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Workshop on Quantum Simulations

Sum rules and collective oscillations in spin-orbit coupled quantum gases

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CNR-INO

- Recent **experimental realization** of syntethic gauge fields is providing new challenging many-body configurations in ultra-cold atomic gases

- **Conservation laws are modified** by spin coupling

- **New quantum phases** (stripes, spin polarized) and new phase transitions in both Bose and Fermi gases

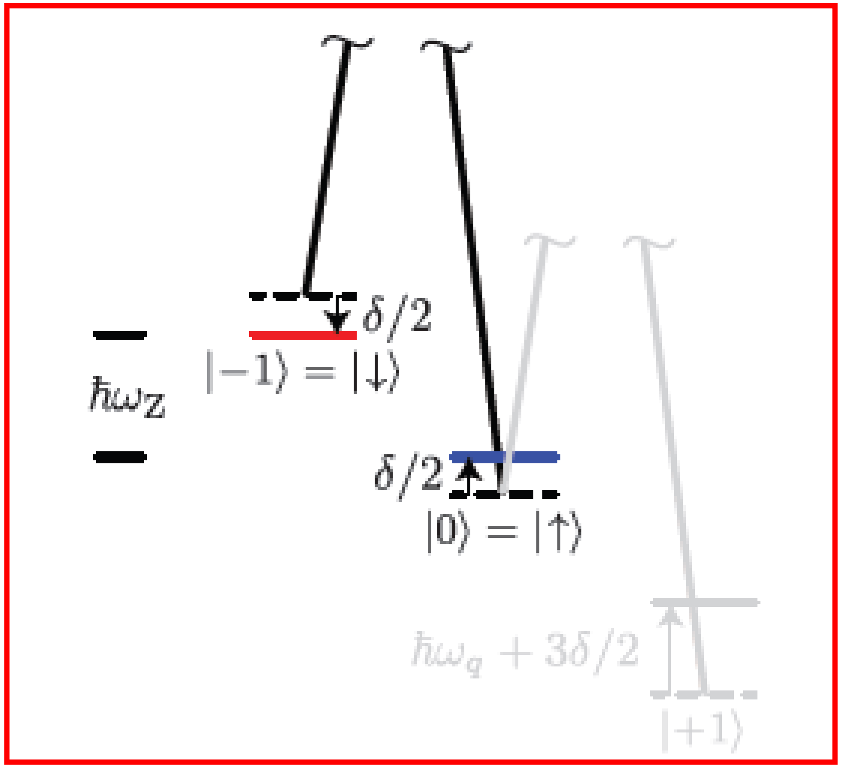
- **New dynamic** properties

- **Sum rules**: well suited tool to emphasize the role of the new conservation laws in the dynamical behavior

- Focus of the present paper: **center of mass oscillation** in in spinor BEC gases with equal Rashba and Dresselhaus coupling. Key role played by **spin polarizability**

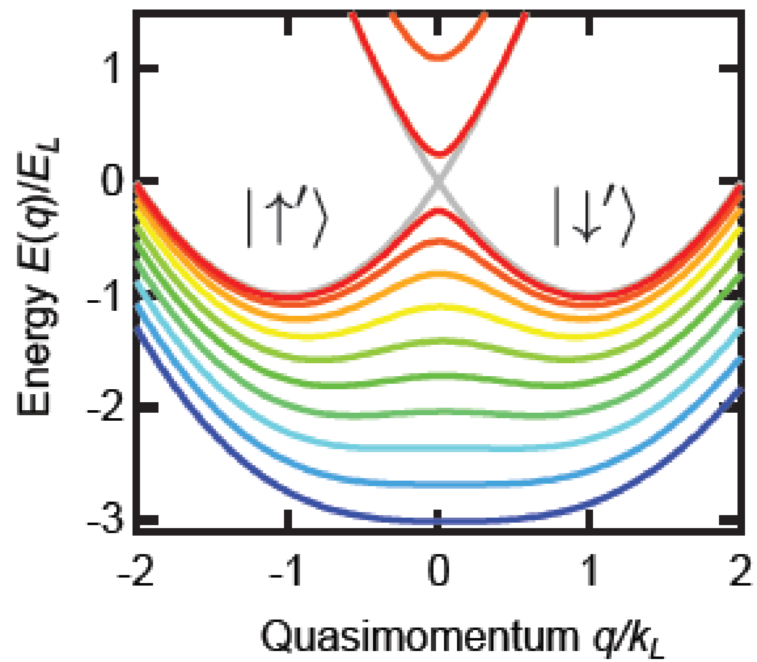
Center of mass oscillation in **spin-orbit** coupled BEC

(Yun Li, Giovanni Martone and S.S, arXive: 1205.6398)



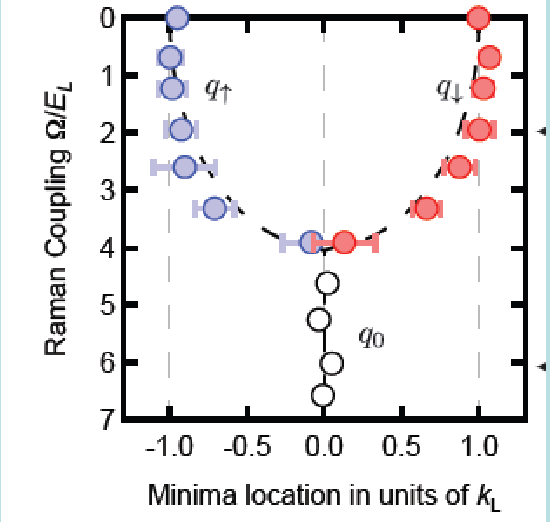
$$h_0 = \frac{1}{2}[(p_x - k_0\sigma_z)^2 + p_\perp^2] + \frac{1}{2}\Omega\sigma_x + \frac{1}{2}\delta\sigma_z + V_{ext}$$

New SO Hamiltonian (equal Rashba and Dresselhaus couplings)



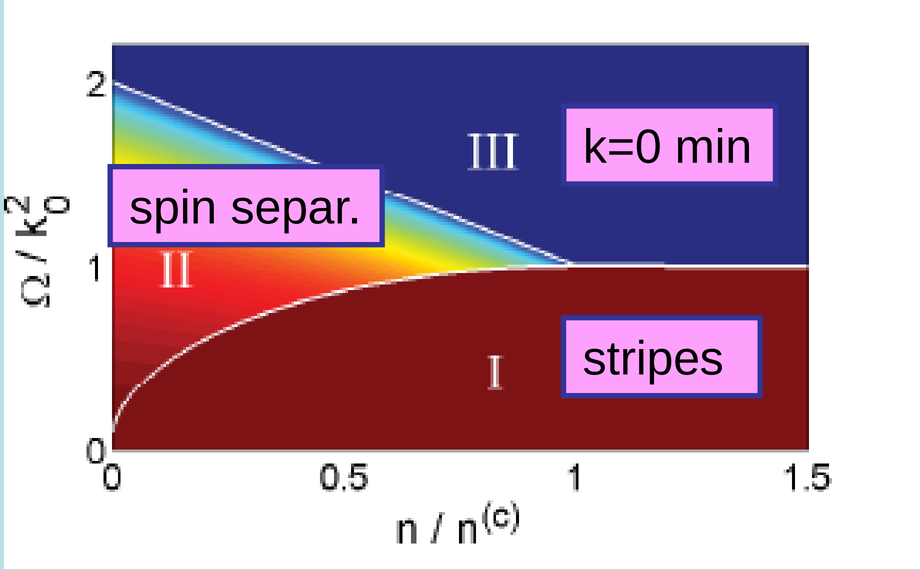
BEC with two minima

First experimental Implementation with BEC's Spielman, Nature 2011)



Theory of new quantum phases:
Ho and Zhang (PRL 2011)
Many theoretical papers (……)

Recent Trento paper (Li, Pitaevskii, S.S, PRL 2012)



Ansatz for ground state order parameter (spinor BEC gas):

$$\Psi = \sqrt{\frac{N}{V}}\left[C_+ \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} e^{ik_1 x} + C_- \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} e^{-ik_1 x} \right]$$

Minimization of energy with

$$H = \sum_i h_0(i) + \sum_{\alpha,\beta} \frac{1}{2} \int d\vec{r}\, g_{\alpha\beta} n_\alpha n_\beta$$

and

$$g_{\alpha\alpha} = g_{\beta\beta} \equiv g$$

$$\delta = 0$$

yields three different phases

$$h_0 = \frac{1}{2}[\left(p_x - k_0\sigma_z\right)^2 + p_\perp^2] + \frac{1}{2}\Omega\sigma_x + V_{ext}$$

$$n_\pm = \frac{1}{2}\sum_i (1 \pm \sigma_i^z)\delta(\vec{r} - \vec{r}_i)$$

$$\cos(k_1 / k_0) = 2\theta$$

Quantum phases of the spin-orbit coupled spinor BEC (weak coupling regime)

$$\Psi = \sqrt{\frac{N}{V}}\left[C_+\begin{pmatrix}\cos\theta\\ -\sin\theta\end{pmatrix}e^{ik_1x} + C_-\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}e^{-ik_1x}\right]$$

Ω

PHASE I (Stripes)

$$C_+ = C_-$$
$$k_1 = \sqrt{k_0^2 - \Omega^2/4}$$

$$\Omega_{I-II} = 2k_0^2\sqrt{2\gamma/(1+2\gamma)}$$

$$\gamma = (g - g_{+-})/(g + g_{+-})$$

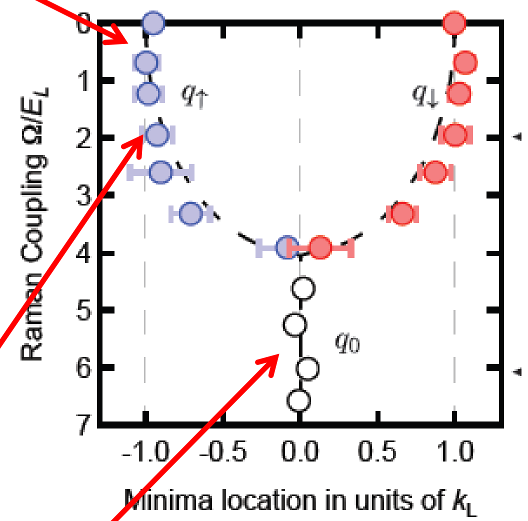
PHASE II (spin separated)

$$C_- = 0 \quad (or\ C_+ = 0)$$
$$k_1 = \sqrt{k_0^2 - \Omega^2/4}$$

$$\Omega_{II-III} = 2k_0^2$$

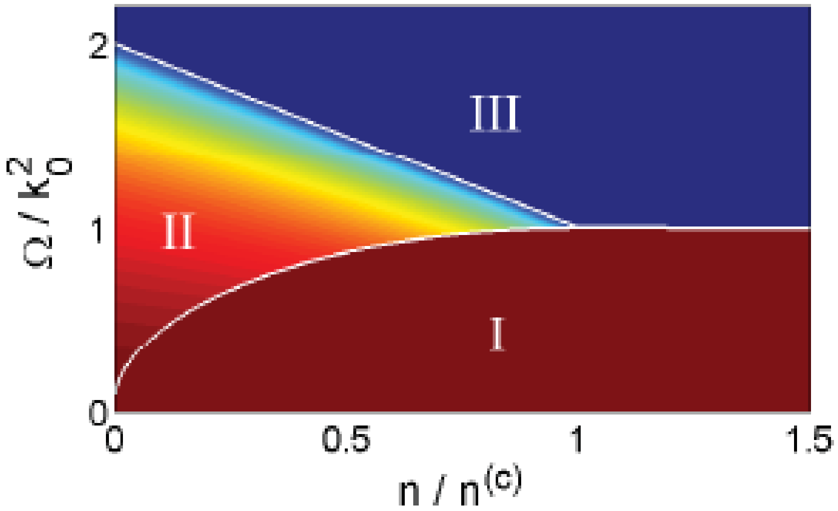
PHASE III (zero momentum)

$$C_- = 0\,;\ k_1 = 0$$



Beyond weak coupling limit

(Li et al. PRL 2012)



Main features:

- Transition frequency between Phase II and Phase III depends on interaction: $\Omega_{II-III} = 2k_0^2 - n(g - g_{+-})/2$

- Jump in the value of k_1 between Phase I and Phase II

- Existence of tri-critical point at density $n_c = \frac{k_0^2}{2g}\frac{g+g_{+-}}{g-g_{+-}}$ above which phase II disappears (transition I-III characterized by sizable jump in k_1)

Dynamics of spin-orbit coupled BEC's in harmonic trap

center of mass oscillation

Li, Martone and S.S. arXiv:1205.6398

Coupling between center of mass ($X = \sum_i x_i$) and
Spin ($\sigma = \sum_i \sigma_i$) degrees of freedom
emphasized by **commutation rule** of dipole operator

$$[H, X] = -i\left(P_x - k_0\sigma_z\right)$$

- Reflects modification of **equation of continuity**
- Implies **new dynamic** behavior of center of mass coordinate

In the absence of spin-orbit coupling dipole operator X excites a single mode with frequency ω_x (Kohn's theorem)

Changes of dynamic behavior of center of mass can be explored using sum rule approach

Relevant sum rules for the dipole operator

$$m_k(X) = \int d\omega\, S_X(\omega)\omega^k = \sum_n |<0|X|n>|(E_n - E_0)^k$$

Energy weigthed (k=1) sum rule (f-sum rule) :

$$m_1(X) = \frac{1}{2} < [X,[H,X]] >= \frac{N}{2}$$

- Comment: Despite the fact that $[H,X]$ depends on spin, the double commutator does not. Universality and model independence of f-sum rule.

Inverse energy weigthed (k=-1) sum rule

(dipole polarizability sum rule) :

The $m_{-1}(X)$ sum rule can be calculated exactly in the presence of harmonic trapping using the commutation relation:

$$[H, P_X] = i\omega_x^2 X \qquad P_X = \sum_i p_{i,x}$$

(follows from translation invariance of two-body interaction)

One finds:

$$m_{-1}(X) = \sum_n \frac{|<0|X|n>|^2}{E_n - E_0} = -\frac{i}{2\omega_x^2} <0|[X, P_X]|0> = \frac{N}{2\omega_x^2}$$

- Comment: both **energy** weighted and **inverse-energy** weighted sum rules are **independent** of **spin-orbit** coupling

Where does spin orbit coupling enter the sum rule approach ?

Cubic inverse energy weigthed (k=-3) sum rule

- Very sensitive to **low energy** part of excitation spectrum
- Can be worked out explicitly using commutation rules

First step: use commutation relation $[H, P_X] = i\omega_x^2 X$

$$m_{-3}(X) = \frac{1}{\omega_x^4}\sum_n \frac{|<0|P_X|n>|^2}{E_n - E_0} =$$

$$= \frac{1}{\omega_x^4}\sum_n \frac{|<0|P_X - k_0\sigma_z|n>|^2}{E_n - E_0} + \frac{k_0^2}{\omega_x^4}\sum_n \frac{|<0|\sigma_z|n>|^2}{E_n - E_0}$$

Second step: use commutation rule $[H, X] = -i\left(P_x - k_0\sigma_z\right)$ and identify **spin polarizability**

Cubic inverse energy weigthed (k=-3) sum rule

$$m_{-3}(X) = \frac{N}{2\omega_x^4}[1 + k_0^2 \chi(\sigma_z)]$$

with $\chi(\sigma_z) = 2m_{-1}(\sigma_z) \equiv$ spin polarizability.

Comments on sum rules:

- Results for k=1, k=-1 and k=-3 sum rules **hold exactly** for RD Hamiltonian + 2-body int. + harmonic trapping

- Hold for both **Bose** and **Fermi** statistics.

- Are **not restricted** to **mean field** regime

- k=-3 sum rule emphasizes key role played by spin-orbit coupling and **spin polarizability**

Sum rules and frequency of lowest dipole mode

Taking the ratio between $m_{-1}(X)$ and $m_{-3}(X)$ sum rules one finds useful estimate for frequency of dipole oscillation:

$$\omega_D = \omega_x \sqrt{\frac{1}{1+k_0^2\chi(\sigma_z)}}$$

Comments:

- ω_D provides rigorous **upper bound** to lowest frequency excited by dipole operator X

- f-sum rule $m_1(X)$ unsuitable to describe lowest frequency mode in the presence of spin-orbit coupling

- result for ω_D expected to be accurate for $\Omega >> \omega_x$.
  For smaller Raman coupling, lowest mode is a pure spin oscillation with no coupling with center of mass oscillation (try optimized choice for excitation operator $F = \eta\sigma_z + P_x$)

Behavior of spin polarizability

- Calculation of $\chi(\sigma_z)$ based on standard definition: evaluate changes of order parameter caused by external field $-h\sigma_z$

$$\Psi = \sqrt{\frac{N}{V}}\left[C_+ \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} e^{ik_1 x} + C_- \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} e^{-ik_1 x} \right] \Longrightarrow$$

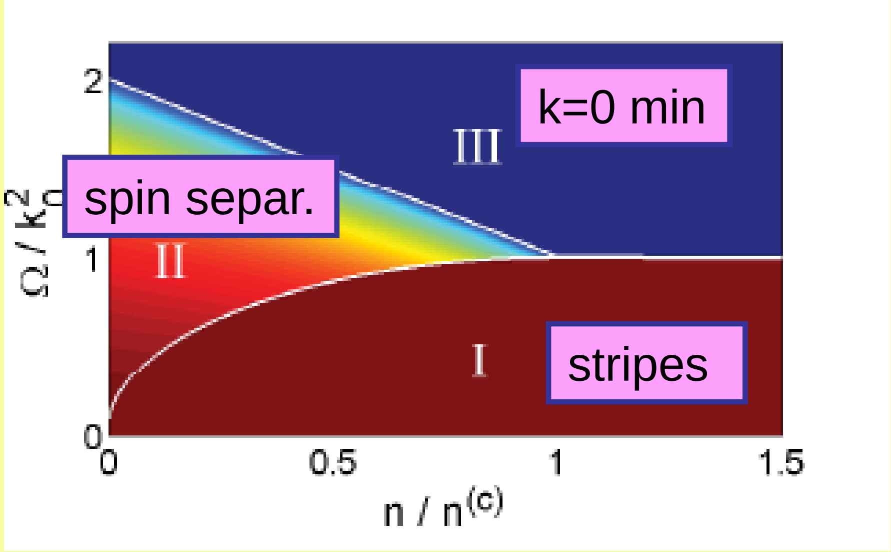
Behavior of spin polarizability

- Calculation of $\chi(\sigma_z)$ based on standard definition: evaluate changes of order parameter caused by external field $-h\sigma_z$

$$\Psi = \sqrt{\frac{N}{V}}\left[C_+ \begin{pmatrix} \cos\theta_+ \\ -\sin\theta_+ \end{pmatrix} e^{ik_+x} + C_- \begin{pmatrix} \sin\theta_- \\ -\cos\theta_- \end{pmatrix} e^{-ik_-x} \right]$$

- results for polarizability depend on the phase considered

- Non trivial results for the behavior of $\chi(\sigma_z)$ at the transition between the quantum phases



Spin polarizability of spin-orbit coupled BEC
(weak coupling limit in uniform matter)

PHASE I (stripes)

$$\chi(\sigma_z) = \frac{\Omega^2 - 4k_0^2}{(G_1 + 2G_2)\Omega^2 - 8G_2 k_0^4}$$

with

$$G_1 = n(g + g_{\uparrow\downarrow})/4$$
$$G_2 = n(g - g_{\uparrow\downarrow})/4$$

spin polarizability becomes larger and larger as $n \to 0$
Reflecting instability of non interacting gas

PHASE II (spin separated)

$$\chi(\sigma_z) = \frac{\Omega^2}{k_0^2(4k_0^2 - \Omega^2)}$$

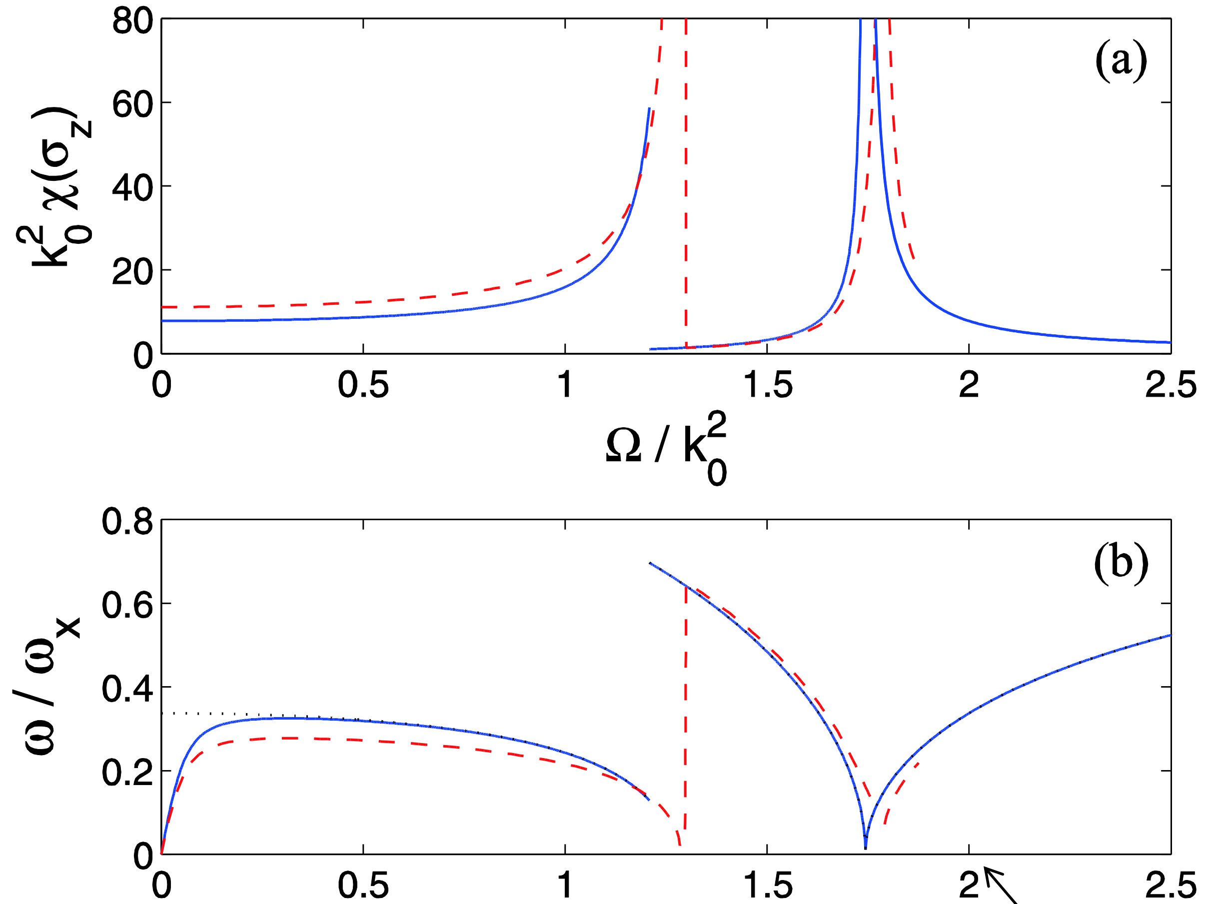
PHASE III (k=0 minimum)

$$\chi(\sigma_z) = \frac{2}{\Omega - 2k_0^2}$$

- Spin polarizability **diverges** at the transition between phases II and III (second order phase transition).
- Spin polarizability in phases II and III are **density independent**

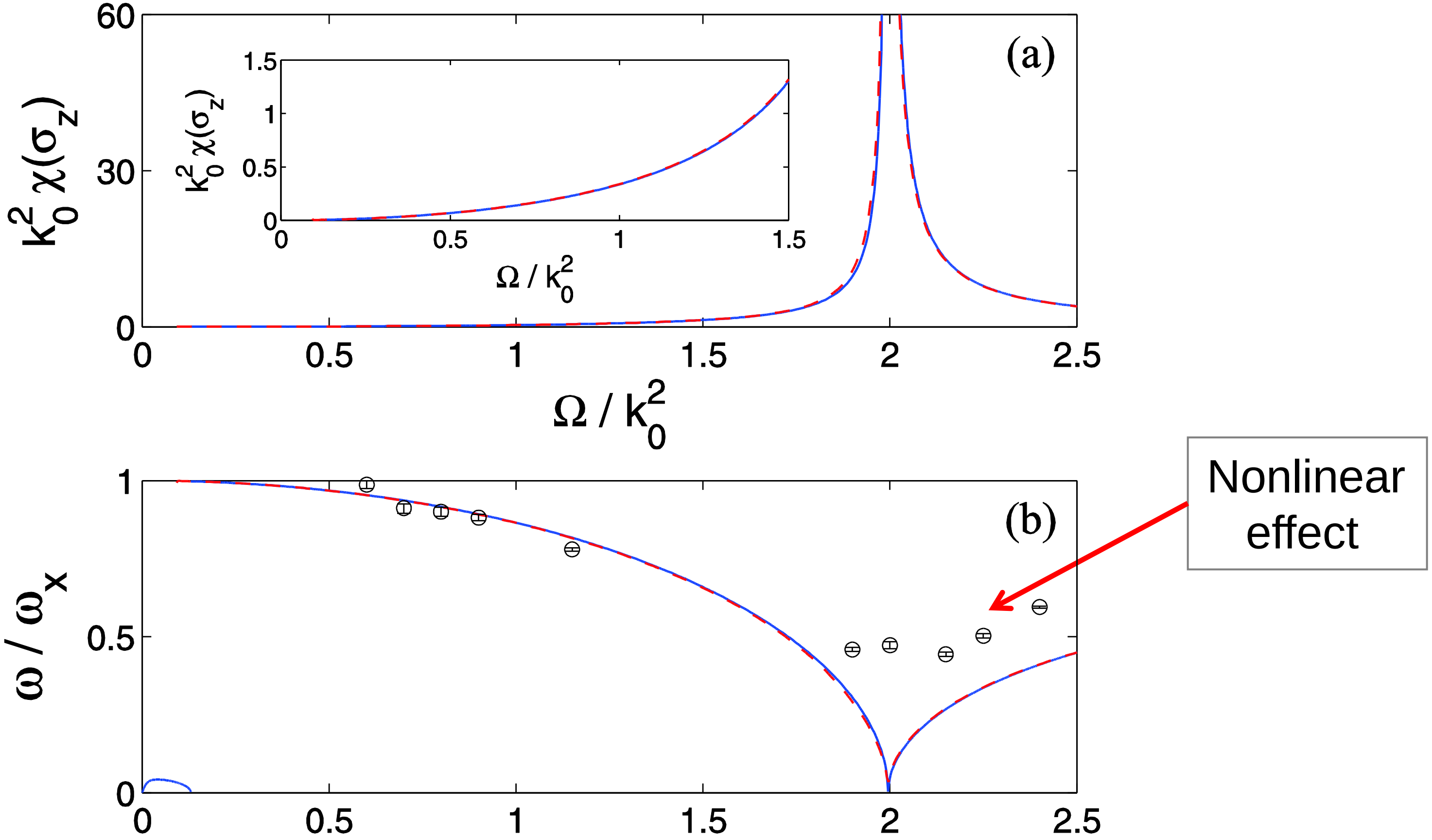
Spin polarizability and dipole frequency

for choice of parameters beyond weak coupling limit



- **shift** of transition (II-III) with respect to value $2k_0^2$
- Dipole frequency **quenched** with respect to oscillator value ω_x
- **Uniform** matter (blue) calculation of χ close to GP in **trap** (red)

Spin polarizability and dipole frequency
in weak coupling regime (exp: Chen et al (2012))



- **Uniform** matter and **GP** calculations **indistinguishible**
- **Stripe** phase I **quenched** due to smallness of ratio G_2 / G_1
- Near transition II-III nonlinear effects in exp are important

Useful analytic formula for dipole frequency in weakly interacting regime ($G_2, G_1 << k_0^2$)

$$\omega_D = \omega_x \sqrt{\frac{1}{1 + k_0^2 \chi(\sigma_z)}}$$

$$\begin{cases} \omega_D = \omega_x \sqrt{1 - \dfrac{\Omega^2}{4k_0^2}} & \text{PHASE II} \\[2ex] \omega_D = \omega_x \sqrt{1 - \dfrac{2k_0^2}{\Omega}} & \text{PHASE III} \end{cases}$$

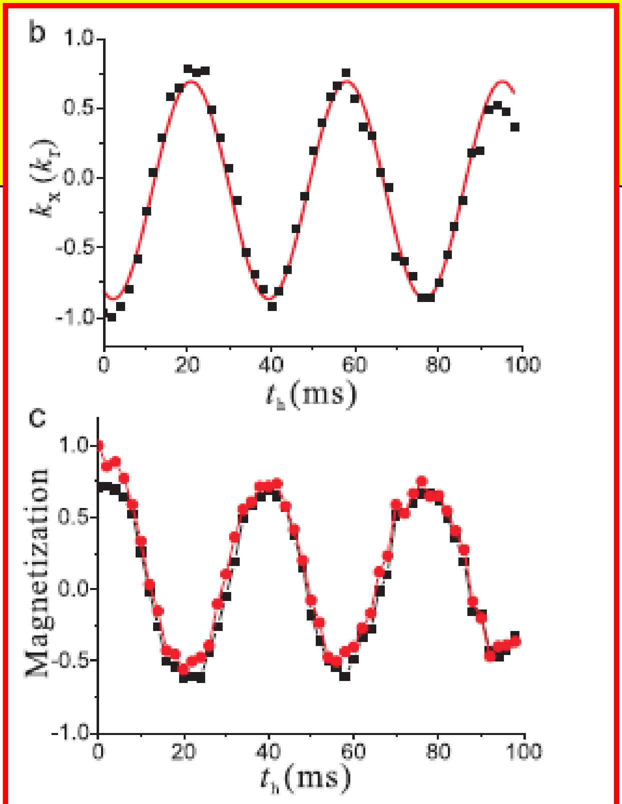
Effect of spin orbit coupling on amplitudes of oscillations

- During dipole oscillation, center of mass position X, momentum P_X and magnetization σ_Z oscillate in time

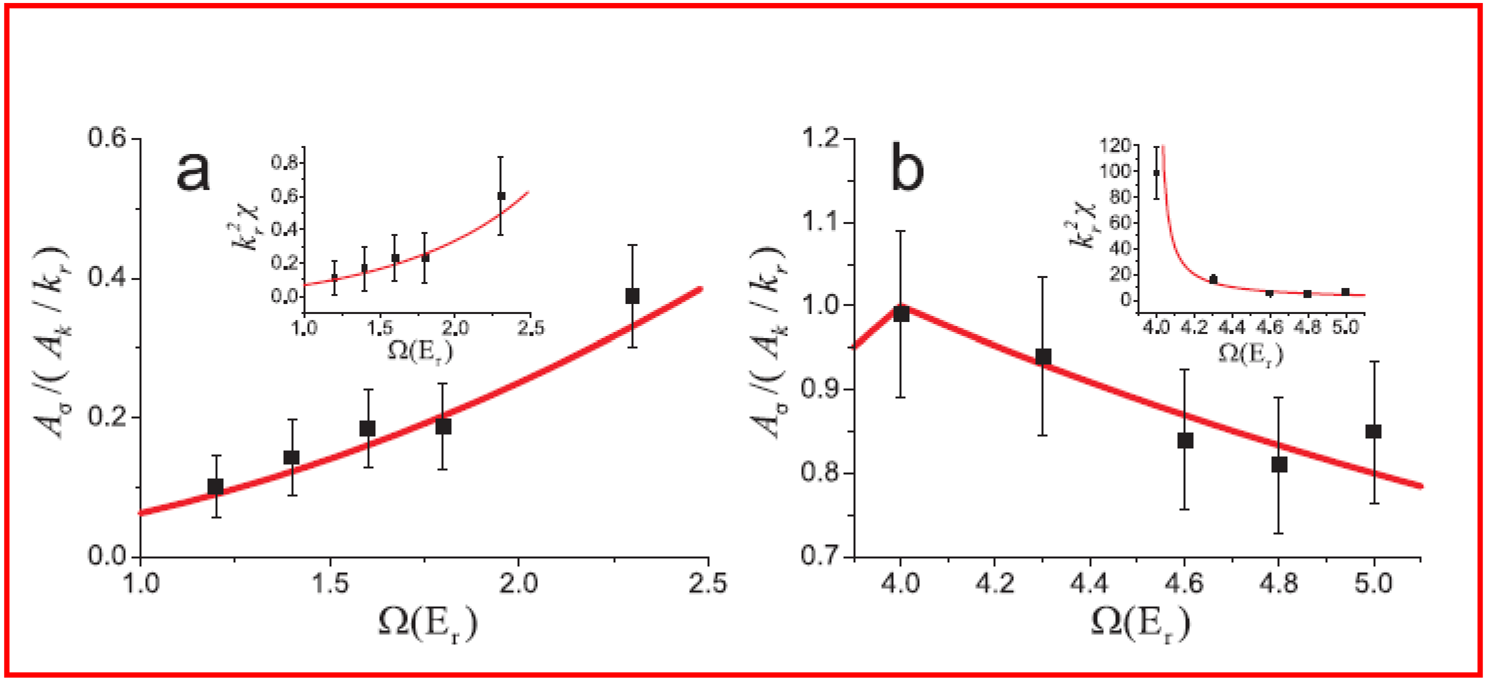
- Coupling between relative amplitudes A_X , A_P , A_σ

$$A_\sigma = A_X k_0 \omega_x \chi(\sigma_z) \frac{1}{\sqrt{1+k_0^2\chi(\sigma_z)}}$$

$$A_\sigma = \frac{A_P}{k_0} \frac{k_0^2\chi(\sigma_z)}{1+k_0^2\chi(\sigma_z)}$$



Experimental evidence of coupling of spin and momentum amplitudes near II-III phase transition (large $k_0^2\chi(\sigma_z)$)- Chern et al 2002



Measured spin polarizability from spin and momentum amplitudes (Zhang et al. 2012)

Conclusions

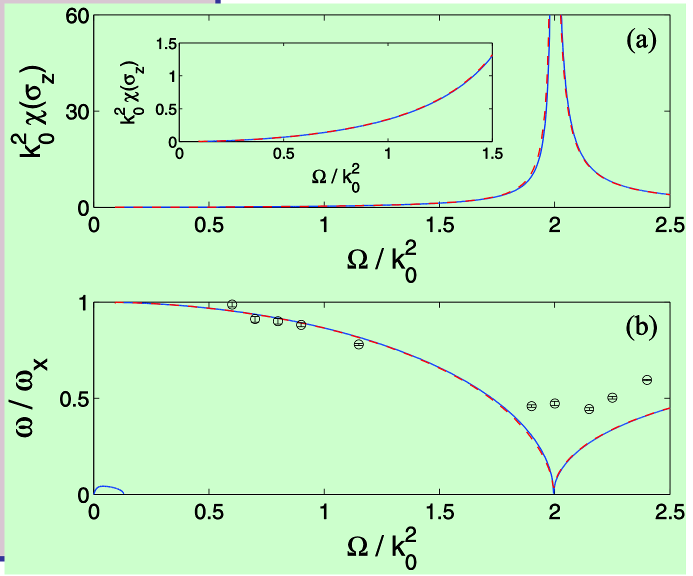
- Calculated **frequency of dipole mode** for Rashba+Dresselhaus spin coupling using **sum rule** approach

$$\omega_D = \omega_x \sqrt{\frac{1}{1 + k_0^2 \chi(\sigma_z)}}$$

- Crucial ingredient: **spin polarizability**

- Spin susceptibility exhibits **divergent** behavior at transition between spin separated and single minimum phase (**second order** phase transition)

- dipole frequency exhibits **deep minimum** at the transition



Perspectives and running projects in Trento

- **Anisotropy** of **sound** propagation and **rotonic structure** in uniform matter (G. Martone et al. In preparation)

- collective oscillations for **trapped superfluid Fermi gases**

- Collective modes for different spin-orbit coupling (ex. **Rashba**)