2s+m-2}$$

The important point is that

$$\{b,a\} = \{a^{\frac{2s}{m}}, a\} = 0,$$

hence $[\mathrm{Op}^w(b), \mathrm{Op}^w(a)] = \mathrm{Op}^w(r)$ and we have the estimate

$$\begin{aligned}|\langle[\mathrm{Op}^w(b), \mathrm{Op}^w(a)]u, u\rangle| &\leq \|\mathrm{Op}^w(r)\, u\|_{-s}\ \|u\|_s \leq C\|u\|_{s+m-3}\|u\|_s \\ &\leq C\|u\|_s^2\end{aligned}$$

provided $m \leq 3$.

So we have obtained

$$\partial_t \langle \mathrm{Op}^w(b)\, u, u\rangle \leq C\|u\|_s^2$$

and integrating in time

$$\langle \mathrm{Op}^w(b(t))\, u(t), u(t)\rangle \leq \langle \mathrm{Op}^w(b(0))\, u(0), u(t0\rangle + \int_0^t \|u(\tau)\|_s^2 \mathrm{d}\tau$$

By the Sobolev continuity, Garding inequality and the L^2 conservation

$$\|u(t)\|_s^2 \leq \|u(0)\|_s^2 + C\int_0^t \|u(\tau)\|_s^2 \mathrm{d}\tau$$

Then apply Gronwall inequality.

As a final comment, remark that the solution is constructed by regularization with the operators J_ϵ defined in (11.3). So one has to substitute $A \rightsquigarrow J_\epsilon A J_\epsilon$ and to compute the commutators. Remark that $J_\epsilon = p(\epsilon\xi)$ and $p(\epsilon\xi) \in \mathcal{S}^{-\infty}$, but its seminorms are uniformly bounded in ϵ only as a symbol in $\mathcal{S}^0$. Thus $\{b, p(\epsilon\xi)\}_{\mathcal{M}} \in \mathcal{S}^{2s-1}$ uniformly in ϵ. Hence

$$\begin{aligned}[\mathrm{Op}(b), J_\epsilon \mathrm{Op}(a)\, J_\epsilon] &= J_\epsilon[\mathrm{Op}(b), \mathrm{Op}(a)]J_\epsilon \\ &\quad + [[J_\epsilon, \mathrm{Op}(b)],\ J_\epsilon \mathrm{Op}(a)] \\ &\quad - [J_\epsilon, \mathrm{Op}(b)]\, [\mathrm{Op}(a), J_\epsilon]\end{aligned}$$

has a symbol in $\mathcal{S}^{2s+m-2}$ uniformly in ϵ. □

11.3 Application: Local smoothing effect

Consider the Cauchy problem

$$\begin{cases} \mathrm{i}\dot{u} = Au \\ u(0) = u_0 \end{cases} \tag{11.14}$$

where $A = \mathrm{Op}^w(a)$ has a real symbol $a(x,\xi) \in \mathcal{S}^m$, for instance $-\partial_{x_j}(a_{jk}(x)\partial_{x_k})$, or $-\Delta$, or $\mathrm{i}\partial_x^3$. Since A is selfadjoint, the operator $-\mathrm{i}A$ is skew-adjoint, hence the L^2 norm of the flow is preserved.

We want to prove the so called (local in time) local smoothing estimates, which says, roughly speaking, that the flow generated by A locally in space regularize a bit the solution, provided one integrate over time. The classical smoothing estimate reads

$$\int_0^T \int_{|x|\le R} \left| \langle D \rangle^{\frac{m-1}{2}} u(t,x) \right|^2 \mathrm{d}x\,\mathrm{d}t \le C_{T,R} \|u_0\|_0^2 \tag{11.15}$$

where the positive constant $C_{T,R}$ depends on T, R.
In particular inequality (11.15) implies that if $u_0 \in L^2(\mathbb{R}^d)$ the solution $e^{-\mathrm{i}tA}u_0 \in H^{\frac{m-1}{2}}_{\mathrm{loc}}$ for almost all t. Notice that this gain of derivatives is a pure dispersive phenomenon, which cannot hold in hyperbolic problems.

A very nice method of proof relies on the so called *positive commutator method*, which we now illustrate.
Let us take a bounded operator B, and consider the usual energy estimate, which reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Bu, u \rangle = \left\langle \frac{1}{\mathrm{i}}[B,A]u, u \right\rangle = \langle \mathrm{i}[A,B]u, u \rangle ,$$

as $-\mathrm{i}A$ is skew-adjoint. Integrating in time from 0 to T and using that B is bounded and the flow is unitary in L^2 we get

$$\int_0^T \langle \mathrm{i}[A,B]u, u \rangle \,\mathrm{d}t \le C\|u_0\|_0^2. \tag{11.16}$$

Now the crucial point: choose B such that $\mathrm{i}[A,B] \ge 0$ is positive.

Let us see some examples.

Case $A = -\Delta$. In this case we have the following result, due to Doi [**?**]:

Lemma 11.5 (Doi)**.** *Let λ be even, radially decreasing, non negative, $\lambda \in L^1([0,\infty))$, smooth. Then there exists a real value symbol $b \in \mathcal{S}^0$ and a constant $0 < \beta < 1$ such that*

$$\{|\xi|^2, b\} \ge \beta\lambda(|x|)|\xi| - \frac{1}{\beta}$$

Proof. Define $f(t) := \int_0^t \lambda(r)\mathrm{d}r$ and let

$$\Phi(x) = (f(x_1), \dots, f(x_d))$$

so that $\Phi(x)$ is smooth and bounded. Let

$$\Phi'_{\mathrm{sym}}(x) := \frac{1}{2}\left(\partial_{x_j} f_i + \partial_{x_i} f_j\right) = \begin{pmatrix} \lambda(x_1) & 0 & \dots & 0 \\ 0 & \lambda(x_2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda(x_d) \end{pmatrix} \ge \lambda(|x|)\mathrm{Id}$$

as λ is even and decreasing. Define

$$b(x,\xi) := \Phi(x) \cdot \frac{\xi}{\langle \xi \rangle} \in \mathcal{S}^0.$$

Then

$$\{|\xi|^2, b\} = 2\frac{\xi}{\langle \xi \rangle} \cdot \nabla_x(\Phi(x)\cdot\xi) = \frac{2}{\langle \xi \rangle}\Phi'_{\rm sym}(x)\xi\cdot\xi \geq 2\lambda(|x|)\frac{|\xi|^2}{\langle \xi \rangle}.$$

$\square$

By Doi's lemma, we have that

$$\{a,b\}_{\mathcal{M}} = \{|\xi|^2, b\} + r_0 \geq \lambda(|x|)\,|\xi| - c_0$$

and therefore by Garding inequality

$$\langle {\rm i}[A,B]u,u\rangle \geq \left\langle {\rm Op}\left(\lambda(|x|)\ \langle\xi\rangle^{m-1}\right)u,u\right\rangle - c_0\|u\|_0^2 - C\|u\|_0^2$$

Finally use that

$$\lambda(|x|)\,|\xi| = \sqrt{\lambda(|x|)\,|\xi|}\ \sqrt{\lambda(|x|)\,|\xi|} = \left(\sqrt{\lambda(|x|)\,|\xi|}\right)^* \# \sqrt{\lambda(|x|)\,|\xi|} + \mathcal{S}^0$$

so that

$$\left\langle {\rm Op}\left(\lambda(|x|)\ \langle\xi\rangle^{m-1}\right)u,u\right\rangle = \|{\rm Op}\left(\lambda(|x|)^{\frac12}\,|\xi|^{\frac12}\right)u\|_0^2 - \langle {\rm Op}\,(r)\,u,u\rangle\,, \qquad r\in\mathcal{S}^0$$

Altogether we have proved that

$$\int_0^T \|{\rm Op}\left(\lambda(|x|)^{\frac12}\,|\xi|^{\frac12}\right)u\|_0^2{\rm d}t \leq C_T\|u_0\|_0^2,$$

which is the smoothing effect.

Case $A = -\partial_{x_j}(a_{jk}(x)\partial_{x_k}$**.** The proof is analogous to the previous one. In this case, so get the analogous statement than Lemma 11.5, one needs to require that

(H1) $A(x) = (a_{jk}(x))$ is real, symmetric and positive definite.

(H2) $a_{jk} \in C_b^\infty$ and

$$||\nabla a_{jk}(x)| = o(|x|^{-1}), \qquad x\to\infty$$

(H3) the Hamiltonian flow of $A(x)\xi\cdot\xi$ is non trapped in one direction, i.e. the orbits of the hamiltonian system with hamiltonian $A(x)\xi\cdot\xi$ are unbounded for any initial datum.

Under these conditions, Doi [**?**] proved that Lemma 11.5 is true with $A(x)\xi\cdot\xi$ replacing $|\xi|^2$.