10 Garding-like inequalities

If $a(D)$ is a Fourier multiplier, and its symbol $a(\xi) \geq 0$, then the operator $a(D)$ is positive, in the sense that

$$\langle a(D)u, u\rangle \geq 0, \qquad \forall u \in \mathcal{S}.$$

We want to extend this kind of result to pseudodifferential operators. These results, which go under the name of Garding inequalities, allow to prove that a pseudodifferential operator whose symbol is positive is bounded from below.

The first result of this type is the weak Garding inequality

Theorem 10.1 (Weak Garding inequality)**.** *Let $a \in \mathcal{S}^m$ with real part is elliptic, i.e.*

$$\operatorname{Re} a(x,\xi) \geq c\,\langle\xi\rangle^m\,, \qquad \forall|\xi| \geq R.$$

Then $\forall N \in \mathbb{N}$, $\exists C_N > 0$ s.t.

$$\operatorname{Re}\langle \operatorname{Op}(a)\,u, u\rangle \geq \frac{c}{4}\|u\|^2_{\frac{m}{2}} - C_N\|u\|^2_{-N}. \tag{10.1}$$

Remark that the theorem does not say that $\operatorname{Re}\langle \operatorname{Op}(a)\,u, u\rangle$ is positive, but that it is possible to sum a arbitrary negative norm of u to this term to make it positive:

$$\operatorname{Re}\langle \operatorname{Op}(a)\,u, u\rangle + C_N\|u\|^2_{-N} \geq \frac{c}{2}\|u\|^2_{\frac{m}{2}}$$

In particular we can choose $N = 0$ and find a constant C_0 such that $\operatorname{Op}(a) + C$ is bounded from below.

Proof. First remark that

$$\begin{aligned}
\operatorname{Re}\langle \operatorname{Op}(a)\,u, u\rangle &= \frac{\langle \operatorname{Op}(a)\,u, u\rangle + \overline{\langle \operatorname{Op}(a)\,u, u\rangle}}{2} = \frac{\langle \operatorname{Op}(a)\,u, u\rangle + \langle u, \operatorname{Op}(a)\,u\rangle}{2} \\
&= \left\langle \frac{\operatorname{Op}(a) + \operatorname{Op}(a^*)}{2}u, u\right\rangle = \left\langle \operatorname{Op}\left(\frac{a+a^*}{2}\right)u, u\right\rangle \\
&= \langle \operatorname{Op}(\operatorname{Re} a)\,u, u\rangle + \langle \operatorname{Op}(b)\,u, u\rangle
\end{aligned}$$

where $b \in \mathcal{S}^{m-1}$.

Let $\chi(\xi) \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ such that $\chi \equiv 0$ for $|\xi| \leq R$ and define

$$p(x,\xi) := \sqrt{\left(\operatorname{Re} a(x,\xi) - \frac{c}{2}\langle\xi\rangle^m\right)(1-\chi(\xi))} \in \mathcal{S}^{\frac{m}{2}}.$$

Note that $p(x,\xi)$ is a symbol as $\operatorname{Re} a(x,\xi) - \frac{c}{2}\langle\xi\rangle^m \geq \frac{c}{2}\langle\xi\rangle^m$ for $|\xi|$ large enough. Now

$$p^*\#p = p^*p + \mathcal{S}^{m-1} = p^2 + \mathcal{S}^{m-1} = \operatorname{Re} a(x,\xi) - \frac{c}{2}\langle\xi\rangle^m + \mathcal{S}^{m-1};$$

quantizing it we find

$$\operatorname{Op}(p)^*\operatorname{Op}(p) = \operatorname{Op}(\operatorname{Re} a) - \frac{c}{2}\langle D\rangle^m + \operatorname{Op}(r)\,, \qquad r \in \mathcal{S}^{m-1}$$

and applying it to u and taking the scalar product with u we get

$$0 \leq \|\operatorname{Op}(p)\,u\|_0 = \langle \operatorname{Op}(\operatorname{Re} a)\,u, u\rangle - \frac{c}{2}\|u\|^2_{\frac{m}{2}} + \langle \operatorname{Op}(r)\,u, u\rangle\,,$$

namely

$$\langle \mathrm{Op}\,(\mathrm{Re}\, a)\, u, u\rangle \geq \frac{c}{2}\|u\|^2_{\frac{m}{2}} - \langle \mathrm{Op}\,(r)\, u, u\rangle$$

Combining it with the first inequality

$$\mathrm{Re}\,\langle \mathrm{Op}\,(a)\, u, u\rangle \geq \frac{c}{2}\|u\|^2_{\frac{m}{2}} + \langle \mathrm{Op}\,(b-r)\, u, u\rangle$$

As $b + r \in \mathcal{S}^{m-1}$, we have

$$|\langle \mathrm{Op}\,(b-r)\, u, u\rangle| \leq \|\mathrm{Op}\,(b-a)\, u\|_{-\frac{m}{2}}\|u\|_{\frac{m}{2}} \leq C\|u\|_{\frac{m}{2}-1}\|u\|_{\frac{m}{2}}.$$

Hence we found

$$\langle \mathrm{Op}\,(\mathrm{Re}\, a)\, u, u\rangle \geq \frac{c}{2}\|u\|^2_{\frac{m}{2}} - C\|u\|_{\frac{m}{2}-1}\|u\|_{\frac{m}{2}}.$$

Finally we interpolate. As it holds that

$$\langle\xi\rangle^{\frac{m}{2}-1} \leq \epsilon^2\langle\xi\rangle^{\frac{m}{2}} + C^2_{\epsilon,N}\langle\xi\rangle^{-N}, \qquad \forall \epsilon, N > 0$$

(indeed for $\langle\xi\rangle \geq \frac{1}{\epsilon^2}$, $\langle\xi\rangle^{\frac{m}{2}-1} \leq \epsilon^2\langle\xi\rangle^{\frac{m}{2}}$, while for $\langle\xi\rangle \leq \frac{1}{\epsilon^2}$, $\langle\xi\rangle^N\langle\xi\rangle^{\frac{m}{2}-1} \leq C^2_{\epsilon,N}$) we get

$$\begin{aligned}\|u\|_{\frac{m}{2}-1}\|u\|_{\frac{m}{2}} &\leq \epsilon^2\|u\|^2_{\frac{m}{2}} + C^2_{\epsilon,N}\|u\|_{-N}\|u\|_{\frac{m}{2}}\\ &\leq \epsilon^2\|u\|^2_{\frac{m}{2}} + C^2_{\epsilon,N}(\frac{\eta}{2}\|u\|^2_{\frac{m}{2}} + \frac{1}{2\eta}\|u\|^2_{-N})\\ &\leq (\epsilon^2 + C^2_{\epsilon,N}\eta)\|u\|^2_{\frac{m}{2}} + \frac{C^2_{\epsilon,N}}{2\eta}\|u\|^2_{-N})\end{aligned}$$

Now choose ϵ and η so small so that $\epsilon^2 + C^2_{\epsilon,N}\eta \leq \frac{c}{4}$ and we obtain the claimed estimate. □

As we already commented, if $a(x,\xi)$ is positive, in the sense that

$$a(x,\xi) \geq 0$$

the operator $\mathrm{Op}\,(a)$ is not nonnegative. Consider for example $a(x,\xi) := a(x)\xi^2$ with $a \in C_0^\infty(\mathbb{R})$ and $a(x) \geq 0$. Then the associated operator is $\mathrm{Op}\,(a) = -a(x)\partial_x^2$, but it is not nonnegative! For instance if $u \in C_0^\infty(\mathbb{R})$ satisfies $u'' = u$ on the support of $a(x)$, then

$$\langle \mathrm{Op}\,(a)\, u, u\rangle = -\int a(x)u^2(x)\mathrm{d}x < 0,$$

On the other end, the operator $-\partial_x a \partial_x$ is nonnegative, and it agrees with $\mathrm{Op}\,(a)$ up to a term of one lower order.

This is true in general for pseudodifferential operators. Specifivally, if $a(x,\xi) \geq 0$, then, up to a lower order correction, $\mathrm{Op}\,(a)$ is also nonnegative. More precisely one has the following result, due to Friedrichs.

Theorem 10.2. *Let $a \in \mathcal{S}^m$ and assume that $a(x,\xi) \geq 0$. Then one can write*

$$a(x,\xi) = a_F(x,\xi) + r(x,\xi)$$

where $r \in \mathcal{S}^{m-1}$ and

$$\langle \mathrm{Op}\,(a_F)\, u, u\rangle \geq 0, \qquad \forall u \in \mathcal{S}.$$

The proof of this result is quite technical, and the reader can find it in [**?**].
However, an easy consequence is the following strong Garding inequality.

Theorem 10.3 (Strong Garding inequality)**.** *Let* $a \in \mathcal{S}^m$ *with* $\operatorname{Re} a(x,\xi) \geq 0$*. Then*

$$\operatorname{Re} \langle \operatorname{Op}(a)\, u, u \rangle \geq -C \|u\|^2_{H^{\frac{m-1}{2}}}$$

for some $C > 0$*.*

Proof. First recall that

$$\operatorname{Re} \langle \operatorname{Op}(a)\, u, u \rangle = \langle \operatorname{Op}(\operatorname{Re} a)\, u, u \rangle + \langle \operatorname{Op}(b)\, u, u \rangle$$

where $b \in \mathcal{S}^{m-1}$. By Theorem 10.2 applied to $\operatorname{Re} a(x,\xi)$,

$$\operatorname{Re} a(x,\xi) = a_F(x,\xi) + r(x,\xi), \qquad r \in \mathcal{S}^{m-1}$$

Then

$$\operatorname{Re} \langle \operatorname{Op}(a)\, u, u \rangle = \langle \operatorname{Op}(a_F)\, u, u \rangle + \langle \operatorname{Op}(b+r)\, u, u \rangle \qquad \geq - |\langle \operatorname{Op}(b+r)\, u, u \rangle| \\ \geq -C \|u\|^2_{H^{\frac{m-1}{2}}},$$

where we used that by Sobolev continuity

$$\|\operatorname{Op}(b+r)\, u\|_{H^{-\frac{m-1}{2}}} \leq C \|u\|_{H^{\frac{m-1}{2}}}$$

□

Remark that for nonnegative symbols of order 2, strong Garding inequality gives

$$\operatorname{Re} \langle \operatorname{Op}(a)\, u, u \rangle \geq -C \|u\|^2_{H^{\frac{1}{2}}}$$

An impressive improvement is given by the Fefferman-Phong inequality:

Theorem 10.4 (Fefferman-Phong inequality)**.** *If* $a \in \mathcal{S}^m$ *fulfills* $a \geq 0$*, then for some* $C > 0$

$$\operatorname{Re} \langle \operatorname{Op}(a)\, u, u \rangle + C \|u\|^2_{\frac{m-2}{2}} \geq 0.$$