6 Stationary phase and oscillatory integrals

In this section we study integrals of the form (5.10) when the function a is not necessarily bounded. We want to give meaning to such integrals and find a way to compute them.

Actually the first step is to study integrals of the form

$$\int_{\mathbb{R}^d} e^{\frac{\mathrm{i}}{\hbar}\varphi(x)}\, a(x)\, \mathrm{d}x \tag{6.1}$$

when $a \in C_0^\infty(\mathbb{R}^d)$, $\varphi \in C^\infty$ is real valued, and $\hbar$ is a small parameter. The idea is that, when $\hbar$ is small, the phase (which is real valued) is fast oscillating and the integral is small.
There are essentially two distinct cases to analyze: the first is when

$$\nabla\varphi \neq 0 \text{ on } \operatorname{supp} a \tag{6.2}$$

and the second one when

$$\exists\ x_0 \in \operatorname{supp} a: \quad \nabla\varphi(x_0) = 0. \tag{6.3}$$

In the first case we will see that the integral is $O(\hbar^N)$ $\forall N$, while in the second one we will get an expansion in powers of $\hbar$.

6.1 Rapid decay

Let us analyze the first case. We have the following result.

Theorem 6.1 (Rapid decay)**.** *Assume $\nabla\varphi \neq 0$ on $\operatorname{supp} a$. Then $\forall N \in \mathbb{N}$ there is a constant $C_N \equiv C_N(\operatorname{supp} a, d, \varphi) > 0$ such that $\forall 0 < \hbar \leq 1$ one has*

$$\left| \int_{\mathbb{R}^d} e^{\frac{\mathrm{i}}{\hbar}\varphi(x)}\, a(x)\, \mathrm{d}x \right| \leq C_N\, \hbar^N \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^d} \left|\partial_x^\alpha a(x)\right|. \tag{6.4}$$

In particular the integral is smaller than any power of $\hbar$.

Proof. Define the differential operator

$$L := \frac{\hbar}{\mathrm{i}} \frac{1}{|\nabla\varphi(x)|^2} \nabla\varphi(x) \cdot \nabla$$

and note that

$$L\left(e^{\frac{\mathrm{i}}{\hbar}\varphi}\right) = e^{\frac{\mathrm{i}}{\hbar}\varphi};$$

then clearly $\forall N \in \mathbb{N}$

$$L^N\left(e^{\frac{\mathrm{i}}{\hbar}\varphi}\right) = e^{\frac{\mathrm{i}}{\hbar}\varphi}.$$

Consequently

$$\int_{\mathbb{R}^d} e^{\frac{\mathrm{i}}{\hbar}\varphi(x)}\, a(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} L^N\left(e^{\frac{\mathrm{i}}{\hbar}\varphi(x)}\right) a(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} e^{\frac{\mathrm{i}}{\hbar}\varphi(x)}\, (L^*)^N a(x)\ \mathrm{d}x$$

Everything is well defined as $\nabla\varphi \neq 0$ on supp a. By the assumptions it follows that

$$|\nabla\varphi(x)| \geq c_0 > 0 \qquad \text{on } \operatorname{supp} a$$

But now note that

$$L^* a = -\frac{\hbar}{\mathrm{i}} \mathrm{div} \left(\frac{a}{|\nabla\varphi(x)|^2} \nabla\varphi \right),$$

thus we get

$$\left|(L^*)^N (a(x))\right| \le C_N \, \hbar^N \sum_{|\alpha| \le N} \sup_{x\in\mathbb{R}^d} |\partial_x^\alpha a(x)|$$

which is the claimed estimate. □

6.2 Stationary phase

We consider now the second case, and just in case the phase is a quadratic function:

$$\varphi(x) = \frac{1}{2} \langle Qx, x\rangle .$$

We also assume that Q is not degenerate, namely $\det Q \neq 0$. This implies that $\nabla\varphi(x) = 0$ iff $x = 0$. We also assume $0 \in \mathrm{supp}\, a$. In this case we have the following result

Theorem 6.2 (Stationary phase)**.** *Let Q be a real $d \times d$ matrix, symmetric, $\det Q \neq 0$. Then for any $u \in C_0^\infty(\mathbb{R}^d)$ and $N \ge 1$ one has*

$$\int_{\mathbb{R}^d} e^{\frac{\mathrm{i}}{\hbar}\frac{\langle Qx,x\rangle}{2}} u(x)\mathrm{d}x = \frac{(2\pi)^{d/2}\, e^{\mathrm{i}\frac{\pi}{4}\mathrm{sign}\, Q}}{|\det Q|^{1/2}} \sum_{k=0}^{N-1} \frac{\hbar^{d/2}}{(2\mathrm{i})^k\, k!} \left[\left\langle Q^{-1}D_x, D_x\right\rangle^k u \right]\Big|_{x=0} + R_N(u,\hbar) \quad (6.5)$$

where

$$\mathrm{sign}\, Q = \#\textit{positive eig} - \#\ \textit{negative eig}$$

and

$$|R_N(u,\hbar)| \le \frac{C\, \hbar^{N+d/2}}{2^N\, N!\, |\det Q|^{1/2}} \sum_{|\alpha|\le d+1} \|\partial_x^\alpha \left\langle Q^{-1}D_x, D_x\right\rangle^N u\|_{L^2}. \quad (6.6)$$

In order to prove the theorem we need the following fact about the Fourier transform of imaginary Gaussian functions:

Lemma 6.3. *If Q is a symmetric $d \times d$ real matrix with $\det Q \neq 0$, then*

$$\mathcal{F}\left(e^{\frac{\mathrm{i}}{\hbar}\frac{\langle Qx,x\rangle}{2}} \right) = e^{\mathrm{i}\frac{\pi}{4}\mathrm{sign}\, Q}\ \hbar^{d/2}\ \frac{(2\pi)^{d/2}}{|\det Q|^{1/2}}\, e^{-\frac{\hbar\mathrm{i}}{2}\langle Q^{-1}\xi,\xi\rangle} \quad (6.7)$$

The lemma is proved in [Zwo12, Lemma 3.7].

Proof of Theorem 6.2. By definition of the Fourier transform on $\mathcal{S}'$, we have

$$\int_{\mathbb{R}^d} e^{\frac{\mathrm{i}}{\hbar}\frac{\langle Qx,x\rangle}{2}} u(x)\mathrm{d}x = \left\langle e^{\frac{\mathrm{i}}{\hbar}\frac{\langle Qx,x\rangle}{2}}, u\right\rangle_{\mathcal{S}',\mathcal{S}} = \left\langle \mathcal{F}\left(e^{\frac{\mathrm{i}}{\hbar}\frac{\langle Qx,x\rangle}{2}}\right), \mathcal{F}^{-1}u\right\rangle_{\mathcal{S}',\mathcal{S}}$$

$$\overset{(6.7)}{=} \frac{\hbar^{d/2}\,(2\pi)^{d/2}\, e^{\mathrm{i}\frac{\pi}{4}\mathrm{sign}\, Q}}{|\det Q|^{1/2}} \frac{1}{(2\pi)^d} \int e^{-\frac{\hbar\mathrm{i}}{2}\langle Q^{-1}\xi,\xi\rangle}\, \widehat{u}(\xi)\, \mathrm{d}\xi \quad (6.8)$$

Now we have $\forall t \in \mathbb{R}$

$$\left| e^{\mathrm{i}t} - \sum_{k=0}^{N-1} \frac{(\mathrm{i}t)^k}{k!} \right| \le \frac{|t|^N}{N!},$$

thus expanding the complex exponential we obtain

$$e^{-\frac{\hbar i}{2}\left\langle Q^{-1}\xi,\xi\right\rangle}=\sum_{k=0}^{N-1}\frac{1}{k!}\left(\frac{\hbar}{2\mathrm{i}}\left\langle Q^{-1}\xi,\xi\right\rangle\right)^{k}+r_N(\xi,\hbar) \tag{6.9}$$

where the remainder is given by

$$r_N(\xi,\hbar)=\frac{\hbar^N}{(N-1)!}\int_0^1(1-t)^{N-1}e^{-\frac{t\hbar i}{2}\left\langle Q^{-1}\xi,\xi\right\rangle}\left(\frac{\hbar}{2\mathrm{i}}\left\langle Q^{-1}\xi,\xi\right\rangle\right)^{N}\mathrm{d}t \tag{6.10}$$

and it fulfills the estimate

$$|r_N(\xi,\hbar)|\leq\frac{\hbar^N}{2^N\,N!}\left|\left\langle Q^{-1}\xi,\xi\right\rangle\right|^N. \tag{6.11}$$

Substituting the expression (6.9) in the integral (6.8) and using that

$$\int\left\langle Q^{-1}\xi,\xi\right\rangle^k\widehat{u}(\xi)\,\mathrm{d}\xi=\int e^{\mathrm{i}x\xi}\left\langle Q^{-1}\xi,\xi\right\rangle^k\widehat{u}(\xi)\,\mathrm{d}\xi\Big|_{x=0}=(2\pi)^d\left[\left\langle Q^{-1}D_x,D_x\right\rangle^k u\right]\Big|_{x=0} \tag{6.12}$$

we get the finite sum of (6.5). To get the estimate of the remainder just use that

$$\left|\int r_N(\xi,h)\widehat{u}(\xi)\mathrm{d}\xi\right|\leq C\hbar^N\int\left|\left\langle Q^{-1}\xi,\xi\right\rangle^N\,\widehat{u}(\xi)\right|\mathrm{d}\xi \tag{6.13}$$

and the estimate

$$\|\widehat{v}\|_{L^1}=\int\langle\xi\rangle^{-d-1}\,\langle\xi\rangle^{d+1}\,|\widehat{v}(\xi)|\,\mathrm{d}\xi\leq C_d\|\,\langle\xi\rangle^{d+1}\,\widehat{v}\|_{L^2}\leq C_d\sum_{|\alpha|\leq d+1}\|\partial_x^\alpha v\|_{L^2}$$

applied to $\widehat{v}=\left\langle Q^{-1}\xi,\xi\right\rangle^N\,\widehat{u}(\xi)=\left(\left\langle Q^{-1}D_x,D_x\right\rangle^N u\right)^{\wedge}(\xi)$. □

Before showing applications of this expansion in some specific case, it is useful to keep in mind the following identity:

Lemma 6.4. *Let A be a $d\times d$, real, symmetric matrix. Then for $u\in\mathcal{S}$ one has*

$$\left[e^{\frac{\mathrm{i}}{2}\left\langle AD_x,D_x\right\rangle}u\right](x)=\frac{e^{\mathrm{i}\frac{\pi}{4}\mathrm{signA}}}{(2\pi)^{d/2}\,|\det A|^{1/2}}\int_{\mathbb{R}^d}e^{-\frac{\mathrm{i}}{2}\left\langle A^{-1}z,z\right\rangle}u(x+z)\,\mathrm{d}z \tag{6.14}$$

Proof. By the definition of Fourier multiplier and using Lemma 6.3

$$\begin{aligned}e^{\frac{\mathrm{i}}{2}\left\langle AD_x,D_x\right\rangle}u&=\frac{1}{(2\pi)^d}\int e^{\mathrm{i}(x-y)\xi}\,e^{\frac{\mathrm{i}}{2}\left\langle A\xi,\xi\right\rangle}u(y)\,\mathrm{d}\xi\mathrm{d}y=\frac{1}{(2\pi)^d}\int u(y)\,\left(e^{\frac{\mathrm{i}}{2}\left\langle A\xi,\xi\right\rangle}\right)^{\wedge}(y-x)\mathrm{d}y\\&=\frac{e^{\mathrm{i}\frac{\pi}{4}\mathrm{signA}}}{(2\pi)^{d/2}\,|\det A|^{1/2}}\int e^{-\frac{\mathrm{i}}{2}\left\langle A^{-1}(y-x),y-x\right\rangle}u(y)\,\mathrm{d}y\\&=\frac{e^{\mathrm{i}\frac{\pi}{4}\mathrm{signA}}}{(2\pi)^{d/2}\,|\det A|^{1/2}}\int e^{-\frac{\mathrm{i}}{2}\left\langle A^{-1}z,z\right\rangle}u(x+z)\,\mathrm{d}z\end{aligned}$$

□

Examples. There are several nice applications of these formula. We will see those appearing in the pseudodifferential calculus.

(i) If $u \in \mathcal{S}(\mathbb{R}^{2n})$, $u = u(x,y)$, $d = 2n$ and

$$A = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

then $\frac{1}{2}\left\langle A\begin{pmatrix} D_x \\ D_\xi \end{pmatrix}, \begin{pmatrix} D_x \\ D_\xi \end{pmatrix}\right\rangle = \langle D_x, D_\xi\rangle$, so we get

$$\left[e^{\mathrm{i}\langle D_x, D_\xi\rangle} u\right](x,\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}\, u(y+x, \xi+\eta)\,\mathrm{d}y\mathrm{d}\eta \tag{6.15}$$

$$= \sum_{k=0}^{N} \frac{1}{\mathrm{i}^k\, k!}\left[(\partial_\eta \cdot \partial_y)^k\, u(\cdot + x, \cdot + \xi)\right]_{\substack{\eta=\xi \\ y=x}} + R_N \tag{6.16}$$

$$= \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\left[\partial_\xi^\alpha\, D_x^\alpha u\right](x,\xi) + R_N \tag{6.17}$$

where R_N fulfills

$$|R_N| \leq C_N \sum_{|\alpha+\beta|\leq 2n+1} \|\partial_\eta^\alpha \partial_y^\beta (\partial_\eta i \cdot \partial_y)^N u\|_{L^2} \tag{6.18}$$

(ii) If $u \in \mathcal{S}(\mathbb{R}^{4n})$, $u = u(z,w)$ and

$$\sigma(z,w) = \langle Jz, w\rangle = \left\langle \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} z, w\right\rangle$$

then writing $z = (x,\xi)$, $w = (y,\eta)$

$$\left[e^{\mathrm{i}\sigma(D_z, D_w)} u\right](z,w) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{4n}} e^{-\mathrm{i}\sigma(\widetilde{z},\widetilde{w})}\, u(\widetilde{z}+z, \widetilde{w}+w)\,\mathrm{d}\widetilde{z}, \mathrm{d}\widetilde{w} \tag{6.19}$$

$$= \sum_{k=0}^{N} \frac{1}{\mathrm{i}^k\, k!}\left[(\sigma(D_{\widetilde{z}}, D_{\widetilde{w}})^k\, u(\cdot + z, \cdot + w)\right]_{\substack{\eta=\xi \\ y=x}} + R_N \tag{6.20}$$

$$= \sum_{k=0}^{N} \frac{1}{\mathrm{i}^k\, k!}\left[(D_\xi \cdot D_y - D_x \cdot D_\eta)^k u\right](x,\xi,y,\eta) + R_N \tag{6.21}$$

6.3 Oscillatory integrals

We are now ready to study integrals of the form

$$\int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot Qx}\, a(x)\,\mathrm{d}x \tag{6.22}$$

with Q a real, symmetric, $d \times d$ matrix and a a function polynomially growing in x. In particular we take a to be in the class of the amplitudes:

$$A_\delta^m(\mathbb{R}^d) = \left\{a \in C^\infty(\mathbb{R}^d, \mathbb{C})\colon \quad \forall \alpha \in \mathbb{N}^d \quad \exists C_\alpha > 0\colon \quad |\partial_x^\alpha a(x)| \leq C_\alpha \langle x\rangle^{m-\delta|\alpha|}\right\}. \tag{6.23}$$

If $\delta = 0$ we simply write $A_0^m \equiv A^m$.

Remark that amplitudes behave similar to symbols, in the sense that

$$a \in A_\delta^{m_1}, \quad b \in A_\delta^{m_2} \quad \Rightarrow \quad ab \in A_\delta^{m_1+m_2}$$

We endow A_δ^m with the seminorms

$$N_k^m(a) := \sum_{|\alpha| \le k} \sup_{x \in \mathbb{R}^d} |\partial_x^\alpha a(x)| \, \langle x \rangle^{-m+\delta|\alpha|} \tag{6.24}$$

which turn the space A_δ^m into Frechet.

Define the linear form

$$I_Q(a) := \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot Qx} \, a(x) \, \mathrm{d}x$$

which is well defined for $a \in A_\delta^m$ when $m < -n$. We want to prolong I_Q continuously to the space A_δ^m also for $m \ge -n$.

Theorem 6.5. *Let $a \in A_\delta^m$, $\delta \in (-1, 1]$. Let Q real, symmetric, $\det Q \neq 0$ and $\varphi \in \mathcal{S}$ with $\varphi(0) = 1$. Then the limit*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot Qx} a(x) \varphi(\epsilon x) \mathrm{d}x$$

exists and is independent of φ, as soon as $\varphi(0) = 1$. We define

$$I_Q(a) := \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot Qx} a(x) \varphi(\epsilon x) \mathrm{d}x \tag{6.25}$$

Moreover

$$|I_Q(a)| \le C_{Q,m,d} \, N^m_{\frac{m+d+1}{1+\delta}}(a) \tag{6.26}$$

Therefore I_Q can be extended with continuity to all A_δ^m. The extension is unique due to the density of $\mathcal{S}$ in this space.

Proof. Take $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi \equiv 1$ in $B_1(0)$, $\chi \equiv 0$ outside $B_2(0)$. Define

$$I_j := \int e^{\mathrm{i}x\cdot Qx} a(x) \, \chi(2^{-j}x) \mathrm{d}x.$$

We prove that $\{I_j\}_{j \ge 1} \subset \mathbb{C}$ is Cauchy. First

$$\begin{aligned} I_j - I_{j-1} &= \int e^{\mathrm{i}x\cdot Qx} a(x) \; \big(\chi(2^{-j}x) - \chi(2^{-j+1}x)\big) \, \mathrm{d}x \\ &\overset{y=2^{-j}x}{=} \int e^{2^{2j}\mathrm{i}y\cdot Qy} a(2^j y) \; \underbrace{(\chi(y) - \chi(2y))}_{\mathrm{supp} \subset \{\frac{1}{2} \le |y| \le 2\}} \, 2^{jd} \, \mathrm{d}y \end{aligned}$$

Since the support of the function is bounded away from zero, we can apply stationary phase with rapid decay, namely Theorem 6.1, with $\hbar = 2^{-2j}$: then $\forall M > 0$ (we will fix it later), $\exists C_M > 0$ s.t.

$$|I_j - I_{j-1}| \le 2^{jd} \, C_M \, \hbar^M \sum_{|\alpha| \le M} \sup_{\frac{1}{2} \le |y| \le 2} \left| \partial_y^\alpha \left[a(2^j y) \; (\chi(y) - \chi(2y)) \right] \right|$$

Now we have

$$\left| \partial_y^\alpha [a(2^j y)] \right| \le 2^{j|\alpha|} \left| (\partial_y^\alpha a)(2^j y) \right| \le 2^{j|\alpha|} \left\langle 2^j y \right\rangle^{m - \delta|\alpha|} \; N^m_{|\alpha|}(a)$$

so we get, using also Leibnitz rule,

$$\begin{aligned}|I_j - I_{j-1}| &\le C_M\, N^m_M(a)\, 2^{jd}\, 2^{-2jM} \sum_{|\alpha|\le M} 2^{j|\alpha|} \sup_{\frac12\le|y|\le 2} \langle 2^j y\rangle^{m-\delta|\alpha|}\\ &\le C_M\, N^m_M(a)\ 2^{jd-2jM+jm} \sum_{k\le M} 2^{j(1-\delta)k}\\ &\le C_M\, N^m_M(a)\ 2^{jd-2jM+jm+j(1-\delta)M}\\ &\le C_M\ N^m_M(a)\, 2^{j(d+m-(1+\delta)M)}\\ &\le C_M\ N^m_M(a)\, 2^{-j}\end{aligned}$$

choosing $M = \frac{d+m+1}{1+\delta}$. Clearly we need $\delta > -1$. The choice of M also fixes the constant C_M.

Thus $\{I_j\}_{j\ge1}$ converges and

$$I_Q(a) = \lim_{j\to\infty} I_j$$

Next we prove that the limit (6.25) exists and is independent of ϵ. Denote

$$I_j(\epsilon) := \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot Qx} a(x)\, \varphi(\epsilon x)\chi(2^{-j}x)\, \mathrm{d}x.$$

Note that by dominated convergence theorem $\epsilon \mapsto I_j(\epsilon) \in C^0([0,1],\mathbb{C})$ and

$$\lim_{\epsilon\to0} I_j(\epsilon) = I_j, \qquad \lim_{j\to\infty} I_j(\epsilon) = \int e^{\mathrm{i}x\cdot Qx} a(x)\varphi(\epsilon x)\mathrm{d}x$$

We prove that $\{I_j(\epsilon)\}_{j\ge1}$ is Cauchy in $(C^0([0,1],\mathbb{C}), \|\cdot\|_\infty)$. Then $\{I_j(\epsilon)\}_{j\ge1}$ converges uniformly in $[0,1]$, and we are allowed to exchange the order of limits, getting

$$\lim_{\epsilon\to0} \int e^{\mathrm{i}x\cdot Qx} a(x)\, \varphi(\epsilon x)\mathrm{d}x = \lim_{\epsilon\to0}\lim_{j\to\infty} I_j(\epsilon) = \lim_{j\to\infty}\lim_{\epsilon\to0} I_j(\epsilon) = \lim_{j\to\infty} I_j = I_Q(a)$$

which proves that the limit does not depend on the regularizing function.

To prove that $\{I_j(\epsilon)\}_{j\ge1}$ is Cauchy in $(C^0([0,1],\mathbb{C}), \|\cdot\|_\infty)$ one adapts the argument above and shows

$$\|I_j(\cdot) - I_{j-1}(\cdot)\|_\infty \ \le C\, N^m_{\frac{d+m+1}{1+\delta}}(a)\, 2^{-j}\ ;$$

we leave the details as an exercise. □

Thanks to the procedure of regularization one checks that "classical" operations are valid for oscillatory integrals:

Proposition 6.6. *Let Q be real, symmetric, $d\times d$, invertible matrix. Let $a \in A^m_\delta$, $\delta\in(-1,1]$. Then the following holds true:*

(i) Linear change of variables*: Let $A \in \mathrm{Mat}(\mathbb{R}^d)$ be real and invertible. Then*

$$\int e^{\mathrm{i}x\cdot Qx}\, a(x)\, \mathrm{d}x = \int e^{\mathrm{i}Ay\cdot QAy}\, a(Ay)\ |\det A|\, \mathrm{d}y \tag{6.27}$$

(ii) Integration by parts: *let $b \in A^m_\delta$, then*

$$\int e^{\mathrm{i}x\cdot Qx}\, a(x)\, \partial^\alpha_x b(x)\mathrm{d}x = \int (-\partial^\alpha_x)[e^{\mathrm{i}x\cdot Qx}\, a(x)]\, b(x)\mathrm{d}x \tag{6.28}$$

(*iii*) Differentiation under $\int$: *if* $a \in A^m_\delta(\mathbb{R}^n \times \mathbb{R}^p)$, *then* $\int e^{\mathrm{i}xQx} a(x,y)\mathrm{d}x \in A^m_\delta(\mathbb{R}^p)$ *and*

$$\partial^\alpha_y \int e^{\mathrm{i}x\cdot Qx}\, a(x,y)\mathrm{d}x = \int e^{\mathrm{i}x\cdot Qx}\, \partial^\alpha_y a(x,y)\mathrm{d}x \tag{6.29}$$

(*iv*) Inversion of $\int$: *if* $a \in A^m_\delta(\mathbb{R}^n \times \mathbb{R}^p)$ *and if* P *is a non degenerate* $p \times p$ *real symmetric matrix, then*

$$\int e^{\mathrm{i}y\cdot Py} \left(\int e^{\mathrm{i}x\cdot Qx} a(x,y)\mathrm{d}x \right) \mathrm{d}y = \int e^{\mathrm{i}y\cdot Py + \mathrm{i}x\cdot Qx} a(x,y)\mathrm{d}x\mathrm{d}y \tag{6.30}$$

(*v*) Passage to the limit under $\int$: *let* $\{a_j\}_{j\in\mathbb{N}} \subset A^m_\delta$ *be bounded in* A^m_δ *and assume that*

$$\partial^\alpha_x a_j(x) \to \partial^\alpha_x a(x) \qquad \textit{pointwisely } \forall \alpha \in \mathbb{N}^d.$$

Then $a \in A^m_\delta$ *and*

$$\int e^{\mathrm{i}x\cdot Qx} a(x)\mathrm{d}x = \lim_{j\to\infty} \int e^{\mathrm{i}x\cdot Qx} a_j(x)\mathrm{d}x$$

Proof. The proof of the proposition consists essentially in writing the integrals as oscillatory integrals, perform the wanted manipulations to the convergent integrals, and then take the limit when $\epsilon \to 0$. The details are in [SR91, Theorem 2.5].

(i) By definition of oscillatory integral

$$\int e^{\mathrm{i}x\cdot Qx}\, a(x)\, \mathrm{d}x = \lim_{\epsilon\to 0} \int e^{\mathrm{i}x\cdot Qx}\, a(x)\, \varphi(\epsilon x)\, \mathrm{d}x$$

Now the integral on the r.h.s. is well defined, so we make the change of variables $x = Ay$ we get

$$\int e^{\mathrm{i}x\cdot Qx}\, a(x)\, \varphi(\epsilon x)\, \mathrm{d}x = \int e^{\mathrm{i}y\cdot(A^*QA)y}\, a(Ay)\, \varphi(\epsilon Ay)\, |\det A|\, \mathrm{d}x$$

Now remark that $\widetilde{\varphi}(y) := \varphi(Ay) \in \mathcal{S}$ and $\widetilde{\varphi}(0) = 1$, while $a(Ay)\,|\det A|$ is an amplitude of order m. Thus the limit

$$\lim_{\epsilon\to 0} \int e^{\mathrm{i}y\cdot(A^*QA)y}\, a(Ay)\, \varphi(\epsilon Ay)\, |\det A|\, \mathrm{d}x$$

exists as oscillatory integral. This proves (i).

(ii) Again we exploit the definition of oscillatory integral and compute

$$\begin{aligned}\int e^{\mathrm{i}x\cdot Qx}\, a(x)\, \partial^\alpha_x b(x)\mathrm{d}x &= \lim_{\epsilon\to 0} \int e^{\mathrm{i}x\cdot Qx}\, a(x)\, \partial^\alpha_x b(x)\, \varphi(\epsilon x)\mathrm{d}x \\ &= \lim_{\epsilon\to 0} \int -\partial^\alpha_x [e^{\mathrm{i}x\cdot Qx}\, a(x)\, \varphi(\epsilon x)]\, b(x)\, \mathrm{d}x\end{aligned}$$

where we integrated by parts in the regularized integral. Using Leinbitz we split

$$\partial^\alpha_x [e^{\mathrm{i}x\cdot Qx}\, a(x)\, \varphi(\epsilon x)] = \sum_{\alpha'+\alpha''=\alpha} C_{\alpha',\alpha''} (\partial^{\alpha'}_x [e^{\mathrm{i}x\cdot Qx}\, a(x)]\, \epsilon^{|\alpha''|} (\partial^{\alpha''}_x \varphi)(\epsilon x)$$

Now one checks that if $\alpha'' \neq 0$, then

$$\lim_{\epsilon\to 0} \int (\partial^{\alpha'}_x [e^{\mathrm{i}x\cdot Qx}\, a(x)]\, \epsilon^{|\alpha''|} (\partial^{\alpha''}_x \varphi)(\epsilon x)\, b(x)\mathrm{d}x = 0,$$

while the limit

$$\lim_{\epsilon\to 0}\int -\partial_x^{\alpha}[e^{\mathrm{i}x\cdot Qx}\,a(x)]\,\epsilon^{|\alpha''|}\varphi(\epsilon x)b(x)\mathrm{d}x$$

exists and gives the r.h.s. of *(ii)*. We leave the details to the reader.

(iii) Consider the oscillatory integral

$$I(y) := \int e^{\mathrm{i}x\cdot Qx}\,a(x,y)\mathrm{d}x.$$

By the previous theorem we know that, provided we can interchange the limit and the derivative

$$\begin{aligned}\partial_y^{\alpha} I(y) &= \partial_y^{\alpha}\lim_{j\to\infty} I_j(y), \qquad I_j(y) := \int e^{\mathrm{i}x\cdot Qx} a(x,y)\chi(2^{-j}x)\mathrm{d}x\\ &= \lim_{j\to\infty}\partial_y^{\alpha} I_j(y)\\ &= \lim_{j\to\infty}\int e^{\mathrm{i}x\cdot Qx}\,\partial_y^{\alpha} a(x,y)\,\chi(2^{-j}x)\,\mathrm{d}x\\ &= \int e^{\mathrm{i}x\cdot Qx}\,\partial_y^{\alpha} a(x,y)\,\mathrm{d}x\end{aligned}$$

and the only passage to justify is the exchange of the limit and the derivative. This is justified provided $\{\partial_y^{\alpha} I_j(y)\}$ converges uniformly (at least on compact sets). But this is true, as arguing as in the previous proof, one shows the punctual estimate

$$\left|\partial_y^{\alpha} I_j(y) - \partial_y^{\alpha} I_{j-1}(y)\right| \preceq 2^{-j}\,\langle y\rangle^{m-\delta|\alpha|} \tag{6.31}$$

which implies uniform convergence on any compact set for the sequence $\{\partial_y^{\alpha} I_j(y)\}_j$. Actually one concludes that $\langle y\rangle^{-m+\delta|\alpha|}\,\partial_y^{\alpha} I_j(y)$ is Cauchy in $(C^0(\mathbb{R}^p,\mathbb{C}), \|\cdot\|_{\infty})$, so in particular the limit fulfills

$$\sup_y \left|\langle y\rangle^{-m+\delta|\alpha|}\,\partial_y^{\alpha} I(y)\right| \le C$$

so $I(y)\in A_{\delta}^{m}$ is an amplitude.

(iv) As above let

$$I(y) := \int e^{\mathrm{i}x\cdot Qx}\,a(x,y)\mathrm{d}x.$$

By the previous proof we already know it is an amplitude. Now

$$\int e^{\mathrm{i}y\cdot Py} I(y)\mathrm{d}y = \lim_{j\to\infty}\int e^{\mathrm{i}y\cdot Py}\,I(y)\,\chi(2^{-j}y)\mathrm{d}y$$

Now, denoting $I_j(y)$ as above

$$\int e^{\mathrm{i}y\cdot Py}\,I(y)\,\chi(2^{-j}y)\mathrm{d}y = \int e^{\mathrm{i}y\cdot Py}\,I_j(y)\,\chi(2^{-j}y)\mathrm{d}y + \int e^{\mathrm{i}y\cdot Py}\,(I(y)-I_j(y))\,\chi(2^{-j}y)\mathrm{d}y$$

Now the first integral is regularized, we can exchange the integrals and pass to the limit for $j\to\infty$, getting that

$$\int e^{\mathrm{i}y\cdot Py}\,I_j(y)\,\chi(2^{-j}y)\mathrm{d}y = \int e^{\mathrm{i}y\cdot Py+\mathrm{i}x\cdot Qx}\,a(x,y)\,\chi(2^{-j}x)\,\chi(2^{-j}y)\mathrm{d}x\,\mathrm{d}y \to \int e^{\mathrm{i}y\cdot Py+\mathrm{i}x\cdot Qx}\,a(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Concerning the second integral, one passes to the limit in (6.31) and proves that

$$\left|\partial_y^\alpha (I(y) - I_j(y))\right| \lesssim 2^{-j} \langle y\rangle^{m-\delta|\alpha|}$$

namely $b_j(y) := (I(y) - I_j(y))\, \chi(2^{-j}y) \in A^m_\delta$ with $N^m_{\frac{m+p+1}{1+\delta}}(b) \lesssim C_0 2^{-j}$. Hence

$$\left|\int e^{\mathrm{i}y\cdot Py}\, b_j(y)\mathrm{d}y\right| \leq C N^m_{\frac{m+p+1}{1+\delta}}(b) \lesssim C_0 2^{-j}$$

which goes to 0 as $j \to \infty$.

(v) The proof that $a \in A^m_\delta$ is easy and we skip it. By the linearity of the oscillatory integral it is enough to show that

$$\int e^{\mathrm{i}x\cdot Qx}\, (a_j(x) - a(x))\, \mathrm{d}x \to 0, \qquad j \to \infty.$$

Introduce the operator

$$L := \frac{1}{\langle x\rangle^2}\left(1 + \frac{1}{2\mathrm{i}} Q^{-1}x \cdot \partial_x\right);$$

then, being Q symmetric and invertible,

$$L^k e^{\mathrm{i}x\cdot Qx} = e^{\mathrm{i}x\cdot Qx} \qquad \forall k \in \mathbb{N}$$

the adjoint operator is given by

$$L^* = \frac{1}{\langle x\rangle^2} + \operatorname{div}\left(\frac{\cdot}{2\mathrm{i}\,\langle x\rangle^2} Q^{-1}x\right)$$

As $b_j := a_j - a \in A^m_\delta$, by integration by parts in the oscillatory integrals we have

$$\int e^{\mathrm{i}x\cdot Qx}\, b_j(x)\, \mathrm{d}x = \int e^{\mathrm{i}x\cdot Qx}\, (L^*)^k[b_j(x)]\, \mathrm{d}x$$

Now

$$(L^* b_j)(x) = \frac{b_j(x)}{\langle x\rangle^2} + \operatorname{div}\left(\frac{b_j(x)}{2\mathrm{i}\,\langle x\rangle^2} Q^{-1}x\right)$$

is bounded by

$$|L^* b_j| \lesssim \frac{|b_j(x)|}{\langle x\rangle^2} + \frac{|\partial_x b_j(x)|}{\langle x\rangle} \lesssim \langle x\rangle^{m-2} + \langle x\rangle^{m-(\delta+1)}$$

In particular, as $\delta + 1 > 0$, we gained decay. With a similar estimate one shows that

$$\left|(L^*)^k b_j(x)\right| \lesssim \langle x\rangle^{m-2k} + \langle x\rangle^{m-(\delta+1)k} \in L^1(\mathbb{R}^d)$$

which is integrable for k large enough. As $b_j(x) \to 0$ punctually, we apply Lebesgue dominated convergence theorem ans prove that

$$\int e^{\mathrm{i}x\cdot Qx}\, b_j(x)\, \mathrm{d}x = \int e^{\mathrm{i}x\cdot Qx}\, (L^*)^k[b_j(x)]\, \mathrm{d}x \to 0, \qquad j \to \infty$$

□

Working with symbols, we deduce the following corollary

Corollary 6.7. *Let $a \in S^m$, $m \in \mathbb{R}$ and $(a_j)_{j\in\mathbb{N}} \subset \mathcal{S}$ such that*

(i) $(a_j)_{j\in\mathbb{N}}$ *is bounded in* $\mathcal{S}^m$*;*

(ii) $\forall \alpha, \beta \in \mathbb{N}_0^d$, $\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi) \to \partial_x^\alpha \partial_\xi^\beta a(x,y)$ *uniformly on compacts.*

Then

$$\langle \mathrm{Op}\,(a_j)\, f, g\rangle \to \langle \mathrm{Op}\,(a)\, f, g\rangle\,, \qquad \forall f, g \in \mathcal{S}. \tag{6.32}$$

Proof. Write

$$\langle \mathrm{Op}\,(a_j)\, f, g\rangle = \int e^{\mathrm{i}x\xi}\, a_j(x,\xi)\, \widehat{f}(\xi)\, \overline{g}(x) \mathrm{d}\xi \mathrm{d}x;$$

the r.h.s. is an oscillatory integral, thus we can apply Proposition 6.6 (v) to conclude that

$$\int e^{\mathrm{i}x\xi}\, a_j(x,\xi)\, \widehat{f}(\xi)\, \overline{g}(x) \mathrm{d}\xi \mathrm{d}x \to \int e^{\mathrm{i}x\xi}\, a(x,\xi)\, \widehat{f}(\xi)\, \overline{g}(x) \mathrm{d}\xi \mathrm{d}x$$

which proves (6.32). □

It is useful to show that one can actually approximate symbols.

Lemma 6.8 (Approximation of symbols)**.** *Let $a \in S^m$, $m \in \mathbb{R}$. Then there exists a sequence $(a_j)_{j\in\mathbb{N}} \subset \mathcal{S}$ such that*

(i) $(a_j)_{j\in\mathbb{N}}$ *is bounded in* $\mathcal{S}^m$*, i.e.*

$$\wp_k^m(a_j) \le C_k\, \wp_k^m(a), \qquad \forall j \in \mathbb{N}.$$

(ii) $\forall \alpha, \beta \in \mathbb{N}_0^d$, $\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi) \to \partial_x^\alpha \partial_\xi^\beta a(x,y)$ *uniformly on compacts.*

(iii) $a_j \to a$ *in* $\mathcal{S}^{m'}$ *as* $j \to \infty$ *for any* $m' > m$*.*

Proof. Let $\chi \in C_0^\infty$, $\chi \equiv 1$ in $B_1(0)$. Set

$$a_j(x,\xi) := a(x,\xi)\chi(2^{-j}x)\chi(2^{-j}\xi) \in C_0^\infty.$$

By Leibnitz rule, $\partial_x^\alpha \partial_\xi^\beta a_j$ equals

$$\chi(2^{-j}x)\chi(2^{-j}\xi)\partial_x^\alpha \partial_\xi^\beta a(x,\xi) + \sum_{\substack{0\neq\alpha'\le\alpha \\ 0\neq\beta'\le\beta}} C_{\alpha',\beta'} 2^{-j|\alpha'+\beta'|} (\partial_x^{\alpha'}\chi)(2^{-j}x)\, (\partial_\xi^{\beta'}\chi)(2^{-j}\xi)\, (\partial_x^{\alpha-\alpha'}\partial_\xi^{\beta-\beta'} a(x,\xi))$$

In particular, using the boundedness of χ and its derivatives, we get for each fixed k and $|\alpha+\beta| \le k$, that

$$\left|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)\right| \le C\, \langle\xi\rangle^m\, \wp_k^m(a)$$

proving (i). Item (ii) follows by taking the punctual limit in the expression above. Finally to prove item (iii) we remark that $a - a_j = (1 - \chi(2^{-j}x)\chi(2^{-j}\xi))a$

$$\begin{aligned}\left|\partial_x^\alpha \partial_\xi^\beta (a_j - a)(x,\xi)\right| &\lesssim \left|(1-\chi(2^{-j}x)\chi(2^{-j}\xi))\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \\ &\quad + \sum_{0\neq\beta'\le\beta} 2^{-j|\beta'|}(\partial_\xi^{\beta'}\chi)(2^{-j}\xi)\left|\partial_x^{\alpha-\alpha'}\partial_\xi^{\beta-\beta'} a(x,\xi)\right|\end{aligned}$$

from which we deduce that, $\forall m' > m$,

$$\begin{aligned}\langle\xi\rangle^{-m'+|\beta|}\left|\partial_x^\alpha\partial_\xi^\beta(a_j-a)(x,\xi)\right| &\lesssim \wp_k^m(a)\sup_{|\xi|\geq 2^j}\langle\xi\rangle^{-m'+m}\\ &\quad+\wp_k^m(a)\sup_{|\xi|\sim 2^j}\langle\xi\rangle^{-m'+m+|\beta'|}2^{-j|\beta'|}\\ &\lesssim \wp_k^m(a)2^{-j(m'-m)}\end{aligned}$$

using that in the first term $|\xi|\geq 2^j$, while in the second term $|\xi|\sim 2^j$. Hence $\wp_k^{m'}(a_j-a)\to 0$ as $j\to\infty$, as claimed. □

This means that, when we work with symbols of some pseudodifferential operators, we can always assume that they are Schwartz and then argue by approximation.

Examples: The method of the proof actually gives a way to compute oscillatory integrals. We give few examples.

(i) If $a\in A_\delta^m(\mathbb{R}^n)$, $\delta\in(-1,1]$, then

$$(2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}a(y)\,\mathrm{d}y\mathrm{d}\eta = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}a(\eta)\,\mathrm{d}y\mathrm{d}\eta = a(0) \tag{6.33}$$

Indeed the integral is of the form (6.22) with $d=2n$, $Q=\frac{1}{2}\begin{pmatrix}0 & -\mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}$, $x=\begin{pmatrix}y\\ \eta\end{pmatrix}$. Indeed take $\varphi\in\mathcal{S}(\mathbb{R}^n)$ with $\varphi(0)=1$. From the theorem

$$\int e^{-\mathrm{i}y\eta}a(y)\mathrm{d}y\mathrm{d}\eta = \lim_{\epsilon\to 0}\int e^{-\mathrm{i}y\eta}\,a(y)\,\varphi(\epsilon y)\,\varphi(\epsilon\eta)\mathrm{d}y\mathrm{d}\eta$$

By the properties of the Fourier transform

$$\int_{\mathbb{R}^n} e^{-\mathrm{i}y\eta}\,\varphi(\epsilon\eta)\mathrm{d}\eta = \frac{1}{\epsilon^n}\widehat{\varphi}(y/\epsilon)$$

hence

$$\int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}\,a(y)\,\varphi(\epsilon y)\,\varphi(\epsilon\eta)\mathrm{d}y\mathrm{d}\eta = \frac{1}{\epsilon^n}\int_{\mathbb{R}^n} a(y)\,\varphi(\epsilon y)\,\widehat{\varphi}(y/\epsilon)\,\mathrm{d}y \overset{y=\epsilon z}{=} \int_{\mathbb{R}^n} a(\epsilon z)\,\varphi(\epsilon^2 z)\,\widehat{\varphi}(z)\,\mathrm{d}z$$

Since $\widehat{\varphi}\in\mathcal{S}$, by dominated convergence we get

$$\lim_{\epsilon\to 0}\int a(\epsilon z)\,\varphi(\epsilon^2 z)\,\widehat{\varphi}(z)\,\mathrm{d}z = a(0)\,\varphi(0)\underbrace{\int\widehat{\varphi}(z)\mathrm{d}z}_{=(2\pi)^n\varphi(0)} = (2\pi)^n a(0)$$

which proves (6.33).

(ii) Let $\alpha,\beta\in\mathbb{N}^n$. Then

$$\int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}\,\frac{y^\alpha}{\alpha!}\,\frac{\eta^\beta}{\beta!}\,\mathrm{d}y\,\mathrm{d}\eta = \begin{cases}0 & \alpha\neq\beta\\ (2\pi)^n\frac{(-\mathrm{i})^{|\alpha|}}{\alpha!} & \alpha=\beta\end{cases} \tag{6.34}$$

Indeed $y^\alpha e^{-\mathrm{i}y\eta} = (-D_\eta)^\alpha e^{-\mathrm{i}y\eta}$, so by integration by parts in oscillatory integrals

$$\begin{aligned}\int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}\,\frac{y^\alpha}{\alpha!}\,\frac{\eta^\beta}{\beta!}\,\mathrm{d}y\,\mathrm{d}\eta = \int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}\,\frac{(D_\eta)^\alpha}{\alpha!}\,\frac{\eta^\beta}{\beta!}\,\mathrm{d}y\,\mathrm{d}\eta = \int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}\,\frac{(-\mathrm{i})^{|\alpha|}}{\alpha!}\,\frac{\eta^{\beta-\alpha}}{(\beta-\alpha)!}\,\mathrm{d}y\,\mathrm{d}\eta \\ = \frac{(2\pi)^n\,(-\mathrm{i})^{|\alpha|}}{\alpha!}\,\frac{\eta^{\beta-\alpha}}{(\beta-\alpha)!}\bigg|_{\eta=0}\end{aligned}$$

which gives the result.