7 Symbolic calculus

We are ready to prove rigorously the results about symbolic calculus. We will use Peetre's inequality

Lemma 7.1 (Peetre's inequality)**.** *For any $s, \in \mathbb{R}$, for any $\xi, \eta \in \mathbb{R}^n$ one has*

$$\langle \xi \rangle^s \leq C_s \, \langle \xi - \eta \rangle^{|s|} \, \langle \eta \rangle^s \tag{7.1}$$

Proof. For $s \geq 0$ is trivial. For $s < 0$ use that

$$\langle \eta \rangle^{-s} \leq C_s \, \langle \xi - \eta \rangle^{-s} \, \langle \xi \rangle^{-s} .$$

□

It is also useful to approximate symbols with Schwartz functions.

Lemma 7.2 (Approximation of symbols)**.** *Let $a \in S^m$, $m \in \mathbb{R}$. Then there exists a sequence $(a_j)_{j \in \mathbb{N}} \subset \mathcal{S}$ such that*

(i) $(a_j)_{j \in \mathbb{N}}$ is bounded in $\mathcal{S}^m$, i.e.

$$\wp_k^m(a_j) \leq C_k \, \wp_k^m(a), \qquad \forall j \in \mathbb{N}.$$

(ii) $\forall \alpha, \beta \in \mathbb{N}_0^d$, $\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi) \to \partial_x^\alpha \partial_\xi^\beta a(x,y)$ uniformly on compacts.

(iii) $a_j \to a$ in $\mathcal{S}^{m'}$ as $j \to \infty$ for any $m' > m$.

Proof. Let $\chi \in C_0^\infty$, $\chi \equiv 1$ in $B_1(0)$. Set

$$a_j(x,\xi) := a(x,\xi)\chi(2^{-j}x)\chi(2^{-j}\xi) \in C_0^\infty .$$

By Leibnitz rule, $\partial_x^\alpha \partial_\xi^\beta a_j$ equals

$$\chi(2^{-j}x)\chi(2^{-j}\xi)\partial_x^\alpha \partial_\xi^\beta a(x,\xi) + \sum_{\substack{0 \neq \alpha' \leq \alpha \\ 0 \neq \beta' \leq \beta}} C_{\alpha',\beta'} 2^{-j|\alpha'+\beta'|} (\partial_x^{\alpha'}\chi)(2^{-j}x) \, (\partial_\xi^{\beta'}\chi)(2^{-j}\xi) \, (\partial_x^{\alpha-\alpha'}\partial_\xi^{\beta-\beta'} a(x,\xi))$$

In particular, using the boundedness of χ and its derivatives, we get for each fixed k and $|\alpha+\beta| \leq k$, that

$$\left| \partial_x^\alpha \partial_\xi^\beta a_j(x,\xi) \right| \leq C \, \langle \xi \rangle^m \, \wp_k^m(a)$$

proving (i). Item (ii) follows by taking the punctual limit in the expression above. Finally to prove item (iii) we remark that $a - a_j = (1 - \chi(2^{-j}x)\chi(2^{-j}\xi))a$

$$\begin{aligned} \left| \partial_x^\alpha \partial_\xi^\beta (a_j - a)(x,\xi) \right| &\lesssim \left| (1 - \chi(2^{-j}x)\chi(2^{-j}\xi))\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| \\ &\quad + \sum_{0 \neq \beta' \leq \beta} 2^{-j|\beta'|} (\partial_\xi^{\beta'}\chi)(2^{-j}\xi) \left| \partial_x^{\alpha-\alpha'}\partial_\xi^{\beta-\beta'} a(x,\xi) \right| \end{aligned}$$

from which we deduce that, $\forall m' > m$,

$$\begin{aligned}\langle\xi\rangle^{-m'+|\beta|}\left|\partial_x^\alpha\partial_\xi^\beta(a_j-a)(x,\xi)\right| &\lesssim \wp_k^m(a)\sup_{|\xi|\geq 2^j}\langle\xi\rangle^{-m'+m}\\ &\quad+\wp_k^m(a)\sup_{|\xi|\sim 2^j}\langle\xi\rangle^{-m'+m+|\beta'|}2^{-j|\beta'|}\\ &\lesssim \wp_k^m(a)2^{-j(m'-m)}\end{aligned}$$

using that in the first term $|\xi|\geq 2^j$, while in the second term $|\xi|\sim 2^j$. Hence $\wp_k^{m'}(a_j-a)\to 0$ as $j\to\infty$, as claimed. □

This means that, when we work with symbols of some pseudodifferential operators, we can always assume that they are Schwartz and then argue by approximation.

7.1 Adjoint

For the adjoint one has the following result

Theorem 7.3 (Adjoint)**.** *Let $a\in\mathcal{S}^m$. Then exists $a^*\in\mathcal{S}^m$ such that $\mathrm{Op}\,(a)^*=\mathrm{Op}\,(a^*)$. In particular*

$$a^*(x,\xi)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}}e^{-\mathrm{i}y\eta}\,\overline{a(x+y,\xi+\eta)}\,\mathrm{d}y\,\mathrm{d}\eta \tag{7.2}$$

and one has the asymptotic expansion

$$a^*(x,\xi)\sim\sum_\alpha\frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha\overline{a(x,\xi)} \tag{7.3}$$

Finally for every $j\in\mathbb{N}_0$ there exist $C,N>0$ such that

$$\wp_j^m(a^*)\leq C\,\wp_{j+N}^m(a) \tag{7.4}$$

The proof of the theorem can be found in [AG91].

Remark 7.4. *We can write*

$$a^*(x,y)=[e^{\mathrm{i}D_x\cdot D_\xi}\overline{a}](x,\xi) \tag{7.5}$$

where the equality is meant in sense of Fourier multipliers. See (6.15) for a proof .

Formula (7.12) gives us a mnemonic way to compute the expansion of the symbol: indeed formally

$$e^{\mathrm{i}D_x\cdot D_\xi}\overline{a(x,\xi)}=\sum_k\frac{\mathrm{i}^k}{k!}(D_x\cdot D_\xi)^k\overline{a(x,\xi)}=\sum_\alpha\frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha\overline{a(x,\xi)}$$

where we used the multinomial theorem

$$(\sum_{j=1}^n x_j)^k=\sum_{\substack{\alpha\in\mathbb{N}^n\\|\alpha|=k}}\binom{k}{\alpha}x^\alpha,\qquad\binom{k}{\alpha}=\frac{k!}{\alpha!}$$

Remark 7.5. *The expansion (7.3) is in decreasing symbols, since $\partial_\xi^\alpha D_x^\alpha\overline{a(x,\xi)}\in\mathcal{S}^{m-|\alpha|}$. In particular symbolic calculus tells us that*

$$a^*(x,\xi)=\overline{a(x,\xi)}+\mathcal{S}^{m-1} \tag{7.6}$$

Remark 7.6. *Since $a \in \mathcal{S}^m$ implies $a^* \in \mathcal{S}^m$, we have* $\mathrm{Op}\,(a)^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$*. In particular we can extend the action of* $\mathrm{Op}\,(a)$ *by duality over* $\mathcal{S}'(\mathbb{R}^n)$*:*

$$\langle \mathrm{Op}\,(a)\, u, v\rangle_{\mathcal{S}',\mathcal{S}} := \left\langle u, \overline{\mathrm{Op}\,(a)^* v}\right\rangle_{\mathcal{S}',\mathcal{S}}, \qquad \forall u \in \mathcal{S}', \quad \forall v \in \mathcal{S} \tag{7.7}$$

where the conjugate operator $\overline{A}$ *is defined by*

$$\overline{A}\, v := \overline{A\, \overline{v}}.$$

To deduce (7.7) *we used that*

$$\langle \mathrm{Op}\,(a)\, u, v\rangle_{\mathcal{S}',\mathcal{S}} = \langle \mathrm{Op}\,(a)\, u, \overline{v}\rangle_{L^2} = \langle u, \mathrm{Op}\,(a)^*\, \overline{v}\rangle_{L^2} = \left\langle u, \overline{\mathrm{Op}\,(a)^*\, \overline{v}}\right\rangle_{\mathcal{S}',\mathcal{S}}$$

Remark that $\overline{\mathrm{Op}\,(a)}$ *is a pseudodifferential operator with symbol*

$$\overline{\mathrm{Op}\,(a(x,\xi))} = \mathrm{Op}\left(\overline{a(x,-\xi)}\right) \tag{7.8}$$

indeed

$$\overline{\mathrm{Op}\,(a)}v = \overline{\mathrm{Op}\,(a)\,\overline{v}} = (2\pi)^{-n}\int e^{-\mathrm{i}(x-y)\xi}\,\overline{a(x,\xi)}\,\overline{v(y)}\,\mathrm{d}y\mathrm{d}\xi = (2\pi)^{-n}\int e^{\mathrm{i}(x-y)\xi}\,\overline{a(x,-\xi)}\,\overline{v(y)}\,\mathrm{d}y\mathrm{d}\xi$$

7.2 Composition

Theorem 7.7 (Composition)**.** *Let $a \in \mathcal{S}^m$, $b \in \mathcal{S}^{m'}$, then exists $c := a\#b \in \mathcal{S}^{m+m'}$ such that*

$$\mathrm{Op}\,(a) \circ \mathrm{Op}\,(b) = \mathrm{Op}\,(c)\,.$$

In particular

$$c(x,\xi) = \frac{1}{(2\pi)^n}\int e^{-\mathrm{i}y\eta}\, a(x,\xi+\eta)\, b(x+y,\xi)\,\mathrm{d}y\mathrm{d}\eta \tag{7.9}$$

and one has the asymptotic expansion

$$c(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!}\partial_\xi^\alpha a\, D_x^\alpha b \tag{7.10}$$

Finally for every $j \in \mathbb{N}_0$ there exist $C, N > 0$ such that

$$\wp_j^{m+m'}(a\#b) \leq C\, \wp_{j+N}^{m}(a)\, \wp_{j+N}^{m'}(b) \tag{7.11}$$

Remark 7.8. *We can write*

$$c(x,y) = e^{\mathrm{i}D_y\cdot D_\eta}(a(x,\eta)b(y,\xi))|_{\substack{y=x\\ \eta=\xi}} \tag{7.12}$$

where the equality is meant in sense of Fourier multipliers. See (6.15) *for a proof.*

Formula (7.12) *gives us a mnemonic way to compute the expansion of the symbol: indeed formally*

$$\begin{aligned} e^{\mathrm{i}D_y\cdot D_\eta}(a(x,\eta)b(y,\xi))|_{\substack{y=x\\ \eta=\xi}} &= \sum_k \frac{\mathrm{i}^k}{k!}(D_y\cdot D_\eta)^k(a(x,\eta)b(y,\xi))|_{\substack{y=x\\ \eta=\xi}} \\ &= \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi) \end{aligned}$$

Remark 7.9. *Note that* $\partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi) \in \mathcal{S}^{m-|\alpha|}$, *hence the expansion is in decreasing symbols. In particular symbolic calculus tells us that*

$$a\#b = ab + \partial_\xi a\, D_x b + \mathcal{S}^{m+m'-2} \tag{7.13}$$

Remark 7.10. *It holds that*

(i) $a\#1 = 1\#a = 1$

(ii) $(a\#b)\#c = a\#(b\#c)$

which follows immediately by the properties of the corresponding quantizations.

Proof of Theorem 7.7. We split the proof in several steps.

Step 1: $c(x,y)$ is a symbol.

Note that $c(x,\xi)$ is defined through an oscillatory integral. So first we verify that

$$(y,\eta) \mapsto c_{x,\xi}(y,\eta) := a(x,\xi+\eta)\, b(x+y,\xi)$$

is an amplitude in $A^m \equiv A_0^m$, namely $\forall \alpha,\beta \in \mathbb{N}^d$

$$\left|\partial_y^\alpha \partial_\eta^\beta c_{x,\xi}\right| \le C_{\alpha,\beta}\, \langle (y,\eta)\rangle^m$$

By the fact that a,b are symbols and Peetre's inequality we have

$$\begin{aligned}
\left|\partial_y^\alpha \partial_\eta^\beta c_{x,\xi}(y,\eta)\right| &\le \left|\partial_\eta^\beta a(x,\xi+\eta)\right| \left|\partial_y^\alpha b(y+x,\xi)\right| \\
&\lesssim \wp^m_{|\beta|}(a)\, \langle \xi+\eta\rangle^{m-|\beta|}\, \wp^{m'}_{|\alpha|}(b)\, \langle\xi\rangle^{m'} \\
&\lesssim \langle \xi+\eta\rangle^{m}\, \langle\xi\rangle^{m'}\, \wp^m_{|\beta|}(a)\, \wp^{m'}_{|\alpha|}(b) \\
&\overset{(7.1)}{\lesssim} \langle\xi\rangle^{m+m'}\, \langle\eta\rangle^{|m|}\, \wp^m_{|\beta|}(a)\, \wp^{m'}_{|\alpha|}(b) \\
&\lesssim \langle\xi\rangle^{m+m'}\, (1+|y|+|\eta|)^{|m|} \wp^m_{|\beta|}(a)\, \wp^{m'}_{|\alpha|}(b)
\end{aligned}$$

In particular $c_{x,\xi} \in A^{|m|}$ so the oscillatory integral is well defined. We can also be more precise: as

$$N^{|m|}_{|m|+2n+1}(c_{x,\xi}) \le \langle\xi\rangle^{m+m'}\, \wp^m_{|m|+2n+1}(a)\, \wp^{m'}_{|m|+2n+1}(b)$$

we get, with $N = |m| + 2n + 1$,

$$|c(x,\xi)| \le C N^{|m|}_{|m|+2n+1}(c_{x,\xi}) \le \langle\xi\rangle^{m+m'}\, \wp^m_N(a)\, \wp^{m'}_N(b)$$

To prove that $c(x,\xi)$ is a symbol we need to compute the derivatives $\partial_x^\alpha \partial_\xi^\beta c(x,\xi)$. We use that we can exchange derivative and oscillatory integral and obtain

$$\partial_x^\alpha \partial_\xi^\beta c(x,\xi) = \sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} \frac{C^{\beta',\beta''}_{\alpha',\alpha''}}{(2\pi)^n} \int e^{-\mathrm{i}y\eta}\, \underbrace{(\partial_x^{\alpha'} \partial_\xi^{\beta'} a(x,\xi+\eta))\, (\partial_x^{\alpha''} \partial_\xi^{\beta''} (b(x+y,\xi))}_{\widetilde{c}_{x,\xi}(y,\eta)}\, \mathrm{d}y\mathrm{d}\eta \tag{7.14}$$

We know that the oscillatory integral is bounded by the seminorms of the amplitude, so we compute them. Take $|\delta+\gamma| \le K$, then

$$
\begin{aligned}
\left|\partial_y^\gamma \partial_\eta^\delta \widetilde{c}_{x,\xi}(y,\eta)\right| &\lesssim \left|\partial_x^{\alpha'} \partial_\eta^{\beta'+\delta} a(x,\xi+\eta)\right| \left|\partial_y^{\alpha''+\gamma} \partial_\xi^{\beta''} b(y+x,\xi)\right| \\
&\lesssim \langle \xi+\eta\rangle^{m-|\beta'|-|\delta|}\ \wp^m_{K+|\alpha+\beta|}(a)\ \langle\xi\rangle^{m'-|\beta''|}\ \wp^{m'}_{K+|\alpha+\beta|}(b) \\
&\lesssim \langle \xi+\eta\rangle^{m-|\beta'|}\ \langle\xi\rangle^{m'-|\beta''|}\ \wp^m_{K+|\alpha+\beta|}(a)\, \wp^{m'}_{K+|\alpha+\beta|}(b) \\
&\overset{(7.1)}{\lesssim} C\ \langle\xi\rangle^{m+m'-|\beta|}\ \langle\eta\rangle^{|m-|\beta'||}\ \wp^m_{K+|\alpha+\beta|}(a)\, \wp^{m'}_{K+|\alpha+\beta|}(b) \\
&\lesssim C\ \langle\xi\rangle^{m+m'-|\beta|}\ (1+|y|+|\eta|)^{|m-|\beta'||}|\ \wp^m_{K+|\alpha+\beta|}(a)\, \wp^{m'}_{K+|\alpha+\beta|}(b)
\end{aligned}
$$

hence each term in the finite sum in (7.14) is an amplitude in $A^{|m-|\beta||} \subset A^{|m|+|\beta|}$. It follows by (6.26) that for $|\alpha+\beta| \le j$ we have

$$
\left|\partial_x^\alpha \partial_\xi^\beta c(x,\xi)\right| \le C\, N^{|m|+j}_{|m|+j+2n+1}(\widetilde{c}_{x,\xi}) \le \langle\xi\rangle^{m+m'-|\beta|}\ \wp^m_{2j+N}(a)\, \wp^{m'}_{2j+N}(b)
$$

where $N = |m|+2n+1$. This because we need to control $|\gamma+\delta| \le K \le |m|+|\beta|+2n+1$ derivatives in the seminorm, thus $K+|\alpha+\beta| \le K+j \le |m|+2j+2n+1$.

This proves that $c(x,\xi)$ is a symbol and the estimate on its seminorms.

Step 2: Asymptotic expansion.
To prove the asymptotic expansion, we use Taylor expansion [2]

$$
a(x,\xi+\eta)\, b(x+y,\xi) = \sum_{|\alpha+\beta|\le 2k-1} \frac{\eta^\alpha}{\alpha!}\, \frac{y^\beta}{\beta!}\, \left(\partial_\xi^\alpha a(x,\xi)\right)\, \left(\partial_x^\beta b(x,\xi)\right) + r_k(x,\xi,y,\eta)
$$

$$
r_k(x,\xi,y,\eta) = \sum_{|\alpha+\beta|=2k} (2k)\, \frac{\eta^\alpha}{\alpha!}\, \frac{y^\beta}{\beta!} r_{\alpha\beta}(x,\xi,y,\eta)
$$

$$
r_{\alpha\beta}(x,\xi,y,\eta) = \int_0^1 (1-t)^{2k-1} \partial_\eta^\alpha a(x,\xi+t\eta)\, \partial_y^\beta b(x+ty,\xi)\, \mathrm{d}t
$$

Insert the expansion in the integral and use that

$$
c(x,\xi) = \sum_{|\alpha+\beta|\le 2k-1} \partial_\xi^\alpha a(x,\xi)\, \partial_x^\beta b(x,\xi) \frac{1}{(2\pi)^n} \int e^{-\mathrm{i}y\eta} \frac{\eta^\alpha}{\alpha!}\, \frac{y^\beta}{\beta!}\, \mathrm{d}y\, \mathrm{d}\eta + R_k(x,\xi)
$$

where

$$
R_k(x,\xi) := (2\pi)^{-n} \int e^{-\mathrm{i}y\eta} r_k(x,\xi,y,\eta) \mathrm{d}y \mathrm{d}\eta. \tag{7.15}
$$

Now recall from (6.34) that

$$
\int_{\mathbb{R}^{2n}} e^{-\mathrm{i}y\eta}\, \frac{y^\alpha}{\alpha!}\, \frac{\eta^\beta}{\beta!}\, \mathrm{d}y\, \mathrm{d}\eta = \begin{cases} 0 & \alpha \ne \beta \\ (2\pi)^n \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!} & \alpha = \beta \end{cases}
$$

[2] Recall multivariable Taylor expansion at order $2k-1$

$$
f(\boldsymbol{x}) = \sum_{|\alpha|\le k} \frac{D^\alpha f(\boldsymbol{a})}{\alpha!} (\boldsymbol{x}-\boldsymbol{a})^\alpha + \sum_{|\beta|=k+1} R_\beta(\boldsymbol{x}) (\boldsymbol{x}-\boldsymbol{a})^\beta,
$$

$$
R_\beta(\boldsymbol{x}) = \frac{|\beta|}{\beta!} \int_0^1 (1-t)^{|\beta|-1} D^\beta f\big(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a})\big)\, dt.
$$

so we get that only the terms with $\alpha = \beta$ survive, and

$$c(x,\xi) = \sum_{|\alpha|\le k-1} \frac{1}{\alpha!} \underbrace{\partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi)}_{\in \mathcal{S}^{m+m'-|\alpha|}} + R_k(x,\xi) \tag{7.16}$$

We get the finite terms of the asymptotic expansion. So have a truly an asymptotic expansion we have to show that

$$c(x,\xi) - \sum_{|\alpha|\le k-1} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi) = R_k(x,\xi) \in \mathcal{S}^{m+m'-k}$$

We begin by proving

$$|R_k(x,\xi)| \lesssim \langle\xi\rangle^{m+m'-k}\,. \tag{7.17}$$

As above, the strategy is to show that R_k is well defined as an oscillatory integral with amplitude r_k, then use the bound (6.26) to prove (7.17). First we remark that

$$\int e^{-\mathrm{i}y\eta} r_k(x,\xi,y,\eta)\mathrm{d}y\mathrm{d}\eta = \underset{|\alpha+\beta|=2k}{\text{lin. com.}} \int e^{-\mathrm{i}y\eta}\, \eta^\alpha\, y^\beta\, r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\mathrm{d}\eta$$

Now we use integration by parts, first in η and then in y (the integration by parts is justified by Proposition 6.6), using that $y^\beta e^{-\mathrm{i}y\eta} = (-\mathrm{i})^{|\beta|}\beta!\,(\partial_\eta^\beta e^{-\mathrm{i}y\eta})$

$$\begin{aligned}
\underset{|\alpha+\beta|=2k}{\text{lin. com.}} \int e^{-\mathrm{i}y\eta}\, \eta^\alpha\, y^\beta\, r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\mathrm{d}\eta &= \underset{|\alpha+\beta|=2k}{\text{lin. com.}} \int \left(\partial_\eta^\beta e^{-\mathrm{i}y\eta}\right) \eta^\alpha\, r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\mathrm{d}\eta \\
&= \underset{\substack{|\alpha+\beta|=2k\\ \gamma\le\beta}}{\text{lin. com.}} \int e^{-\mathrm{i}y\eta}\, \eta^{\alpha-\gamma}\, \partial_\eta^{\beta-\gamma} r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\mathrm{d}\eta \\
&= \underset{\substack{|\alpha+\beta|=2k\\ \gamma\le\beta,\alpha}}{\text{lin. com.}} \int \left(\partial_y^{\alpha-\gamma} e^{-\mathrm{i}y\eta}\right) \partial_\eta^{\beta-\gamma} r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\mathrm{d}\eta \\
&= \underset{\substack{|\alpha+\beta|=2k\\ \gamma\le\beta,\alpha}}{\text{lin. com.}} \int e^{-\mathrm{i}y\eta}\, \partial_y^{\alpha-\gamma}\partial_\eta^{\beta-\gamma} r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\mathrm{d}\eta
\end{aligned}$$

Note that we have the conditions $\gamma \le \beta$, $\gamma \le \alpha$, otherwise one gets zero in the second passage. In particular $2|\gamma| \le |\alpha+\beta| = 2k$, hence $|\gamma| \le k$ and

$$|\alpha+\beta-\gamma| \ge k. \tag{7.18}$$

Now, using that

$$\partial_\eta^{\beta-\gamma}\partial_\xi^\alpha a(x,\xi+t\eta) = t^{|\beta-\gamma|}\partial_\xi^{\alpha+\beta-\gamma} a(x,\xi+t\eta),$$

we prove that R_k is linear combination of terms of the form

$$\int e^{-\mathrm{i}y\eta} \int_0^1 f(t) \left(\partial_\xi^{\alpha+\beta-\gamma} a(x,\xi+t\eta)\right) \left(\partial_x^{\alpha+\beta-\gamma} b(x+ty,\xi)\right) \mathrm{d}t\mathrm{d}y\mathrm{d}\eta$$

Now we show that

$$(y,\eta) \mapsto \left(\partial_\xi^{\alpha+\beta-\gamma} a(x,\xi+t\eta)\right) \left(\partial_x^{\alpha+\beta-\gamma} b(x+ty,\xi)\right) \equiv r_{x,\xi}^{\alpha,\beta,\gamma}(y\,\eta)$$

in an amplitude in $A^{m-k} \subseteq A^{|m-k|}$. In particular we verify that

$$N_{|m-k|+2n+1}^{|m-k|}\left(r_{x,\xi}^{\alpha,\beta,\gamma}\right) \lesssim \langle\xi\rangle^{m+m'-k} \tag{7.19}$$

Indeed for $|\delta+v| \le |m-k|+2n+1$ we have, recalling $|\alpha+\beta| \le 2k$,

$$\begin{aligned}
\left|\partial_y^\delta \partial_\eta^v \, r_{x,\xi}^{\alpha,\beta,\gamma}(y,\eta)\right| &\lesssim \int_0^1 f(t) \left|\partial_\xi^{\alpha+\beta-\gamma+v} a(x,\xi+t\eta)\right| \ \left|\partial_x^{\alpha+\beta-\gamma+\delta} b(x+ty,\xi)\right| \mathrm{d}t \\
&\lesssim \int_0^1 f(t) \, \langle \xi+t\eta\rangle^{m-|\alpha+\beta-\gamma|-|v|} \ \langle\xi\rangle^{m'} \ \wp_{2k+|v|}^m(a) \, \wp_{2k+|\delta|}^m(b) \, \mathrm{d}t \\
&\lesssim \int_0^1 f(t) \ \langle \xi+t\eta\rangle^{m-k} \ \langle\xi\rangle^{m'} \ \wp_{2k+|v|}^m(a) \, \wp_{2k+|\delta|}^m(b) \, \mathrm{d}t \\
&\lesssim \langle\xi\rangle^{m+m'-k} \ \wp_{2k+|v|}^m(a) \, \wp_{2k+|\delta|}^m(b) \int_0^1 f(t) \, \langle t\eta\rangle^{|m-k|} \ \mathrm{d}t \\
&\lesssim \langle\xi\rangle^{m+m'-k} \ \langle\eta\rangle^{|m-k|} \ \wp_{2k+|v|}^m(a) \, \wp_{2k+|\delta|}^m(b)
\end{aligned}$$

where in the last step we used $\langle t\eta\rangle \le \langle t\rangle \, \langle\eta\rangle$. Estimate (7.19) implies

$$|R_k(x,\xi)| \le \sum_{\substack{|\alpha+\beta|\le 2k \\ \gamma\le\beta,\alpha}} N^{|m-k|}_{|m-k|+2n+1}\left(r_{x,\xi}^{\alpha,\beta,\gamma}\right) \lesssim C \ \langle\xi\rangle^{m+m'-k} \ \wp^m_{|m|+2n+1+3k}(a) \, \wp^{m'}_{|m|+2n+1+3k}(b)$$

One argues similarly for the derivatives of R_k, substituting r_k by $\partial_x^\gamma \partial_\xi^\delta r_k$.

Step 3: Composition formula.
We prove that $\mathrm{Op}\,(a) \circ \mathrm{Op}\,(b) = \mathrm{Op}\,(a\sharp b)$ when $a \in \mathcal{S}^m$, $b \in \mathcal{S}^{m'}$. By Lemma 7.2, we approximate $a \in \mathcal{S}^m$ and $b \in \mathcal{S}^{m'}$ with two sequences of symbols $(a_j)_{j\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}} \subset \mathcal{S}$. Recall that $(a_j)_{j\in\mathbb{N}}$ is bounded in $\mathcal{S}^m$, a_j converges pointwise to a with all its derivatives, and $a_j \to a$ in $\mathcal{S}^{m+\epsilon}$ as $j\to\infty$ for any $\epsilon > 0$; the same holds for $(b_j)_{j\in\mathbb{N}}$.
Then $\mathrm{Op}\,(b_j)\,f \to \mathrm{Op}\,(b)\,f$ in $\mathcal{S}$ by Proposition 5.3. By the same proposition we have that

$$\mathrm{Op}\,(a_j)\,\mathrm{Op}\,(b_j)\,f \to \mathrm{Op}\,(a)\,\mathrm{Op}\,(b)\,f \quad \text{in } \mathcal{S} \quad \text{as } j\to\infty.$$

Now $a_j,\ b_j \in \mathcal{S}$, so the formal computation of section 5.2, we deduce that $\mathrm{Op}\,(a_j)\,\mathrm{Op}\,(b_j)\,f = \mathrm{Op}\,(a_j\sharp b_j)\,f$. But now, by the proof before, we know that $a\sharp b$ is bilinear and continuous, so $a_j\sharp b_j \to a\sharp b \in \mathcal{S}^{m+m'+2\epsilon}$. Hence $\mathrm{Op}\,(a_j\sharp b_j)\,f \to \mathrm{Op}\,(a\sharp b)\,f$ in $\mathcal{S}$. By the uniqueness of the limit we deduce that $\mathrm{Op}\,(a)\circ\mathrm{Op}\,(b) = \mathrm{Op}\,(a\sharp b)$. □

Example: let us consider in dimension $d=1$ the operator

$$A = b(x)\partial_{xx}.$$

Then its adjoint is

$$A^* := \partial_{xx}\circ\bar{b}(x);$$

note that, as operator, this means

$$A^*u = \partial_{xx}(\bar{b}(x)u(x)) = \bar{b}(x)\partial_{xx}u + 2\bar{b}_x(x)\partial_x u + \bar{b}_{xx}u$$

i.e. we have $A^* = \bar{b}(x)\partial_{xx} + 2\bar{b}_x(x)\partial_x + \bar{b}_{xx}$. Let us compute A^* by symbolic calculus. We have $A = \mathrm{Op}\left(-b\xi^2\right)$, hence $A^* = \mathrm{Op}\,(a^*)$ where

$$a^* = [e^{\mathrm{i}D_x\cdot D_\xi}\bar{a}](x,\xi) = \sum_k \frac{1}{k!}\partial_\xi^k D_x^k \bar{a}(x,\xi) = \sum_k \frac{1}{k!}\partial_\xi^k D_x^k(-\bar{b}\xi^2) = -\bar{b}\xi^2 - \frac{1}{\mathrm{i}}\bar{b}_x 2\xi - \frac{1}{2}\frac{\bar{b}_{xx}}{\mathrm{i}^2}2$$

so we see, using also $\mathrm{Op}\,(\xi) = D_x = \frac{1}{\mathrm{i}}\partial_x$, that $\mathrm{Op}\,(a^*) = b\partial_{xx} + 2\overline{b}_x\partial_x + \overline{b}_{xx}$.

We can also compute A^* by composition theorem. Indeed

$$\partial_{xx} \circ \overline{b} = \mathrm{Op}\left(-\xi^2\right) \circ \mathrm{Op}\left(\overline{b}\right) = -\mathrm{Op}\left(\xi^2 \sharp \overline{b}\right)$$

and by symbolic calculus

$$\xi^2 \sharp \overline{b} = \sum_k \frac{1}{k!}\partial_\xi^k(\xi^2)(D_x^k \overline{b}) = \xi^2\overline{b} + 2\xi\frac{1}{\mathrm{i}}\overline{b}_x + \frac{1}{2}2\frac{1}{\mathrm{i}^2}\overline{b}_{xx}$$

hence again we find

$$-\mathrm{Op}\left(\xi^2 \sharp \overline{b}\right) = \overline{b}\partial_{xx} + 2\overline{b}_x\partial_x + \overline{b}_{xx}$$

which, not surprinsingly, coincides with the expression that we already found.