8 Applications of symbolic calculus and different quantizations

In this section we describe some applications of symbolic calculus and we also describe different quantizations.

8.1 Commutators

An immediate application of the results obtained so far is the following one, regarding commutators.

Theorem 8.1. *Let $a \in \mathcal{S}^m$, $b \in \mathcal{S}^{m'}$, then exists $c \in \mathcal{S}^{m+m'-1}$ such that*

$$[\mathrm{Op}\,(a)\,,\ \mathrm{Op}\,(b)] = \mathrm{Op}\,(c)\ .$$

The symbol c is given by

$$c := a\#b - b\#a \sim \sum_\alpha \frac{1}{\alpha!}(\partial_\xi^\alpha a\, D_x^\alpha b - \partial_\xi^\alpha b\, D_x^\alpha a) \tag{8.1}$$

$$= \frac{1}{\mathrm{i}}\{a,b\} + S^{m+m'-2} \tag{8.2}$$

and one has the quantitative estimate: $\forall j \in \mathbb{N}$, there exist $C, N > 0$ such that

$$\wp_j^{m+m'-1}(c) \le C\, \wp_N^m(a)\, \wp_N^{m'}(b)\ . \tag{8.3}$$

Proof. It is sufficient to note that, by composition of pseudodifferential operators

$$[\mathrm{Op}\,(a)\,,\ \mathrm{Op}\,(b)] = \mathrm{Op}\,(a)\ \mathrm{Op}\,(b) - \mathrm{Op}\,(b)\ \mathrm{Op}\,(a) = \mathrm{Op}\,(a\#b - b\#a)\ .$$

Furthermore (7.13) gives

$$\begin{aligned} a\#b - b\#a &= ab + \partial_\xi a D_x b + \mathcal{S}^{m+m'-2} - ba - \partial_\xi b D_x a + \mathcal{S}^{m+m'-2} \\ &= \frac{1}{\mathrm{i}}\{a,b\} + \mathcal{S}^{m+m'-2} \end{aligned}$$

□

We define the *Moyal bracket* by

$$\begin{aligned} \{a,b\}_{\mathcal{M}} &:= \mathrm{i}(a\#b - b\#a) \\ &= \{a,b\} + S^{m+m'-2} \end{aligned} \tag{8.4}$$

Then

$$\mathrm{i}\,[\mathrm{Op}\,(a)\,, \mathrm{Op}\,(b)] = \mathrm{Op}\,(\{a,b\}_{\mathcal{M}}) \tag{8.5}$$

$$= \mathrm{Op}\,(\{a,b\}) + \mathrm{Op}\left(\mathcal{S}^{m+m'-2}\right) \tag{8.6}$$

8.2 Symbol recovery

Given a pseudodifferential operator $\mathrm{Op}\,(a)$ with symbol $a \in \mathcal{S}$, we can recover its symbol by the formula

$$a(x,\xi) := e^{-\mathrm{i}x\xi}\mathrm{Op}\,(a)\,[e^{\mathrm{i}x\xi}]. \tag{8.7}$$

Remark that such a formula is well defined (in the distributional sense), because we have proved that $\mathrm{Op}\,(a) : \mathcal{S}' \to \mathcal{S}'$, and $e^{\mathrm{i}x\xi} \in \mathcal{S}'$. To prove (8.7), use the calculus of oscillatory integrals to compute

$$\begin{aligned}
\mathrm{Op}\,(a)\,[e^{\mathrm{i}x\xi}](x) &= (2\pi)^{-n}\int e^{\mathrm{i}(x-y)\eta}\,a(x,\eta)\,e^{\mathrm{i}y\xi}\mathrm{d}\eta\mathrm{d}y \\
&= \lim_{\epsilon\to 0}(2\pi)^{-n}\int e^{\mathrm{i}(x-y)\eta}\,a(x,\eta)\,e^{\mathrm{i}y\xi}\,\chi(\epsilon y)\,\chi(\epsilon\eta)\,\mathrm{d}\eta\mathrm{d}y \\
&= \lim_{\epsilon\to 0}(2\pi)^{-n}\int e^{\mathrm{i}x\eta}\,a(x,\eta)\,\chi(\epsilon\eta)\,\left(\int e^{\mathrm{i}y(\xi-\eta)}\,\chi(\epsilon y)\mathrm{d}y\right)\mathrm{d}\eta \\
&= \lim_{\epsilon\to 0}(2\pi)^{-n}\int e^{\mathrm{i}x\eta}\,a(x,\eta)\,\chi(\epsilon\eta)\,\frac{1}{\epsilon^n}\widehat{\chi}\left(\frac{\eta-\xi}{\epsilon}\right)\mathrm{d}\eta \\
&\overset{\eta=\epsilon\zeta+\xi}{=} \lim_{\epsilon\to 0}(2\pi)^{-n}\int e^{\mathrm{i}x(\xi+\epsilon\zeta)}\,a(x,\epsilon\zeta+\xi)\chi(\epsilon^2\zeta+\epsilon\xi)\,\widehat{\chi}(\zeta)\mathrm{d}\zeta \\
&= e^{\mathrm{i}x\xi}a(x,\xi)\chi(0)\,(2\pi)^{-n}\int\widehat{\chi}(\zeta)\mathrm{d}\zeta \\
&= e^{\mathrm{i}x\xi}\,a(x,\xi)
\end{aligned}$$

Exercise 8.2. *Prove that if $(a_j)_{j\geq 1} \subset \mathcal{S}$ approximates $a \in \mathcal{S}^m$ in the usual sense, then for any $u \in \mathcal{S}'$, $\mathrm{Op}\,(a_j)\,u \to \mathrm{Op}\,(a)\,u$ in $\mathcal{S}'$. Deduce equality* (8.7).

Consider now an operator $A\colon \mathcal{S}' \to \mathcal{S}'$, and put

$$\sigma_A(x,\xi) := e^{-\mathrm{i}x\xi}A[e^{\mathrm{i}x\xi}]; \tag{8.8}$$

then we can recover A from its symbol σ_A. Indeed for $u, v \in \mathcal{S}$ write $(\mathcal{F}^{-1}v)(\xi) = (2\pi)^{-n}\langle e_\xi, v\rangle_{\mathcal{S}',\mathcal{S}}$, where $e_\xi(x) := e^{\mathrm{i}x\xi}$, and recalling the definition of transposed operator, write

$$\begin{aligned}
\langle Au, v\rangle_{\mathcal{S}',\mathcal{S}} &= \langle u, A^t v\rangle_{\mathcal{S}',\mathcal{S}} = \langle \widehat{u}, \mathcal{F}^{-1}(A^t v)\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \left\langle \widehat{u}, (2\pi)^{-n}\langle e_\xi, A^t v\rangle_{\mathcal{S}',\mathcal{S}}\right\rangle_{\mathcal{S}',\mathcal{S}} = \left\langle \widehat{u}, (2\pi)^{-n}\langle A e_\xi, v\rangle_{\mathcal{S}',\mathcal{S}}\right\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \left\langle \widehat{u}, (2\pi)^{-n}\langle e_\xi\sigma_A(\cdot,\xi), v\rangle_{\mathcal{S}',\mathcal{S}}\right\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{(2\pi)^n}\int \widehat{u}(\xi)e^{\mathrm{i}x\xi}\sigma_A(x,\xi)v(x)\mathrm{d}\xi\mathrm{d}x \\
&= \left\langle \frac{1}{(2\pi)^n}\int \widehat{u}(\xi)e^{\mathrm{i}x\xi}\sigma_A(x,\xi)\mathrm{d}\xi, v\right\rangle_{\mathcal{S}',\mathcal{S}}
\end{aligned}$$

so in particular, if σ_A is a symbol, we get that $A = \mathrm{Op}\,(a)$, hence A is a pseudodiff.
Thus, given an operator A, if we want to prove that it is a pseudodifferential operator, it is sufficient to compute (8.8) and then check that it is a symbol in some appropriate class.

8.3 Parametrix

We shall now construct an approximate inverse to an elliptic operator.

Theorem 8.3. *Let $a \in \mathcal{S}^m$, $m > 0$ be elliptic, namely $\exists c > 0$ such that*

$$|a(x,\xi)| \geq c \langle \xi \rangle^m \ , \qquad \forall |\xi| \geq \rho. \tag{8.9}$$

Then there exists $b \in \mathcal{S}^{-m}$ such that

$$\mathrm{Op}\,(a)\,\mathrm{Op}\,(b) = \mathbb{1} + R, \qquad \mathrm{Op}\,(b)\,\mathrm{Op}\,(a) = \mathbb{1} + R', \qquad R, R' \in \mathrm{Op}\left(\mathcal{S}^{-\infty}\right) \tag{8.10}$$

Proof. By symbolic calculus, we will construct $b, b' \in \mathcal{S}^{-m}$ such that $a\#b - 1$, $b'\#a - 1 \in \mathcal{S}^{-\infty}$. Quantizing them, we obtain two operators $\mathrm{Op}\,(b)$, $\mathrm{Op}\,(b')$ such that

$$\mathrm{Op}\,(a)\,\mathrm{Op}\,(b) = \mathbb{1} + R, \qquad \mathrm{Op}\,(b')\,\mathrm{Op}\,(a) = \mathbb{1} + R', \qquad R, R' \in \mathrm{Op}\left(\mathcal{S}^{-\infty}\right),$$

so they are right and left approximate inverses. Before diving into the proof, notice that

$$\mathrm{Op}\,(b) - \mathrm{Op}\,(b') = \mathrm{Op}\,(b)\underbrace{\Big(\mathbb{1} - \mathrm{Op}\,(a)\,\mathrm{Op}\,(b')\Big)}_{\in \mathrm{Op}(\mathcal{S}^{-\infty})} + \underbrace{\Big(\mathrm{Op}\,(b)\,\mathrm{Op}\,(a) - \mathbb{1}\Big)}_{\in \mathrm{Op}(\mathcal{S}^{-\infty})}\mathrm{Op}\,(b') \in \mathrm{Op}\left(\mathcal{S}^{-\infty}\right),$$

which implies that

$$\mathrm{Op}\,(b)\,\mathrm{Op}\,(a) = \underbrace{\Big(\mathrm{Op}\,(b) - \mathrm{Op}\,(b')\Big)}_{\in \mathrm{Op}(\mathcal{S}^{-\infty})}\mathrm{Op}\,(a) + \underbrace{\mathrm{Op}\,(b')\,\mathrm{Op}\,(a)}_{\mathbb{1} + \mathrm{Op}(\mathcal{S}^{-\infty})} = \mathbb{1} + \mathrm{Op}\left(\mathcal{S}^{-\infty}\right),$$

i.e., a left approximate inverse is also a right approximate inverse, and viceversa.

We show now how to construct b such that $a\#b - 1 \in \mathcal{S}^{-\infty}$, the construction of b' is analogous. We proceed iteratively. Let $\chi \in C_0^\infty$, $\chi \equiv 1$ in $|\xi| \leq \rho$. Then define

$$b_0(x,\xi) := \frac{1 - \chi(\xi)}{a(x,\xi)} \in \mathcal{S}^{-m}$$

Now by symbolic calculus (see (7.13))

$$a\#b_0 = a\,b_0 + \mathcal{S}^{-1} = 1 - \chi + \mathcal{S}^{-1} = 1 - r_{-1} \ , \qquad r_{-1} \in \mathcal{S}^{-1}.$$

Now we iterate. We look for $b = b_0 + b_1 + l.o.t$ in such a way that $a\sharp b$ becomes a symbol in a class of lower and lower order. Note how the construction is perturbative in nature, where the perturbative parameter is the order of the symbol. So now let us look for $b_1 \in \mathcal{S}^{-m-1}$ in such a way that $a\sharp(b_0 + b_1)$ is in $\mathcal{S}^{-2}$. We have

$$\begin{aligned} a\#(b_0 + b_1) &= a\#b_0 + a\#b_1 = 1 - r_{-1} + a\#b_1 \\ &= 1 - r_{-1} + ab_1 + r_{-2} = 1 - r_{-2} \end{aligned}$$

provided we choose, for example,

$$b_1 := r_{-1}\frac{1 - \chi(\xi)}{a(x,\xi)} = r_{-1}b_0 \in \mathcal{S}^{-m-1}.$$

We can iterate this process at any order: assume to have found $b_0, \ldots, b_K$, with $b_j \in \mathcal{S}^{-m-j}$, $\forall j$, such that

$$a\sharp(b_0 + \ldots + b_K) = 1 - r_{-K-1}, \qquad r_{-K-1} \in \mathcal{S}^{-K-1},$$

then, defining

$$b_{K+1} := r_{-K-1}\frac{1-\chi(\xi)}{a(x,\xi)} \in \mathcal{S}^{-m-K-1}$$

we have

$$\begin{aligned} a\#(b_0+\ldots+b_K+b_{K+1}) &= a\#(b_0+\ldots+b_K)+a\#b_{K+1} = 1-r_{-K-1}+a\#b_{K+1} \\ &= 1-r_{-K+1}+ab_{K+1}+r_{-K-2} = 1-r_{-K-2}. \end{aligned}$$

The construction can be performed at any order, then we define $b \sim \sum_j b_j$ as asymptotic series. Lemma 4.11 produces a symbol $b \in \mathcal{S}^{-m}$ asymptotic to $\sum_j b_j$, which fulfills $a\#b - 1 \in \mathcal{S}^{-M}$ $\forall M \in \mathbb{R}$. □

The theorem allows us to construct approximative inverses to general elliptic variable coefficients operators.

Remark 8.4. *The existence of a right approximate inverse of* $\mathrm{Op}\,(a)$*(or equivalently of a left one), implies that the symbol* a *is elliptic. Indeed, by symbolic calculus,* $\mathrm{Op}\,(a)\,\mathrm{Op}\,(b) - 1 \in \mathcal{S}^{-\infty}$ *implies* $a\sharp b - 1 \in \mathcal{S}^{-\infty}$*, hence*

$$a(x,\xi)b(x,\xi) = 1 + r_{-1}(x,\xi), \qquad r_{-1} \in \mathcal{S}^{-1}.$$

This implies that for ξ *large enough*

$$\frac{1}{2} \le |a(x,\xi)|\ |b(x,\xi)| \le C\ |a(x,\xi)|\,\langle\xi\rangle^{-m}, \qquad \forall|\xi| \gg 1.$$

This shows that a *is elliptic. So* a *elliptic is equivalent to* $\mathrm{Op}\,(a)$ *having a right (or left) approximate inverse.*

Remark 8.5. *If* $\mathrm{Op}\,(a)$ *has a left approximate inverse, it* does not mean *that* $\mathrm{Op}\,(a)$ *is injective, but that its kernel is composed by smooth functions. Indeed assume that* a *is an elliptic symbol and that* $u \in \ker \mathrm{Op}\,(a)$ *with only* $u \in L^2$*. We are going to show that actually* $u \in H^s\ \forall s$*. Indeed let* $\mathrm{Op}\,(b)$ *be a parametrix, and since* $\mathbb{1} = \mathrm{Op}\,(b)\,\mathrm{Op}\,(a) - R$ *with* R *smoothing, we have*

$$u = \mathrm{Op}\,(b)\,\mathrm{Op}\,(a)\,u - Ru = -Ru.$$

Since R *is smoothing, it follows that* $u \in H^s\ \forall s$*. In particular it is smooth.*
For example $\frac{\mathrm{d}}{\mathrm{d}x}$ *is elliptic, so it admits an approximate left inverse. Its kernel is composed by constant functions, which are smooth.*

8.4 Different quantization

So far we have worked with the so called "standard" quantization. However it is possible to define a whole family of quantizations in the following way: for any $0 \le t \le 1$ define

$$[\mathrm{Op}_t(a)u](x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}\,a\,(tx+(1-t)y,\xi)\ u(y)\,\mathrm{d}y\mathrm{d}\xi \tag{8.11}$$

The values $t = 0, \frac{1}{2}, 1$ play a special role and they are called

- $t = 0$: right quantization, $\mathrm{Op}_0(a(x)\xi) = \frac{1}{\mathrm{i}}\partial_x\,a$
- $t = \frac{1}{2}$: Weyl quantization, $\mathrm{Op}_{\frac{1}{2}}(a(x)\xi) = \frac{1}{2\mathrm{i}}\,(a\partial_x + \partial_x a)$

- $t=1$: left quantization (standard), $\mathrm{Op}_1(a(x)\xi) = a(x)\frac{1}{\mathrm{i}}\partial_x$

Note that the symbol is the same, but the operators are different. However, the operators are the same at highest order, and the difference is at lower order.

It is also possible to pass from one quantization to a different one:

Theorem 8.6. *Fix $t \in [0,1]$, and let $a \in \mathcal{S}^m$. Then $\forall s \in [0,1]$, there exists $b \in \mathcal{S}^m$ such that*

$$\mathrm{Op}_t(a) = \mathrm{Op}_s(b). \tag{8.12}$$

In particular

$$b(x,\xi) = \frac{1}{(2\pi)^n}\int e^{-\mathrm{i}y\eta}\ a(x+(s-t)y, \xi+\eta)\,\mathrm{d}y\mathrm{d}\eta, \tag{8.13}$$

namely one has the asymptotic expansion

$$b = e^{\mathrm{i}(s-t)D_x\cdot D_\xi}a \sim \sum_\alpha \frac{(s-t)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha D_x^\alpha a.$$

Proof. Assume first that $a \in \mathcal{S}$. The case $a \in \mathcal{S}^m$ is obtained as usual by approximation with Schwarts symbols. We want the integral kernels of $\mathrm{Op}_t(a)$ and $\mathrm{Op}_s(b)$ to be equal. Hence

$$K_a(x,y) = \int e^{\mathrm{i}(x-y)\xi}a(tx+(1-t)y,\xi)\,\mathrm{d}\xi = \int e^{\mathrm{i}(x-y)\eta}b(sx+(1-s)y,\eta)\,\mathrm{d}\eta = K_b(x,y)$$

We change variables in the second integral by putting

$$\begin{cases} x-y=\theta \\ z = sx+(1-s)y = y+s\theta \end{cases} \Rightarrow \begin{cases} x = z+(1-s)\theta \\ y = z-s\theta \end{cases}$$

and get

$$\int e^{\mathrm{i}\theta\xi}a(z+(t-s)\theta,\xi)\,\mathrm{d}\xi = \int e^{\mathrm{i}\theta\eta}b(z,\eta)\,\mathrm{d}\eta \equiv (2\pi)^n\ \ \mathcal{F}_\eta^{-1}(b(z,\cdot)\big|_\theta\,,$$

hence we can invert the Fourier transform and obtain that

$$\begin{aligned} b(z,\eta) &= \frac{1}{(2\pi)^n}\ \mathcal{F}_\theta\left(\int e^{\mathrm{i}\theta\xi}a(z+(t-s)\theta,\xi)\,\mathrm{d}\xi\right)\Big|_\eta \\ &= \frac{1}{(2\pi)^n}\int e^{\mathrm{i}\theta\xi}\,e^{-\mathrm{i}\theta\eta}\,a(z+(t-s)\theta,\xi)\,\mathrm{d}\xi\mathrm{d}\theta \\ &\overset{\substack{\xi=\eta+\zeta \\ \theta=-y}}{=} \frac{1}{(2\pi)^n}\int e^{-\mathrm{i}y\zeta}\,a(z+(s-t)y,\zeta+\eta)\,\mathrm{d}y\mathrm{d}\zeta \end{aligned}$$

as claimed. The proof that $b \in \mathcal{S}^m$ and the asymptotic expansion follows as in the proof of the adjoint theorem. □

Remark 8.7. *In particular one has $\mathrm{Op}\,(a) = \mathrm{Op}^w\,(b)$ with*

$$b \sim \sum_\alpha \left(-\frac{1}{2}\right)^{|\alpha|}\frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha a\ .$$

For example the formula tells us that if $a(x,\xi) = b(x)\xi$ (dimension 1), then

$$b(x)\frac{1}{\mathrm{i}}\partial_x = \mathrm{Op}\,(a) = \mathrm{Op}^w\left(b(x)\xi - \frac{1}{2\mathrm{i}}b_x\right) \tag{8.14}$$

Similarly one has $\mathrm{Op}^w(b) = \mathrm{Op}(a)$ *with*

$$a \sim \sum_\alpha \left(\frac{1}{2}\right)^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha b \ .$$

For example if $b(x,\xi) = \mathtt{b}(x)\xi$ *(dimension 1), then*

$$\mathrm{Op}^w(b) = \mathrm{Op}\left(\mathtt{b}(x)\xi + \frac{1}{2\mathrm{i}}\mathtt{b}_x(x)\right) = \mathtt{b}(x)\frac{1}{\mathrm{i}}\partial_x + \frac{1}{2\mathrm{i}}\mathtt{b}_x(x). \tag{8.15}$$

8.5 Weyl quantization

A particular important quantization is the Weyl quantization, which we recall is given explicitly by the formula

$$[\mathrm{Op}^w(a)\, u](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{\mathrm{i}(x-y)\xi}\, a\left(\frac{x+y}{2}, \xi\right) u(y)\, \mathrm{d}y \mathrm{d}\xi \tag{8.16}$$

Let us check how some common symbols are transformed, in dimension $n = 1$:

- $a(x,\xi) = a(\xi)$, then $\mathrm{Op}^w(a) = a(D)$.
- $a(x,\xi) = a(x)$, then $\mathrm{Op}^w(a) = a(x)$. This can be obtained by oscillatory integrals techniques.
- $a(x,\xi) = \mathtt{a}(x)\xi$, then $\mathrm{Op}^w(a) = \frac{1}{2}(\mathtt{a}D_x + D_x\mathtt{a})$. Indeed, for $u, a \in \mathcal{S}$, then

$$\begin{aligned}
\mathrm{Op}^w(a)[u](x) &= (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\eta}\, \mathtt{a}\left(\frac{x+y}{2}\right) \xi\, u(y)\, \mathrm{d}y \mathrm{d}\xi \\
&= \frac{1}{\mathrm{i}}(2\pi)^{-n} \int e^{\mathrm{i}(x-y)\eta} \left(\frac{1}{2}\mathtt{a}'\left(\frac{x+y}{2}\right) u(y) + \mathtt{a}\left(\frac{x+y}{2}\right) u'(y)\right) \mathrm{d}y \mathrm{d}\xi \\
&= \frac{1}{2\mathrm{i}}\mathtt{a}_x(x) u(x) + \frac{1}{\mathrm{i}}\mathtt{a}(x) u_x(x) \\
&= \left[\frac{1}{\mathrm{i}}\mathtt{a}\partial_x + \frac{1}{2\mathrm{i}}\mathtt{a}_x\right] u = \frac{1}{2\mathrm{i}}\left(\mathtt{a}\partial_x + \partial_x\mathtt{a}\right) u
\end{aligned}$$

  If a is just bounded with bounded derivatives, or a symbol in a different class, one justifies the computation above via oscillatory integrals.

Exercise 8.8. *Compute* $\mathrm{Op}^w(\mathtt{a}(x)\xi)$ *in dimension* n *arbitrary.*

There is an important remark about the last computation: if $\mathtt{a}(x)$ is real, then $\mathrm{Op}^w(a)$ is a selfadjoint operator:

$$\left(\frac{1}{2\mathrm{i}}(\mathtt{a}\partial_x + \partial_x\mathtt{a})\right)^* = \frac{-1}{2\mathrm{i}}\left(-\partial_x\mathtt{a} - \mathtt{a}\partial_x\right) = \frac{1}{2\mathrm{i}}(\mathtt{a}\partial_x + \partial_x\mathtt{a}).$$

This is a general fact, and the main reason why Weyl quantization is important:

Theorem 8.9 (Adjoint with Weyl)**.** *Let* $a \in \mathcal{S}^m$. *Then*

$$\mathrm{Op}^w(a)^* = \mathrm{Op}^w(\overline{a})\,.$$

In particular if a *is real valued, then* $\mathrm{Op}^w(a)$ *is symmetric on its domain.*

Proof. It is sufficient to recall that if $K(x,y)$ is the kernel of A, then $K^*(x,y) = \overline{K(y,x)}$ is the kernel of A^*. Since $\mathrm{Op}^w(a)$ has kernel

$$K^w(x,y) = (2\pi)^{-n} \int e^{\mathrm{i}(x-y)\xi}\, a\left(\frac{x+y}{2}, \xi\right) \mathrm{d}\xi, \tag{8.17}$$

by a direct computation we get

$$K^*(x,y) = \overline{K^w(y,x)}(2\pi)^{-n} \int e^{\mathrm{i}(x-y)\xi}\, \overline{a\left(\frac{x+y}{2}, \xi\right)} \mathrm{d}\xi,$$

but this is the integral kernel of $\mathrm{Op}^w(\overline{a})$ as claimed. □

Remark 8.10. *From* (8.17) *it follows that*

$$K^w\left(x + \frac{t}{2}, x - \frac{t}{2}\right) = (2\pi)^{-n} \int e^{\mathrm{i}t\xi}\, a(x,\xi)\, \mathrm{d}\xi = (\mathcal{F}_{\xi\to t} a)(x,t) \tag{8.18}$$

By taking inverse Fourier transform we get

$$a(x,\xi) = \int e^{-\mathrm{i}t\xi}\, K^w\left(x + \frac{t}{2}, x - \frac{t}{2}\right) \mathrm{d}t \tag{8.19}$$

and we recover the symbol from the kernel.

We state also the theorem about composition in Weyl quantization

Theorem 8.11 (Composition with Weyl)**.** *Let* $a \in \mathcal{S}^m$, $b \in \mathcal{S}^{m'}$, *then*

$$\mathrm{Op}^w(a) \circ \mathrm{Op}^w(b) = \mathrm{Op}^w(c)$$

with $c := a \#^w b \in \mathcal{S}^{m+m'}$. *Moreover, defining the symplectic form*

$$\sigma(x,\xi;y,\eta) := \xi \cdot y - x \cdot \eta,$$

we have that

$$c(x,\xi) = \frac{1}{\pi^{2n}} \int e^{2\mathrm{i}\sigma(t,\tau;z,\zeta)}\, a(x+z, \xi+\zeta)\, b(x+t, \xi+\tau)\, \mathrm{d}t\mathrm{d}z\mathrm{d}\zeta\mathrm{d}\tau \tag{8.20}$$

and one has the asymptotic expansion

$$c(x,\xi) \sim \sum_{j\geq 0} c_j, \qquad c_j(x,\xi) := \sum_{|\alpha+\beta|=j} \frac{1}{\alpha!\,\beta!} \left(\frac{1}{2}\right)^{|\alpha|} \left(-\frac{1}{2}\right)^{|\beta|} \left(\partial_\xi^\alpha D_x^\beta a\right) \left(\partial_\xi^\beta D_x^\alpha b\right) \tag{8.21}$$

Finally for every $j \in \mathbb{N}_0$ *there exist* $C, N > 0$ *such that*

$$\wp_j^{m+m'}(a \#^w b) \leq C\, \wp_{j+N}^m(a)\, \wp_{j+N}^{m'}(b) \tag{8.22}$$

Remark 8.12. *We can write*

$$(a\#^w b)(x,\xi) = e^{\frac{\mathrm{i}}{2}(D_\xi D_y - D_x D_\eta)} a(x,\xi) b(y,\eta)|_{\substack{y=x\\ \eta=\xi}}, \tag{8.23}$$

compare with (6.19).

Remark 8.13. *For Weyl quantization one has*

$$a\#^w b = ab + \frac{1}{2}\left(\partial_\xi a\, D_x b - D_x a\, \partial_\xi b\right) + \mathcal{S}^{m+m'-2} = ab + \frac{1}{2\mathrm{i}}\{a,b\} + \mathcal{S}^{m+m'-2}. \tag{8.24}$$

Moreover all the terms of odd degree are antisymmetric in a, b, while all the even ones are symmetric. Hence we have

$$a\#^w b - b\#^w a = \frac{1}{\mathrm{i}}\{a,b\} + \mathcal{S}^{m+m'-3} \tag{8.25}$$

which implies

$$\left[\mathrm{Op}^w(a), \mathrm{Op}^w(b)\right] = \mathrm{Op}^w\left(\frac{1}{\mathrm{i}}\{a,b\}\right) + \mathrm{Op}^w\left(\mathcal{S}^{m+m'-3}\right) \tag{8.26}$$

Proof of Theorem 8.11. Let K_a and K_b be the intergal kernels of $\mathrm{Op}^w(a)$ and $\mathrm{Op}^w(b)$ respectively. Then $\mathrm{Op}^w(a)\,\mathrm{Op}^w(b)$ has integral kernel

$$\begin{aligned} K_c(x,y) &= \int_{\mathbb{R}^n} K_a(x,z)\; K_b(z,y)\mathrm{d}z \\ &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{3n}} e^{\mathrm{i}(x-z)\zeta}\; a\left(\frac{x+z}{2}, \zeta\right)\; e^{\mathrm{i}(z-y)\tau}\; b\left(\frac{z+y}{2}, \tau\right)\; \mathrm{d}\tau\mathrm{d}\zeta\mathrm{d}z \end{aligned}$$

Using formula (8.19) we get that

$$\begin{aligned} c(x,\xi) &= \int_{\mathbb{R}^n} e^{-\mathrm{i}t\xi} K_c\left(x+\frac{t}{2}, x-\frac{t}{2}\right)\mathrm{d}t \\ &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{4n}} e^{\mathrm{i}[(x-z+t/2)(\zeta-\xi)+(z-x+t/2)(\tau-\xi)]}\; a\left(\frac{x+z+t/2}{2}\zeta\right)\; b\left(\frac{z+x-t/2}{2}, \tau\right)\; \mathrm{d}z\mathrm{d}\zeta\mathrm{d}\tau\mathrm{d}t \end{aligned}$$

Now make the linear change of variables

$$\begin{cases} t = 2(z'-t') \\ z = x + z' + t' \\ \zeta = \zeta' + \xi \\ \tau = \tau' + \xi \end{cases} \qquad \mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\zeta\,\mathrm{d}\tau = 4^n\,\mathrm{d}t'\,\mathrm{d}z'\,\mathrm{d}\zeta'\,\mathrm{d}\tau'$$

and one gets (8.20). For the asymptotic formula, one proceeds as for the proof of composition in the standard quantization: by Taylor expansion

$$a(x+z,\xi+\zeta)\,b(x+t,\xi+\tau) \sim \sum_{\alpha,\beta,\gamma,\delta} (\partial_x^\alpha\partial_\xi^\beta a(x,\xi))\,(\partial_x^\gamma\partial_\xi^\delta b(x,\xi)\,\frac{z^\alpha}{\alpha!}\,\frac{\zeta^\beta}{\beta!}\,\frac{t^\gamma}{\gamma!}\,\frac{\tau^\delta}{\delta!}$$

Then one has to study the oscillatory integral

$$\int e^{2\mathrm{i}(\tau\cdot z - t\cdot\zeta)}\,\frac{z^\alpha}{\alpha!}\,\frac{\zeta^\beta}{\beta!}\,\frac{t^\gamma}{\gamma!}\,\frac{\tau^\delta}{\delta!}\,\mathrm{d}t\,\mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}\zeta$$

and by using integration by parts in the oscillatory integrals, one verifies that one must have $\alpha = \delta$ and $\gamma = \beta$. In such a way one obtains the asymptotic expansion. The details are let to the reader. $\square$