4 Towards pseudodifferential operators: Symbols

Consider a differential operator with variable coefficients:

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha .$$

Then

$$\begin{aligned} Pu &= \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha u = \sum_{|\alpha| \leq m} \frac{a_\alpha(x)}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\xi} \xi^\alpha \widehat{u}(\xi) \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\xi} \sum_\alpha a_\alpha(x) \xi^\alpha \widehat{u}(\xi) \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathrm{i}x\xi} p(x,\xi) \widehat{u}(\xi) \mathrm{d}\xi \end{aligned} \tag{4.1}$$

where

$$p(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

is the symbol of the operator. Now we notice that the last integral makes sense not only when $p(x,\xi)$ is a polynomial in ξ, but for more general functions, which we will call *symbols*.

In the next section we will study some classes of functions which will allow us to define operators as in (4.1) and to study their properties.

4.1 Symbols

In this section we define some classes of symbols which we will use to develop the theory of pseudodifferential operators. The first class of symbols models the differential operators with variable coefficients.

Definition 4.1 (Symbols on $\mathbb{R}^n$)**.** *Let $m \in \mathbb{R}$. A smooth function $a(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$ is a* symbol of order m *if $\forall \alpha, \beta \in \mathbb{N}_0^d$, there exists $C_{\alpha\beta} > 0$ such that*

$$\left| \partial_x^\alpha \, \partial_\xi^\beta a(x,\xi) \right| \leq C_{\alpha\beta} \, \langle \xi \rangle^{m-|\beta|} , \qquad \forall x,\xi \in \mathbb{R}^d. \tag{4.2}$$

We denote the set of symbols of order m by $\mathcal{S}^m$.

Moreover we define $\mathcal{S}^{-\infty} := \cap_{m\in\mathbb{R}} \mathcal{S}^m$.

Remark 4.2. *Symbols in this class gain decay in ξ every time we take a derivative in the ξ variable, while do not improve decay when we derive w.r.t. the x variable. The model that we have in mind is the one of linear variable coefficients differential operators. Sometimes, as we shall see, it is useful to define different classes of symbols..*

We endow $\mathcal{S}^m$ with the family of seminorms

$$\wp_k^m(a) := \sum_{|\alpha|+|\beta| \leq k} \sup_{x,\xi\in\mathbb{R}^d} \left(\langle \xi \rangle^{-m+|\beta|} \left| \partial_x^\alpha \, \partial_\xi^\beta a(x,\xi) \right| \right) \tag{4.3}$$

With these seminorms $\mathcal{S}^m$ becomes a Fréchet space, and the convergence is defined by

$$a_n \xrightarrow[n\to\infty]{\mathcal{S}^m} a \quad \Leftrightarrow \quad \wp_k^m(a_n - a) \xrightarrow[n\to\infty]{} 0 \quad \forall k \in \mathbb{N}$$

Example of symbols: We give now several examples of functions which are symbols.

1. *Differential operators*: let $m \in \mathbb{N}$ and define

$$a(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha,$$

where the $a_\alpha \in C^\infty(\mathbb{R}^d)$ are bounded with all their derivatives:

$$|\partial_x^\gamma a_\alpha(x)| \le C_{\alpha\gamma}, \qquad \forall \gamma \in \mathbb{N}_0^n.$$

Then $a \in \mathcal{S}^m$.

2. *Multiplication operators by smooth bounded functions*: let $a \in C^\infty(\mathbb{R}^d)$ bounded with its derivatives. Then $a \in \mathcal{S}^0$.

3. *Japanese bracket*: let $a(\xi) := \langle\xi\rangle$. Then $a \in \mathcal{S}^1$. More generally $\langle\xi\rangle^m \in \mathcal{S}^m$.
To see it, use Faà di Bruno formula: let $f(t) = (1+t)^{m/2}$ and $b(\xi) = |\xi|^2$; then $\langle\xi\rangle^m = f(b(\xi))$. Using that $\left|f^{(\ell)}(t)\right| \le C_\ell(1+t)^{m/2-\ell}$ and that $b \in \mathcal{S}^2$ we get

$$\begin{aligned}\partial_\xi^\beta\left[f(b(\xi))\right] &= \sum_{1\le\ell\le|\beta|} \sum_{\substack{\gamma_1+\ldots+\gamma_\ell=\beta \\ |\gamma_1|\ge1,\ldots,|\gamma_\ell|\ge1}} f^{(\ell)}(b(\xi))\,(\partial_\xi^{\gamma_1}b(\xi))\cdots(\partial_x^{\gamma_\ell}b(\xi)) \\ &\le C(1+|\xi|^2)^{m/2-\ell}\langle\xi\rangle^{2-|\gamma_1|}\cdots\langle\xi\rangle^{2-|\gamma_n|} \le C\langle\xi\rangle^{2\ell-|\beta|-m-2\ell} \le C\langle\xi\rangle^{m-|\beta|}\end{aligned}$$

4. *Schwartz functions:* let $a \in \mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$; then $a \in \mathcal{S}^{-\infty}$.
To prove it take $m \in \mathbb{R}$ arbitrary, then $\forall\alpha,\beta \in \mathbb{N}_0^d$ one has

$$\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| = \langle\xi\rangle^{m-|\beta|}\,\langle\xi\rangle^{-m+|\beta|}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \le \langle\xi\rangle^{m-|\beta|}\,\wp_{-m+|\alpha|+2|\beta|}(a)$$

which shows $a \in \mathcal{S}^m$. Since m is arbitrary, then $a \in \mathcal{S}^{-\infty}$.
In particular $C_0^\infty \subset \mathcal{S}^{-\infty}$.

5. *Positive homogeneous functions:* let $a(\xi)$ be a positive homogeneous function of order m, i.e. $a \in C^\infty(\mathbb{R}^d\setminus\{0\},\mathbb{C})$ and

$$a(t\xi) = t^m a(\xi), \qquad \forall t > 0. \tag{4.4}$$

Typical examples are

$$a(\xi) = |\xi|^m, \qquad a(\xi) = \xi^m, \qquad m \in \mathbb{R}$$

where m can also be negative. Remark that the condition $t > 0$ is necessary, otherwise (4.4) would give only polynomial functions.

This is not strictly a symbol according to our definition, since a it is not smooth. However, we can use a typical procedure of symbolic calculus, which is to regularize the function in a neighbourhood of zero (where the function is singular) in order to get a symbol. The error produced by this procedure will be a smoothing symbol.
Concretely we take χ a smooth cut-off function, $\chi \equiv 1$ in $B_1(0)$, $\chi \equiv 0$ outside $B_2(0)$ and define

$$\widetilde{a}(\xi) := (1-\chi(\xi))\,a(\xi). \tag{4.5}$$

Let us prove that $\widetilde{a} \in \mathcal{S}^m$. By Leibnitz

$$\partial_\xi^\beta \widetilde{a}(\xi) = \sum_{\gamma \le \beta} C_{\gamma\beta} \left(\partial_\xi^{\beta-\gamma}(1-\chi(\xi)\right) (\partial_\xi^\gamma a(\xi));$$

each term with $\gamma \neq \beta$ have compact support due to $1-\chi$, hence they belong to $\mathcal{S}^{-\infty}$. It remains to check the term $(1-\chi(\xi))\partial_\xi^\beta a$. In particular we need estimates on $\partial_x^\beta a$, and we shall obtain them exploiting homogeneity. Consider first the case $\beta = 0$: then we rewrite (4.4) as

$$a(\xi) = t^{-m} a(t\xi), \tag{4.6}$$

put $t = |\xi|^{-1}$ and get

$$a(\xi) = |\xi|^m \ a(\xi/\,|\xi|).$$

In particular $(1-\chi(\xi))a(\xi)$ is supported only for $|\xi| \ge 1$, and it is bounded by

$$(1-\chi(\xi))\,|a(\xi)| \le (1-\chi(\xi))\ |\xi|^m \left| a\left(\frac{\xi}{|\xi|}\right)\right| \le \langle\xi\rangle^m \sup_{|\eta|=1} |a(\eta)| \le C \langle\xi\rangle^m .$$

The higher derivatives are treated similarly. Taking derivatives of (4.6) one gets

$$\partial_\xi^\beta a(\xi) = t^{-m+|\beta|} \, (\partial_\xi^\beta a)(t\xi),$$

hence putting again $t = |\xi|^{-1}$ we get

$$\partial_\xi^\beta a(\xi) = |\xi|^{m-|\beta|} \, (\partial_\xi^\beta a)(\xi/|\xi|)$$

and as above

$$\left|(1-\chi(\xi))\partial_\xi^\beta a(\xi)\right| \le \langle\xi\rangle^{m-|\beta|} \sup_{|\eta|=1} |\partial_\xi^\beta a(\eta)| \le C \langle\xi\rangle^{m-|\beta|} .$$

6. *Inverse of elliptic symbols:* let $p \in \mathcal{S}^m$ be *elliptic of order* m in $\Omega \times B_R(0) \subset \mathbb{R}^d \times \mathbb{R}^d$, meaning that

$$p(x,\xi) \ge c|\xi|^m \qquad \forall x \in \Omega, \ \ \forall |\xi| \ge R.$$

We define an "approximate inverse" of $p(x,\xi)$ in the following way: take $\chi \equiv 1$ in $B_{R+1}(0)$, $\chi \equiv 0$ outside $B_{R+2}(0)$ and $\eta \in C_0^\infty(\Omega)$. Define

$$a(x,\xi) := \eta(x)\,\frac{1-\chi(\xi)}{p(x,\xi)};$$

then $a \in \mathcal{S}^{-m}$. Indeed a is well defined since $p(x,\xi) \neq 0$ where the cut-off functions are not identically zero, and on this set

$$|a(x,\xi)| \le C \langle\xi\rangle^{-m} .$$

Then we compute the derivatives. Again by Leibnitz one has

$$\partial_x^\alpha \partial_\xi^\beta a = \sum_{\substack{\alpha' \le \alpha \\ \beta' \le \beta}} C_{\alpha\alpha'\beta\beta'} (\partial_x^{\alpha-\alpha'}\eta)\,(\partial_\xi^{\beta-\beta'}(1-\chi))\,(\partial_x^{\alpha'}\partial_\xi^{\beta'}\frac{1}{p(x,\xi)})$$

The only term which has not compact support is when $\alpha' = \alpha$, $\beta' = \beta$, namely

$$\eta(x)\,(1-\chi(\xi))\,\partial_x^\alpha \partial_\xi^\beta \frac{1}{p(x,\xi)}.$$

Thus we compute $\partial_x^\alpha \partial_\xi^\beta \frac{1}{p(x,\xi)}$. We use Faà di Bruno formula (1.1) with $f(t) = \frac{1}{t}$ and variables $z = (x,\xi) \in \mathbb{R}^{2d}$, multindex given by $\delta = (\alpha, \beta) \in \mathbb{N}^{2d}$. Since $f^{(\ell)}(t) = C_\ell t^{-1-\ell}$ we get

$$\begin{aligned}
\partial_z^\delta\,[f(p(z)] &= \sum_{1\le \ell \le |\alpha|+|\beta|} \sum_{\substack{\gamma_1+\ldots+\gamma_\ell=\delta \\ |\gamma_1|\ge 1,\ldots,|\gamma_\ell|\ge 1}} f^{(\ell)}(p(x,\xi))\,(\partial_z^{\gamma_1} p(z)) \cdots (\partial_z^{\gamma_\ell} p(z)) \\
&= \sum_{\substack{(\alpha_1,\beta_1)+\ldots+(\alpha_\ell,\beta_\ell)=(\alpha,\beta) \\ |\gamma_1|\ge 1,\ldots,|\gamma_\ell|\ge 1}} \frac{1}{(p(x,\xi))^{\ell+1}}\,(\partial_x^{\alpha_1}\partial_\xi^{\beta_1} p(x,\xi)) \cdots (\partial_x^{\alpha_\ell}\partial_\xi^{\beta_\ell} p(x,\xi))
\end{aligned}$$

We estimate using that $p(x,\xi) \in \mathcal{S}^m$. We get

$$\begin{aligned}
\left|\eta(x)\,(1-\chi(\xi))\,\partial_x^\alpha \partial_\xi^\beta \frac{1}{p(x,\xi)}\right| &\le C \frac{1}{\langle \xi \rangle^{m(\ell+1)}}\, \langle \xi \rangle^{m-|\beta_1|} \cdots \langle \xi \rangle^{m-|\beta_\ell|} \\
&\le C \langle \xi \rangle^{\ell m - |\beta| - m(\ell+1)} \le C \langle \xi \rangle^{-m-|\beta|}
\end{aligned}$$

which proves that this last term belongs to $\mathcal{S}^{-m}$.

7. *Cut-off functions in cones:* It is possible to construct symbols supported in cones. For example let us take $\mathbb{R}^2$. We claim that there exists a symbol $\chi \in \mathcal{S}^0(\mathbb{R}^2)$ such that $\operatorname{supp}\chi \subset \{\xi\colon |\xi_2| \le C\langle \xi_1\rangle\}$, namely it is supported in a cone. We construct χ as following: pick $\widetilde{\chi} \in C_0^\infty(\mathbb{R})$, $\widetilde{\chi} \equiv 1$ in $B_1(0)$, $\widetilde{\chi} \equiv 0$ outside $B_2(0)$. Define

$$\chi(\xi_1,\xi_2) = \widetilde{\chi}\left(\frac{\xi_2}{\langle \xi_1 \rangle}\right)$$

Then

$$\partial_{\xi_1}\chi(\xi_1,\xi_2) = c\widetilde{\chi}^{(1)}\left(\frac{\xi_2}{\langle \xi_1\rangle}\right)\frac{\xi_2}{\langle \xi_1\rangle^2}\partial_{\xi_1}\langle \xi_1\rangle \Longrightarrow |\partial_{\xi_1}\chi(\xi_1,\xi_2)| \le C\widetilde{\chi}^{(1)}\left(\frac{\xi_2}{\langle \xi_1\rangle}\right)\langle \xi_1\rangle^{-1} \le C\langle \xi\rangle^{-1}$$

Since on the support of $\widetilde{\chi}$ one has $\langle \xi \rangle \le C\langle \xi_1\rangle$. The other derivative is similar. The general case is done with Faà di Bruno formula, and left as an exercise.

Examples of not symbols: it is important to have in mind what is not a symbol.

1. *Functions unbounded in x:* for example $a(x,\xi) = |x|^2 + |\xi|^2$ is not a symbol, since it is unbounded in the x variable.

2. *Complex exponential:* $e^{i\xi}$ is not a symbol, since its derivatives do not gain decay in ξ.

3. *Functions depending on a subset of the variables ξ:* let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $n \ge 2$, have the form $a(x,\xi) \equiv a(x,\xi_1)$ (namely it depends only on the variable ξ_1). Then

$$a \in \mathcal{S}^m \Longleftrightarrow a \text{ is a differential operator in } \xi_1 \text{ of order } m \text{ with smooth bounded coeff}$$

Indeed if a is a differential operator of order m, it is a symbol in $\mathcal{S}^m$.
Assume now that $a \in \mathcal{S}^m$. Then by definition of the class we have

$$\left|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} a(x,\xi_1)\right| \leq C \langle \xi \rangle^{m-\beta_1} \leq C \frac{1}{(1+\xi_1^2+\xi_2^2+\ldots+\xi_n^2)^{(m-\beta_1)/2}}$$

Hence if $\beta_1 > m$, we can take the limit when $\xi_2 \to \infty$ in the r.h.s. of the formula above, and find that $\partial_{\xi_1}^{\beta_1} a(x,\xi_1) \equiv 0$. This shows that a is a polynomial in ξ_1 with smooth bounded coefficients.

Note however that we can transform such functions in symbols by appropriate cut-off. It is sufficient to define

$$\widetilde{a}(x,\xi) := \widetilde{\chi}\left(\frac{\xi_2}{\langle \xi_1 \rangle}\right) a(x,\xi)$$

with $\widetilde{\chi} \in C_0^\infty(\mathbb{R})$, $\widetilde{\chi} \equiv 1$ in $B_1(0)$, $\widetilde{\chi} \equiv 0$ outside $B_2(0)$. Then using the properties of the cut-off functions on cone, one checks that $\widetilde{a}$ is a symbol.
Indeed

$$\left|\partial_x^\alpha \partial_\xi^\beta \widetilde{a}\right| \leq \sum_{\gamma \leq \beta} C_{\gamma\beta} \left|\partial_x^\alpha \partial_\xi^\gamma a\right| \left|\partial_\xi^{\beta-\gamma} \widetilde{\chi}\right| \leq C' \langle \xi_1 \rangle^{m-|\gamma|} \langle \xi \rangle^{-|\beta|+|\gamma|} \widetilde{\chi}\left(\frac{\xi_2}{\langle \xi_1 \rangle}\right) \leq C'' \langle \xi \rangle^{m-|\beta|}$$

Symbol calculus. We want to know that symbols are closed with respect to some basic algebraic operations.

We define the Poisson bracket between functions a, b as

$$\{a,b\} := \partial_\xi a \, \partial_x b - \partial_x a \, \partial_\xi b \equiv \sum_j (\partial_{\xi_j} a)\,(\partial_{x_j} b) - (\partial_{x_j} a)\,(\partial_{\xi_j} b) \tag{4.7}$$

The next lemma is easy to verify, but important:

Lemma 4.3. *Let $a \in \mathcal{S}^m$, $b \in \mathcal{S}^{m'}$. Then*

(i) The product $ab \in \mathcal{S}^{m+m'}$ with the estimate: $\forall k \in \mathbb{N}$, $\exists C_k > 0$ such that

$$\wp_k^{m+m'}(a\,b) \leq C_k \, \wp_k^m(a) \, \wp_k^{m'}(b). \tag{4.8}$$

(ii) For any $\alpha, \beta \in \mathbb{N}^n$ one has $\partial_x^\alpha \partial_\xi^\beta a \in \mathcal{S}^{m-|\beta|}$ with the estimate: $\forall k \in \mathbb{N}$, $\exists C_k > 0$ such that

$$\wp_k^{m-|\beta|}(a) \leq C_k \, \wp_{k+|\alpha+\beta|}^m(a). \tag{4.9}$$

(iii) The Poisson bracket $\{a,b\} \in \mathcal{S}^{m+m'-1}$ with the estimate: $\forall k \in \mathbb{N}$, $\exists C_k > 0$ such that

$$\wp_k^{m+m'-1}(\{a,b\}) \leq C_k \, \wp_{k+1}^m(a) \, \wp_{k+1}^{m'}(b). \tag{4.10}$$

(iv) Let $f \in C^\infty(\mathbb{C},\mathbb{C})$ and $a \in \mathcal{S}^0$. Then $f(a) \in \mathcal{S}^0$.

(v) Let $f \in C^\infty(\mathbb{R},\mathbb{C})$ such that for some $\rho \in \mathbb{R}$, one has

$$\left|\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} f(t)\right| \leq C_\ell \,(1+|t|)^{\rho-\ell}, \qquad \forall \ell \in \mathbb{N}.$$

Let $a \in \mathcal{S}^m$ real valued. Then $f(a(x,\xi)) \in \mathcal{S}^{\rho m}$.

Proof. (*i*) just use Leibnitz rule:

$$\left|\partial_x^\alpha \partial_\xi^\beta (ab)\right| \leq \sum_{\substack{\alpha'\leq\alpha\\ \beta'\leq\beta}} C_{\alpha\beta}^{\alpha'\beta'} \left|\partial_x^{\alpha-\alpha'}\partial_\xi^{\beta-\beta'} a\right| \left|\partial_x^{\alpha'}\partial_\xi^{\beta'} b\right|$$
$$\leq \left(\sum_{\substack{\alpha'\leq\alpha\\ \beta'\leq\beta}} C_{\alpha\beta}^{\alpha'\beta'}\right) \wp^m_{|\alpha+\beta|}(a)\, \wp^{m'}_{|\alpha+\beta|}(b)\, \langle\xi\rangle^{m+m'-|\beta|}$$

Then taking $|\alpha+\beta|\leq k$ we prove

$$\wp_k^{m+m'}(ab) \leq \sum_{|\alpha+\beta|\leq k} \widetilde{C}_{\alpha\beta}\, \wp^m_{|\alpha+\beta|}(a)\, \wp^{m'}_{|\alpha+\beta|}(b) \leq C_k \wp_k^m(a)\, \wp_k^{m'}(b)$$

(*ii*) it is sufficient to note that

$$\left|\partial_x^\gamma \partial_\xi^\delta (\partial_x^\alpha \partial_\xi^\beta a)\right| \leq C\, \langle\xi\rangle^{m-|\beta|-|\delta|}\ \wp^m_{|\alpha+\beta+\gamma+\delta|}(a)$$

from which the estimate follows. follows by Leibnitz rule, while (*ii*) by taking derivatives.
(*iii*) remark that

$$\underbrace{(\partial_{\xi_j} a)}_{\in\mathcal{S}^{m-1}}\ \underbrace{(\partial_{x_j} b)}_{\in\mathcal{S}^{m'}} - \underbrace{(\partial_{x_j} a)}_{\in\mathcal{S}^{m}}\ \underbrace{(\partial_{\xi_j} b)}_{\in\mathcal{S}^{m'-1}} \in \mathcal{S}^{m+m'-1}$$

by item (*i*) and (*ii*).
(*iv*) use Faà di Bruno formula (1.1) and observe that $f^{(\ell)}(a(x,\xi))$ is bounded $\forall\ell$, being $a\in\mathcal{S}^0$.
(*v*) By Faà di Bruno

$$\partial_x^\alpha\partial_\xi^\beta f(a) = \sum_{1\leq\ell\leq|\beta|}\ \sum_{\substack{(\alpha_1,\beta_1)+\ldots+(\alpha_\ell,\beta_\ell)=(\alpha,\beta)\\ |\gamma_1|\geq 1,\ldots,|\gamma_\ell|\geq 1}} f^{(\ell)}(a(x,\xi))\,(\partial_x^{\alpha_1}\partial_\xi^{\beta_1} a(x,\xi))\cdots(\partial_x^{\alpha_\ell}\partial_\xi^{\beta_\ell} a(x,\xi))$$

Using the properties of f

$$\left|f^{(\ell)}(a(x,\xi))\right| \leq C_\ell (1+|a(x,\xi)|)^{\rho-\ell} \leq C_\ell C\ \langle\xi\rangle^{m(\rho-\ell)},$$

hence

$$\left|\partial_x^\alpha\partial_\xi^\beta f(a)\right| \lesssim \langle\xi\rangle^{m(\rho-\ell)}\, \langle\xi\rangle^{m-|\beta_1|}\ldots\langle\xi\rangle^{m-|\beta_\ell|} \lesssim \langle\xi\rangle^{m(\rho-\ell)}\,\langle\xi\rangle^{\ell m-|\beta|} \lesssim \langle\xi\rangle^{m\rho-|\beta|}$$

□

Remark 4.4. *For any $m\leq m'$, $\mathcal{S}^m\subset\mathcal{S}^{m'}$.*

Remark 4.5. *If $a\in\mathcal{S}^m$, then $a\in\mathcal{S}'$.*

Different classes of symbols. According to the problem you have in front, you might want to change the classes of symbols, in order to include one or more of the functions that are not "classical" symbols. We shall see two different classes of symbols:

Definition 4.6 (Symbols "à la Hörmander")**.** *For $\rho,\delta\in[0,1]$, $m\in\mathbb{R}$ define $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{C})$ the set of smooth functions $a(x,\xi)\in\mathbb{C}^\infty(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{C})$ such that for any $\alpha,\beta\in\mathbb{N}_0^d$, there exists $C_{\alpha\beta}>0$ such that*

$$\left|\partial_x^\alpha\partial_\xi^\beta a(\xi,\xi)\right| \leq C_{\alpha\beta}\,\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|}$$

Remark that $S^m \equiv S^m_{1,0}$.

Example 4.7. *The function* $e^{\mathrm{i}\xi} \in \mathcal{S}^0_{0,0}$. *The function* $e^{\mathrm{i}x\xi}\chi(x)$ *(χ smooth cut off) belongs to* $\mathcal{S}^0_{0,1}$. *The function* $e^{\mathrm{i}x\sqrt{\xi}}\chi(x) \in \mathcal{S}^0_{\frac{1}{2},\frac{1}{2}}$.
Such classes are useful in the theory of water waves. For example $S^0_{\frac{1}{2},\frac{1}{2}}$ *has been used by T. Alazard, P. Baldi, D. Han-Kwan for the control of water waves.*

Another classes that we will use is adapted to include $|x|^2 + |\xi|^2$ as symbol; this is useful when studying the quantum harmonic oscillator $H = -\Delta + |x|^2$, whose symbol is $|x|^2 + |\xi|^2$.

Definition 4.8 (Isotropic symbols)**.** *For* $m \in \mathbb{R}$, *define* $\mathcal{S}^m_{ho}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$ *the set of smooth functions* $a(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{C})$ *such that for any* $\alpha, \beta \in \mathbb{N}_0^d$, *there exists* $C_{\alpha\beta} > 0$ *such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(\xi,\xi)\right| \leq C_{\alpha\beta} \langle x,\xi\rangle^{m-|\beta|-|\alpha|}$$

where $\langle x,\xi\rangle = \left(1 + |x|^2 + |\xi|^2\right)^{1/2}$.

Example 4.9. *The function* $x_j \in \mathcal{S}^1_{ho}$. *The function* $|x|^2 + |\xi|^2 \in \mathcal{S}^2_{ho}$.

Remark that if $a \in \mathcal{S}^m_{ho}$, then $\partial_x a, \partial_\xi a \in \mathcal{S}^{m-1}_{ho}$. As a consequence if $a \in \mathcal{S}^m_{ho}$, $b \in \mathcal{S}^{m'}_{ho}$, then $ab \in \mathcal{S}^{m+m'-2}_{ho}$.

4.2 Asymptotic sums

Let us consider a sequence of symbols $a_j \in \mathcal{S}^{m_j}$, $j \in \mathbb{N}$, with $m_j \searrow -\infty$. We want to give a meaning to the expression

$$\sum_{j\geq 0} a_j$$

knowing well that we have no hope for the infinite sum to be convergent. The sum is to be understood in an asymptotic sense:

Definition 4.10. *A symbol a is asymptotic to* $\sum_j a_j$ *(we shall write* $a \sim \sum_j a_j$*) if*

$$a - \sum_{j=0}^{k} a_j \in \mathcal{S}^{m_{k+1}}, \qquad \forall k \geq 0$$

The idea of asymptotic calculus is that $\sum_j a_j$ approximates better and better the symbol a, in the sense that the remainder $a - \sum_{j=0}^k a_j$ is a symbol more and more decaying at infinity in the ξ variable.

Lemma 4.11. *Let* $a_j \in \mathcal{S}^{m_j}$, $m_j \searrow -\infty$. *Then* $\exists\, a \in \mathcal{S}^m$ *(unique up to a smoothing symbol) such that* $a \sim \sum_j a_j$.

Proof. Let $\chi \in C_0^\infty$, $\chi \equiv 1$ in $B_1(0)$. Pick up a sequence $\{\epsilon_j\}_{j\geq 1}$, $\epsilon_j \in (0,1)$, $\epsilon_j \to 0$ sufficiently fast (to be fixed later). We put

$$b_j(x,\xi) := (1 - \chi(\epsilon_j \xi))\, a_j(x,\xi) \in \mathcal{S}^{m_j}$$

and

$$a(x,\xi) := \sum_j b_j(x,\xi) = \sum_j (1 - \chi(\epsilon_j\xi))\, a_j(x,\xi)$$

The sum is well defined: indeed fix $\xi \in \mathbb{R}^n$, then $b_j \equiv 0$ if $|\epsilon_j \xi| \leq 1$, i.e. $|\epsilon_j| \leq \frac{1}{|\xi|}$. Since $\epsilon_j \to 0$, for any ξ fixed there is just a finite number of terms in the sum which are not zero; thus a is a well defined smooth function.

Fix $k > 0$ arbitrary: we want to prove that $a - \sum_{j \leq k} a_j \in \mathcal{S}^{m_{k+1}}$. So fix $\alpha, \beta \in \mathbb{N}^n$: we want to find $C_{\alpha\beta} > 0$ so that

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(a - \sum_{j \leq k} a_j \right) \right| \leq C_{\alpha\beta} \langle \xi \rangle^{m_{k+1} - |\beta|} . \tag{4.11}$$

The difficulty in proving this estimate is that we want uniformity in ξ, so we must be careful with the sum which defines a. We proceed in the following way: having fixed α, β, we choose $N \in \mathbb{N}$ so that

$$|\alpha| + |\beta| \leq N, \qquad m_N + 1 \leq m_{k+1}$$

As $a = \sum_j b_j$, we write

$$a - \sum_{j \leq k} a_j = \sum_{j \leq k} (b_j - a_j) \tag{4.12}$$

$$+ \sum_{k+1 \leq j \leq N} b_j \tag{4.13}$$

$$+ \sum_{N+1 \leq j} b_j \tag{4.14}$$

Consider first (4.12): since $b_j \equiv a_j$ outside a compact set, $b_j - a_j \in \mathcal{S}^{-\infty}$ and therefore it fulfills an estimate as (4.11).

Coming to (4.13), it is a finite sum and $b_j \in \mathcal{S}^{m_j} \subset \mathcal{S}^{m_{k+1}}$ (since $m_j \leq m_{k+1}$).

We are left with (4.14): this is the difficult term, since the number of terms in this sum depends on ξ, and we want the estimate to be uniform in ξ.

It is time to fix the sequence $\epsilon_j \to 0$. We claim that it is possible to choose them in order for this condition to be fulfilled:

$$\forall j : \quad \left| \partial_x^\alpha \partial_\xi^\beta b_j(x, \xi) \right| \leq 2^{-j} \langle \xi \rangle^{1 + m_j - |\beta|} \quad \forall |\alpha + \beta| \leq j. \tag{4.15}$$

Assume for the moment this is true. Then we estimate $\sum_{N+1 \leq j} b_j$ as following:

$$\begin{aligned}
\left| \partial_x^\alpha \partial_\xi^\beta \sum_{j \geq N+1} b_j \right| &\leq \sum_{j \geq N+1} 2^{-j} \langle \xi \rangle^{1 + m_j - |\beta|} && \text{as } |\alpha + \beta| \leq N \leq j \\
&\leq \sum_{j \geq N+1} 2^{-j} \langle \xi \rangle^{1 + m_N - |\beta|} && \text{as } m_j \leq m_N \\
&\leq \sum_{j \geq N+1} 2^{-j} \langle \xi \rangle^{m_{k+1} - |\beta|} && \text{as } m_N + 1 \leq m_{k+1} \\
&\leq C \langle \xi \rangle^{m_{k+1} - |\beta|}
\end{aligned}$$

which proves that (4.14) belongs to $\mathcal{S}^{m_{k+1}}$.
All we have left is to prove (4.15). We distinguish three cases:

- If $|\xi| \leq 1/\epsilon_j$, then $1 - \chi(\epsilon_j \xi) \equiv 0$, hence $b_j \equiv 0$ and the estimate is true.

- If $\frac{1}{\epsilon_j} \le |\xi| \le \frac{2}{|\epsilon_j|}$ then, using $\epsilon_j^{|\gamma|} \langle\xi\rangle^{|\gamma|} \le 4^{|\gamma|}$ and $\epsilon_j \langle\xi\rangle \ge 1$

$$\begin{aligned}\left|\partial_x^\alpha \partial_\xi^\beta b_j\right| &\le \sum_{\gamma\le\beta} C_\gamma \left|\partial_\xi^\gamma (1-\chi(\epsilon_j\xi))\right| \left|\partial_x^\alpha \partial_\xi^{\beta-\gamma} a_j\right| \le \sum_{\gamma\le\beta} C_\gamma \epsilon_j^{|\gamma|} \left|\partial_x^\alpha \partial_\xi^{\beta-\gamma} a_j\right| \\ &\le C'_{\alpha\beta} \sum_{\gamma\le\beta} \epsilon_j^{|\gamma|} \langle\xi\rangle^{m_j-|\beta|+|\gamma|} \le C'_{\alpha\beta}\, 4^{|\beta|}\, \epsilon_j \langle\xi\rangle^{1+m_j-|\beta|}\end{aligned}$$

- If $|\xi| \ge \frac{2}{|\epsilon_j|}$ then $1-\chi(\epsilon_j\xi) \equiv 1$ and, using $2 \le |\xi|\epsilon_j$,

$$\left|\partial_x^\alpha \partial_\xi^\beta b_j\right| = \left|\partial_x^\alpha \partial_\xi^\beta a_j\right| \le C''_{\alpha\beta} \langle\xi\rangle^{m_j-|\beta|} \le C''_{\alpha\beta}\epsilon_j \langle\xi\rangle^{1+m_j-|\beta|}$$

In conclusion is sufficient to choose the sequence ϵ_j so that

$$\epsilon_j \le 2^{-j} \min_{|\alpha+\beta|\le j} \left(\frac{1}{C''_{\alpha\beta}}, \frac{1}{C'_{\alpha\beta}\, 4^{|\beta|}}\right)$$

to have (4.15). This concludes the proof. □

An example of such a construction which is very much used in the literature is the following: take a sequence of $a_j(x,\xi)$, homogeneus of degree $m-j$ in ξ, smooth for $|\xi| \ne 0$, localized in x. Then we can construct b_j as above, and we already proved at the beginning of the section that $b_j \in \mathcal{S}^{m-j}$. Lemma 4.11 gives us a symbol in $\mathcal{S}^m$ with $a \sim \sum_j a_j$.

The class of such symbols is called *classic*. If $a \sim \sum_j a_{m-j}$, then the first term of the expansion, namely a_m, is called the *principal symbol*.

The importance of classical symbols is that the symbols that one meets in applications are classical.

Exercises:

1. Let $b(x)$ be smooth and bounded with its derivatives. Prove that $b \in \mathcal{S}^0_{\frac{1}{2},\frac{1}{2}}$.
2. Assume that $a(x) \in \mathcal{S}^m_{ho}$. Prove that a is a polynomial in x.
3. Let $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\chi \in C^\infty(\mathbb{R}^n)$ supported in $\frac{1}{2} \le |\xi| \le 2$, $\chi \ne 0$ on its support. For $\lambda \ge 1$, put $a_\lambda(x,\xi) = \chi(\xi)a(x,\lambda\xi)$. Prove that $a \in \mathcal{S}^m$ if and only if $\forall k \in \mathbb{N}$, $\|a_\lambda\|_{C^k} \le c_k \lambda^m$.