5 Pseudodifferential operators

We are ready to define pseudodifferential operators.

Definition 5.1. *Let $a \in \mathcal{S}^m(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C})$. The* pseudodifferential operator with symbol a *is the linear operator*

$$[\mathrm{Op}\,(a)\,u]\,(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\xi}\, a(x,\xi)\, \widehat{u}(\xi)\, \mathrm{d}\xi \tag{5.1}$$

We will say that $\mathrm{Op}\,(a)$ has order m, and we will write $\mathrm{Op}\,(a) \in \mathrm{Op}\,(\mathcal{S}^m)$. Sometimes we will write $\mathrm{Op}\,(a) = a(x, D)$.

Example 5.2. *We give few example of quantization of symbols*

- *If $a = a(x)$, then $[\mathrm{Op}\,(a)\,u](x) = a(x)u(x)$, the multiplication operator.*
- *If $a = a(\xi)$, then $\mathrm{Op}\,(a)\,u = a(D)u$ is a Fourier multiplier.*
- *If $a = f(x)g(\xi)$, then $\mathrm{Op}\,(a)\,u = f(x)g(D)u$.*

We start by showing that pseudodifferential operators are well defined on the Schwartz class:

Proposition 5.3. *Let $a \in \mathcal{S}^m$. Then the following holds true:*

(i) If $u \in \mathcal{S}$, then $\mathrm{Op}\,(a)\,u \in \mathcal{S}$. The map

$$\mathcal{S}^m \times \mathcal{S} \to \mathcal{S}, \quad (a,u) \mapsto \mathrm{Op}\,(a)\,u$$

is continuous, in the sense that $\forall k \geq 0$, there exist $M \in \mathbb{N}$, $C > 0$ such that

$$\wp_k(\mathrm{Op}\,(a)\,u) \leq C \wp^m_M(a)\, \wp_M(u).$$

(ii) If $\mathrm{Op}\,(a) = \mathrm{Op}\,(b)$ as operator in $\mathcal{S}$, then $a = b$.

(iii) One has the commutation rules

$$[\mathrm{Op}\,(a)\,, D_j] = \mathrm{i}\mathrm{Op}\left(\partial_{x_j} a\right)\ , \qquad [\mathrm{Op}\,(a)\,, x_j] = -\mathrm{i}\mathrm{Op}\left(\partial_{\xi_j} a\right). \tag{5.2}$$

Proof. Since $\widehat{u} \in \mathcal{S}$, the integral in (5.1) is convergent. Consequently the map $x \mapsto [\mathrm{Op}\,(a)\,u](x)$ is a continuous function and it fulfills

$$\begin{aligned} |[\mathrm{Op}\,(a)\,u](x)| &\leq (2\pi)^{-n} \int \langle\xi\rangle^m\, |\widehat{u}(\xi)|\, \mathrm{d}\xi \sup_{x,\xi\in\mathbb{R}^n} \left|\langle\xi\rangle^{-m} a(x,\xi)\right| \\ &\leq (2\pi)^{-n} \wp^m_0(a) \int \langle\xi\rangle^{m+n+1}\, |\widehat{u}(\xi)|\, \langle\xi\rangle^{-n-1}\, \mathrm{d}\xi \\ &\leq C \wp^m_0(a)\, \wp_{m+n+1}(\widehat{u}) \leq C \wp^m_0(a)\, \wp_{m+3n+1}(u) \end{aligned}$$

Thus $\mathrm{Op}\,(a)\,u \in L^\infty \cap C^0$.

Let us prove immediately item (iii). We have

$$\mathrm{Op}\,(a)\, D_j u(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\xi}\, a(x,\xi)\, (D_j u)^\wedge(\xi) \mathrm{d}\xi = (2\pi)^{-n} \int e^{\mathrm{i}x\xi}\, a(x,\xi)\, \xi_j \widehat{u}(\xi) \mathrm{d}\xi$$

while

$$D_j \mathrm{Op}\,(a)\,u(x) = (2\pi)^{-n}\int (D_j e^{\mathrm{i}x\xi})\,a(x,\xi)\,\widehat{u}(\xi)\mathrm{d}\xi + (2\pi)^{-n}\int e^{\mathrm{i}x\xi}\,(D_j a(x,\xi))\,\widehat{u}(\xi)\mathrm{d}\xi$$
$$= (2\pi)^{-n}\int \xi_j e^{\mathrm{i}x\xi}\,a(x,\xi)\,\xi_j\widehat{u}(\xi)\mathrm{d}\xi + (2\pi)^{-n}\int e^{\mathrm{i}x\xi}\,(\frac{1}{\mathrm{i}}\partial_{x_j} a(x,\xi))\,\widehat{u}(\xi)\mathrm{d}\xi$$

Hence $[\mathrm{Op}\,(a)\,, D_j]u = \mathrm{Op}\left(-\frac{1}{\mathrm{i}}\partial_{x_j}a\right)u$ and one gets the first of (5.2) immediately. The second identity is proven similarly.

In particular we have that $x_j\mathrm{Op}\,(a)\,u = \mathrm{Op}\,(a)\,x_j u + \mathrm{i}\mathrm{Op}\left(\partial_{\xi_j}a\right)u$, and in general

$$x^\alpha D^\beta[\mathrm{Op}\,(a)\,u](x) = \underset{\substack{\alpha'\leq\alpha\\ \beta'\leq\beta}}{\mathrm{lin.\ com.}}\left[\mathrm{Op}\left(\partial_\xi^{\alpha'}\partial_x^{\beta'}a\right)\,x^{\alpha-\alpha'}\,D^{\beta-\beta'}u\right](x)$$

But now $\partial_\xi^{\alpha'}\partial_x^{\beta'}a \in \mathcal{S}^{m-|\alpha'|}$ and $x^{\alpha-\alpha'}\,D^{\beta-\beta'}u \in \mathcal{S}$, thus we can proceed as above and prove that $x \mapsto x^\alpha D^\beta[\mathrm{Op}\,(a)\,u](x) \in L^\infty \cap C^0$ with

$$\left|x^\alpha D^\beta[\mathrm{Op}\,(a)\,u](x)\right| \leq C_{\alpha\beta}\,\wp_M^m(a)\,\wp_M(u)$$

for some M sufficiently large.

Finally we prove (ii). We show that $\mathrm{Op}\,(a) = 0$ implies $a = 0$. By contradiction assume that $a \neq 0$. Then take $v \in \mathcal{S}$, M large (fix it later), and define

$$\widehat{u} = \langle\xi\rangle^{-M}\,\overline{v}(\xi).$$

With this choice, we have that

$$(\mathrm{Op}\,(a)\,u)(x) = \int \frac{a(x,\xi)}{\langle\xi\rangle^M}e^{\mathrm{i}x\xi}\overline{v}(\xi)\mathrm{d}\xi;$$

now if M is sufficiently large the function $\xi \mapsto \frac{\alpha(x,\xi)}{\langle\xi\rangle^M} \in L^2(\mathbb{R}^n)$, thus we write

$$(\mathrm{Op}\,(a)\,u)(x) = \langle b(x,\cdot), \widetilde{v}\rangle = 0, \qquad b(x,\xi) := \frac{\alpha(x,\xi)}{\langle\xi\rangle^M}, \quad \widetilde{v} = e^{-\mathrm{i}x\xi}v(\xi)$$

Since the equality above holds for all $\tilde{v}$ of that form, which form a dense set in $\mathcal{S}$, we conclude that $b = 0$, hence $a = 0$. □

Remark 5.4. *The commutation relations* (5.2) *tell us something interesting immediately:* $[\mathrm{Op}\,(a)\,, D_j]$ *is a pseudodifferetial operator of order* m *(as* $\partial_{x_j}a \in \mathcal{S}^m$*), while* $[\mathrm{Op}\,(a)\,, x_j]$ *is a pseudodifferetial operator of order* $m-1$ *(as* $\partial_{\xi_j}a \in \mathcal{S}^{m-1}$*).*

5.1 Kernels

Assume that $a \in \mathcal{S}^{-\infty}$. Then, for $u \in \mathcal{S}$, using Fourier transform and Fubini theorem

$$[\mathrm{Op}\,(a)\,u]\,(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\xi}\,a(x,\xi)\,\widehat{u}(\xi)\,\mathrm{d}\xi$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}x\xi}\,a(x,\xi)\int_{\mathbb{R}^n} u(y)e^{-\mathrm{i}\xi y}\,\mathrm{d}y\,\mathrm{d}\xi$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}\,a(x,\xi)\,u(y)\,\mathrm{d}\xi\,\mathrm{d}y$$

Thus we can write $\mathrm{Op}\,(a)$ as an operator with integral kernel, namely in the form

$$[Au](x) = \int_{\mathbb{R}^n} K(x,y)u(y)\mathrm{d}y.$$

In our case we have

$$K(x,y) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\mathrm{i}(x-y)\xi} a(x,\xi)\mathrm{d}\xi. \tag{5.3}$$

Remark that such expression makes sense for $a \in \mathcal{S}^m$ with $m < -n$, which gives in particular that $K \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$.

In case $a \in \mathcal{S}^m$ with $m \geq -n$, we need to give a meaning to the expression

$$(\mathrm{Op}\,(a)\,u)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}\, a(x,\xi)\, u(y)\, \mathrm{d}\xi\, \mathrm{d}y$$

and to the kernel. There are several possibilities, we show some of them:
1) Note that the kernel K in (5.3) is the inverse Fourier transform of $\xi \mapsto a(x,\xi)$ evaluated at the point $(x-y)$, namely

$$K(x,y) = \mathcal{F}_\xi^{-1}(a(x,\cdot))|_{x-y} \quad \Rightarrow \quad K(x,x-y) = \mathcal{F}_\xi^{-1}(a(x,\cdot))|_y \tag{5.4}$$

So the first possibility is to view (5.3) as a distributional identity, which allows us also to recover the symbol from the integral by taking the inverse Fourier transform. In particular we get

$$a(x,\xi) = \mathcal{F}_y\left(K(x,x-y)\right) = \int_{\mathbb{R}^n} e^{-\mathrm{i}\xi y} K(x,x-y)\mathrm{d}y \tag{5.5}$$

So in this way I see $\mathrm{Op}\,(a)$ as a linear operator with distributional kernel acting on Schwartz functions.
2) As an oscillatory integral. We will study these objects in details later on, but the idea is the following: Let $\chi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$, $\chi = 1$ around 0, and define

$$(I_{\chi,\epsilon}u)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}\, \chi(\epsilon\xi)\, a(x,\xi)\, u(y)\, \mathrm{d}\xi\, \mathrm{d}y.$$

Now the integral is well defined. Using that

$$-\Delta_y e^{\mathrm{i}(x-y)\xi} = |\xi|^2 e^{\mathrm{i}(x-y)\xi},$$

we find that for any N integer

$$\langle\xi\rangle^{-2N} \left(1-\Delta_y\right)^N e^{\mathrm{i}(x-y)\xi} = e^{\mathrm{i}(x-y)\xi}.$$

Hence, integrating by parts, one gets

$$(I_{\chi,\epsilon}u)(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}\, \langle\xi\rangle^{-2N}\, \chi(\epsilon\xi)\, a(x,\xi)\, (1-\Delta_y)^N u(y)\, \mathrm{d}\xi\, \mathrm{d}y$$

Now if N is sufficiently large we have that the integrand is L^1. We can therefore apply Lebesgue dominated convergence theorem and get that (provided N is large enough) the limit exists and is well defined; in particular

$$\lim_{\epsilon\to 0}(I_{\chi,\epsilon}u)(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}\, \langle\xi\rangle^{-2N}\, a(x,\xi)\, (1-\Delta_y)^N u(y)\, \mathrm{d}\xi\, \mathrm{d}y$$

Moreover such limit does not depend on χ. So we put

$$(\mathrm{Op}\,(a)\,u)(x) := \lim_{\epsilon\to 0}(I_{\chi,\epsilon}u)(x)$$

An important property of pseudodifferential operators is that their integral kernel is smooth outside the diagonal and rapidly decaying as $|x-y|\to\infty$:

Proposition 5.5. *Let $a\in\mathcal{S}^m$. Then the integral kernel $K(x,y)$ of $\mathrm{Op}\,(a)$ (given by (5.3)) satisfies*

$$\left|\partial_{x,y}^{\beta}K(x,y)\right| \le C_{N,\beta}\;|x-y|^{-N} \tag{5.6}$$

for $N>m+n+|\beta|$ and $x\neq y$. Thus $K(x,y)$ is smooth outside the diagonal and rapidly decaying as $|x-y|\to\infty$.

Proof. Exploiting that $K(x,\cdot)$ is the inverse Fourier transform of $a(x,\cdot)$, evaluated at the point $x-y$, one gets (in a distributional sense)

$$\begin{aligned}(x-y)^{\alpha}K(x,y) &= (x-y)^{\alpha}\mathcal{F}_{\xi}^{-1}[a(x,\cdot)](x-y) = \mathcal{F}_{\xi}^{-1}[(-D_{\xi})^{\alpha}a(x,\cdot)](x-y)\\ &= (2\pi)^{-n}\int e^{\mathrm{i}(x-y)\xi}(-D_{\xi})^{\alpha}a(x,\xi)\mathrm{d}\xi\end{aligned}$$

But since $a\in\mathcal{S}^m$, it follows that $\left|D_{\xi}^{\alpha}a(x,\xi)\right|\le C_{\alpha}\,\langle\xi\rangle^{m-|\alpha|}$, hence provided $m-|\alpha|<-n$, the integrand is in L^1_{ξ}, hence converging. Thus

$$|(x-y)^{\alpha}K(x,y)| \le \int \langle\xi\rangle^{m-|\alpha|} \le C<\infty$$

This proves (5.6) with $\beta=0$.
One argues similarly for the derivatives, the detail are left to the reader. □

We have also the following lemma, which connects smoothing operators and smooth kernels.

Lemma 5.6. *If $a\in\mathcal{S}^{-\infty}$ then the kernel $K(x,y)$ of $\mathrm{Op}\,(a)$ is everywhere smooth. Viceversa, if the kernel of the operator is Schwartz, then $a\in\mathcal{S}^{-\infty}$.*

Proof. Assume that $a\in\mathcal{S}^{-\infty}$. Then $K(x,y)$ is well defined and continuous everywhere, since $a(x,\cdot)\in L^1$. Now

$$\partial_x^{\beta_1}\partial_y^{\beta_2}K(x,y) = \sum_{\alpha_1\le\beta_1} C_{\alpha_1}\int e^{\mathrm{i}(x-y)\xi}\,\xi^{\alpha_1+\beta_2}\,\left(\partial_x^{\beta_1-\alpha_1}a(x,\xi)\right)\mathrm{d}\xi$$

Since a is smoothing, then the integral is always convergent. This proves that the kernel is C^∞ everywhere.

To prove the converse statement just use (5.5) and argue as above. □

5.2 Symbolic calculus: formal results

We work formally and try to understand the basic properties of pseudodifferential operators: in particular, is the class closed under adjoint and composition?

Adjoint. Consider $A\colon \mathcal{S} \to \mathcal{S}$. We look for an operator $A^*\colon \mathcal{S} \to \mathcal{S}$ such that

$$\langle A^* u, v\rangle = \langle u, Av\rangle \ , \qquad \forall u, v \in \mathcal{S}$$

By a density argument, if A^* exists, it is unique. We call A^* the (formal) adjoint of A (with respect to the L^2 scalar product).[1]

If A^* exists one can easily verify that it is unique. Remark that if A^* exists then we can extend A by duality as a map $\mathcal{S}' \to \mathcal{S}'$, by defining

$$\forall u \in \mathcal{S}',\ \forall v \in \mathcal{S}, \qquad (Au)(v) := u(\overline{A^*\overline{v}});$$

the strange definition is due, as usual, to the fact that if $u \in \mathcal{S}$, then

$$(Au)(v) = \int (Au)(x)\, v(x) \mathrm{d}x = \langle Au, \overline{v}\rangle = \overline{\langle \overline{v}, Au\rangle} = \overline{\langle A^*\overline{v}, u\rangle} = \langle u, A^*\overline{v}\rangle = \int u(x)\, \overline{(A^*\overline{v})(x)} \mathrm{d}x = u(\overline{A^*\overline{v}}).$$

Consider now A a linear operator with integral kernel $K(x,y)$, namely $(Au)(x) = \int K(x,y) u(y) \mathrm{d}y$. Then

$$\begin{aligned}\langle A^*u, v\rangle = \langle u, Av\rangle &= \int u(x)\, \overline{[Av](x)} \mathrm{d}x = \iint u(x) \overline{K(x,y)v(y)} \mathrm{d}x\mathrm{d}y \\ &= \left\langle \int \overline{K(x,y)} u(x) \mathrm{d}x, v \right\rangle = \left\langle \int \overline{K(y,x)} u(y) \mathrm{d}y, v \right\rangle\end{aligned}$$

This shows that A^* is an operator with integral kernel

$$K^*(x,y) := \overline{K(y,x)}. \tag{5.7}$$

Since $\mathrm{Op}\,(a)$ has kernel $K(x,y) = (2\pi)^{-n} \int e^{\mathrm{i}(x-y)} a(x,\xi) \mathrm{d}\xi$, it follows that $\mathrm{Op}\,(a)^*$ has kernel

$$K^*(x,y) = \overline{K(y,x)} = (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\eta}\, \overline{a(y,\eta)}\, \mathrm{d}\eta.$$

Now thanks to (5.5) we can reconstruct a symbol from the kernel, from which we find that the symbol of $\mathrm{Op}\,(a)^*$ can be written as

$$\mathrm{Op}\,(a)^*\, u = \int K^*(x,y) u(y) \mathrm{d}y = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}\, a^*(x,\xi)\, u(y)\, \mathrm{d}\xi\, \mathrm{d}y$$

provided we define a^* as the Fourier transform of K^*, i.e. we put

$$\begin{aligned} a^*(x,\xi) &= \int e^{-\mathrm{i}\xi y} K^*(x, x-y) \mathrm{d}y = (2\pi)^{-n} \int e^{-\mathrm{i}\xi y}\, e^{\mathrm{i}y\eta}\, \overline{a(x-y,\eta)}\, \mathrm{d}\eta \mathrm{d}y \\ &\overset{\zeta=\xi-\eta}{=} (2\pi)^{-n} \int e^{-\mathrm{i}\zeta y}\, \overline{a(x-y, \xi-\zeta)}\, \mathrm{d}\zeta \mathrm{d}y \end{aligned} \tag{5.8}$$

This computation is rigorous if the symbol a is fast decaying at infinity (indeed in such a case Fubini theorem justifies all the computations), but what about the general case? And even if it is well defined, is it a symbol?

[1] Note that in general we can define A^* as an operator from $\mathcal{S}' \to \mathcal{S}'$ as $(A^*u)(v) := u(\overline{Av})$. This is always well defined, but for $u \in \mathcal{S}$ it restricts only to an operator $\mathcal{S} \to \mathcal{S}'$ for which $(A^*u)(v) := \int u\overline{Av} = \langle u, Av\rangle$; so the non trivial requirement is that $A^*\colon \mathcal{S} \to \mathcal{S}$.
For example if $A\colon u \mapsto u(0) f$, where $f \in \mathcal{S}$, then $A\colon \mathcal{S} \to \mathcal{S}$. However

$$(A^*u)(v) = \int u(x) \overline{Av}(x) = \langle u, f\rangle\, \overline{v(0)},$$

hence $A^* u = \langle u, f\rangle\, \overline{\delta_0}$, which is not a Schwartz function, but a distribution.

Composition. Let us proceed formally also for the composition of pseudodifferential operators and find the candidate symbol. We have

$$\begin{aligned}
[\mathrm{Op}\,(a)\,\mathrm{Op}\,(b)\,u](x) &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{\mathrm{i}(x-y)\xi}\,a(x,\xi)[\mathrm{Op}\,(b)\,u](y)\,\mathrm{d}y\mathrm{d}\xi\\
&= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{\mathrm{i}(x-y)\xi}\,a(x,\xi)(2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{\mathrm{i}(y-z)\eta}\,b(y,\eta)\,u(z)\,\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi\\
&= (2\pi)^{-2n}\int_{\mathbb{R}^{4n}} e^{\mathrm{i}x\xi}\,e^{-\mathrm{i}y(\xi-\eta)}\,e^{-\mathrm{i}z\eta}\,a(x,\xi)\,b(y,\eta)\,u(z)\,\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}z\mathrm{d}y\\
&= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{\mathrm{i}(x-z)\eta}\,c(x,\eta)\,u(z)\mathrm{d}z\mathrm{d}\eta
\end{aligned}$$

provided

$$\begin{aligned}
c(x,\eta) &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{\mathrm{i}(x-y)(\xi-\eta)}\,a(x,\xi)\,b(y,\eta)\,\mathrm{d}y\mathrm{d}\xi\\
&\overset{\substack{\xi=\zeta+\eta\\ y=x+z}}{=} (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-\mathrm{i}z\zeta}\,a(x,\zeta+\eta)\,b(x+z,\eta)\,\mathrm{d}\zeta\mathrm{d}z
\end{aligned} \tag{5.9}$$

So this is the candidate symbol. We note an interesting pattern: both the candidate symbol of the adjoint (5.8) and of the composition (5.9) have a precise structure: they are integrals of the form

$$\int e^{\mathrm{i}w\cdot Qw}\,a(w)\,\mathrm{d}w \tag{5.10}$$

with Q a real, symmetric, $n\times n$ matrix and a a function polynomially growing. Indeed in both cases Q has the form

$$Q = \frac{1}{2}\begin{pmatrix} 0 & -\mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}$$

acting on the variables $\begin{pmatrix} z\\ \zeta\end{pmatrix}$.

We will devote some time to the study of such integrals, which are called oscillatory integrals.